Laterally Driven Stochastic Motions in the Tropics

MAN-KIN MAK

Massachusetts Institute of Technology, Cambridge

(Manuscript received 22 August 1968)

ABSTRACT

A two-layer dry atmospheric model containing non-geostrophic effects and parametrized dissipation is used to investigate the large-scale tropical eddies as perturbations on a basic state which has empirical static stability and symmetric zonal current. The perturbed motion is treated as a stationary random process driven by stochastic forcing at ±30° latitude. The latter is inferred from observations at 30N. In the predicted internal statistics, 1) variance of horizontal velocity components is much larger at the upper level and decreases equatorward, 2) variances of horizontal divergence and of temperature decrease markedly equatorward, and 3) the eddies transport sensible heat and wave energy equatorward, and transport zonal momentum poleward. The eddies gain kinetic energy from pressure work on the boundaries and lose it by dissipation and conversions to zonal kinetic energy and eddy available potential energy, which is in turn depleted by radiative cooling and conversion to zonal available potential energy. The observed tropical statistics which are available agree with the corresponding theoretical values. Therefore, the large-scale eddies in low latitudes seem to be driven by the unstable baroclinic processes in higher latitudes.

Much of the predicted meridional velocity variance in the equatorial region at the upper level arises from motions of periods around 5 days, wavelength around 10,000 km, with westward phase propagation. These are similar to the disturbances in the equatorial lower stratosphere recently discovered by Yanai and Maruyama.

1. Introduction

This study is a theoretical investigation of the way in which large-scale eddies in the tropics are related to the flow in middle latitudes. Previous studies (Rosenthal, 1965; Matsuno, 1966) have treated the tropical motions essentially as having originated in situ, and the mathematical analyses have been confined to seeking solutions compatible with highly idealized poleward boundary conditions. On the other hand, the available data analyses are fragmentary (Starr and White, 1954; Obasi, 1963; Peixoto, 1960) and thus can offer only little guidance. The latter do indicate that asymmetric motions have a progressively smaller role toward the equator in the poleward transports of momentum and heat, but we have yet to determine how far the influence of eddies extends into the tropics, and we understand very little about the energetics associated with the asymmetric tropical motions.

Although the available data is sparser than in higher latitudes, it has posed some interesting questions. Riehl (1954, 1963) noted that the flow in the lower tropical troposphere is relatively steady, whereas that at higher altitudes has considerably more variability and day-to-day changes in the large-scale disturbances. Furthermore, the low-level systems in the Marshall Island region of the western Pacific and in the Caribbean Sea have often been found to move quite independently of the upper-level systems, in marked contrast to the behavior in middle latitudes. Such observations suggest that the vertical scale of asymmetric motions in the tropics could be so small that motions at one level exert very little influence upon the motions at another level. A theoretical basis for this conjecture was established by Charney’s (1963) scale analysis in which he concluded that the large-scale tropical motions, in the absence of condensation, should be even more quasi-horizontal and horizontally nondivergent than those in middle latitudes, and with little coupling in the vertical. Such large-scale eddy motions must (again, in the absence of condensation) derive their energy either locally from barotropic instability or through lateral coupling with the motions in higher latitudes. This type of lateral coupling would be associated with an equatorward flux of wave energy. This aspect of the coupling has been hinted at by Eliassen and Palm (1960) for stationary waves. Should this also be true for the transient waves, the presumed lateral coupling would supply considerable wave energy into the tropics. How far equatorward such a flux of wave energy can penetrate depends upon the rate of dissipation, the variability of the Coriolis parameter, and the interaction with the mean flow.

In view of this suggestive evidence concerning the plausibility of significant lateral coupling between the middle and low latitude circulation patterns, it is certainly of some interest to test this concept quantita-

1 Present affiliation: Department of Transport, Meteorological Branch, Toronto, Ontario, Canada.
tively. This paper does so with a model tropics that explicitly incorporates this mechanism. Specifically, we shall examine the statistics of the circulation in a model tropics driven by stochastic lateral forcing that is prescribed in terms of second-moment statistics. The latter will be obtained from actual data at 30N.

Section 2 describes the model tropics. Section 3 is an exposition of the stochastic aspect of the analysis. The problem of deducing any second-moment statistic can be resolved into two separate steps: 1) determination of the “fundamental solutions,” which in turn can be combined to obtain the response functions of the system; and 2) formulation of the statistical boundary conditions. These results can then be combined to give unique solutions for such statistics within the tropics. Section 4 describes the hypothesis of statistical independence and similarity which is used to derive the unavailable boundary statistics at 30S from those at 30N, and the method by which these are computed from data at 30N. In Section 5, the fundamental solutions are solved analytically for the special case of no dissipation and no shear in the basic state, and numerically for the more general case in which these effects are present. Finally, the various predicted second-moment statistics for the model tropics and the corresponding observed statistics are presented in Section 6. It will be seen that the forced stochastic motions in the model tropics have statistical properties which are similar to those of the real tropics, thereby establishing quantitative evidence for lateral coupling as an important energy source of tropical motions.

2. Description of the model tropics

For the rather limited objective of this study we will use a simple dry atmospheric model with the hydrostatic approximation. The horizontal coordinates are those of a Mercator projection covering the tropical region from 30S to 30N. The effects of the spherical geometry of the earth are approximated by an equatorial β-plane representation. Pressure is used as the vertical coordinate. For simplicity we write the governing equations in nondimensional forms by using \((2\Omega)^{-1}\), \(a\) and \(p_0\) (1000 mb) as time, length and pressure units, where \(\Omega\) and \(a\) are the earth’s rotation rate and radius, respectively. Thus, we define:

\[
(u', v') = (2\Omega a)^{-1} (u, v), \quad t' = 20t \]
\[
\omega' = (2\Omega p_0)^{-1} \omega, \quad p' = p_0^{-1} p \]
\[
\phi' = (2\Omega a)^{-2} \phi, \quad Q' = K(8\pi a^2)^{-1} Q \]
\[
F' = (4\Omega^2 a)^{-1} F, \quad \epsilon = -RT \frac{\partial \ln \Theta}{\partial p} p_0 \]

where \((u, v, \omega)\) are three velocity components, \(\phi\) the geopotential, \(F\) the frictional force, \(T\), \(\Theta\) temperature and potential temperature, \(Q\) the rate of heating per unit mass, \(R\) the universal gas constant, and \(K = R/C_p = 0.286\) for dry air.

For clarity we will omit the prime superscript in the following complete set of equations of motion:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{\partial \phi}{\partial x} & = -\frac{\partial \psi}{\partial y} + F_x \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \frac{\partial \phi}{\partial y} & = -\frac{\partial \psi}{\partial x} + F_y \\
\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} & = 0 \\
\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} & = \frac{4}{\rho} - \frac{Q}{\rho} \\
\end{align*}
\]  \( (2.1) \)

The ranges of the independent variables are \(0 \leq x \leq 2\pi\), \(-Y \leq y \leq Y\), where

\[
Y = \ln \left( \frac{1 + \sin 30^\circ}{\cos 30^\circ} \right) = 0.548, 
\]

\(0 \leq \rho \leq 1\), and \(-\infty < t < \infty\), where \(t = 1\) is equivalent to 12 hr.

In particular, we use a two-layer model consisting of five pressure levels: 0, 250, 500, 750 and 1000 mb where \(\omega\) is assumed to vanish at the top and bottom levels. This simplification, amounting to no net divergence in a vertical column, eliminates the fast moving “external gravity” waves. Since we are only interested in motions with a long time scale, this assumption is reasonable.

A unique feature of the model is the use of statistical lateral forcing at 30N and 30S. We consider the total circulation in the model tropics as consisting of a time and zonally averaged state, and a deviation from it which arises from the lateral forcing. The deviation component is assumed to be governed by a linearized form of the equations. The mathematical analysis will be made in terms of the meridional velocity \(v\) at the 250- and 750-mb levels. The lateral forcing will be given in terms of the second-moment statistics of \(v\) at these levels at both 30N and 30S.

Some type of dissipative mechanism must be incorporated, or else the possibility of having resonance may render impossible the existence of a statistically stationary state. We will use three simple types of parameterized dissipation. Two are frictional, i.e., an internal friction at the middle level and surface friction at the lower level. The third is a simple radiational cooling, proportional to the temperature perturbation. Three empirical proportional constants must therefore be chosen.

We finally come to the problem of choosing a time and zonally averaged basic state. Let us first consider the static stability defined as \(- (T/\Theta)(\partial \Theta/\partial p)\). It is gen-
generally recognized that this quantity does not vary significantly from low to middle latitudes. Its annual average value for the atmospheric layer 750–250 mb was found to be 0.0508 K mb⁻¹ in the United States (Gates, 1961), 0.0515 in the West Indies (Jordan, 1958), and 0.0493 for the U. S. Standard Atmosphere. The value 0.050 K mb⁻¹ is chosen for the model. The two-layer model to be developed in Section 3 contains internal gravity waves as one mode of oscillation. In the absence of rotation and zonal current, their phase speed is given by

\[ C_g = \left( -\frac{RT}{\Theta} \frac{\partial \Theta}{\partial \varphi} \right)^{-1} = 60 \text{ m sec}^{-1}. \]

Table 1 shows the average zonal velocities obtained by Palmén (1963) and by Obasi (1963) for the Northern and Southern Hemispheres, respectively.

Within the uncertainty margin of the values for the two hemispheres, it appears that the zonal wind is by large symmetrical about the equator. It is therefore reasonable to choose basic zonal currents in the model as being parabolic in \( \frac{\varphi}{\varphi} \). We use, in particular

\[ \tilde{u}_1 = b_1 + c_1 \left( \frac{\varphi}{\varphi} \right)^2 \]

\[ \tilde{u}_2 = b_2 + c_2 \left( \frac{\varphi}{\varphi} \right)^2 \]  

(2.2)

with

\[ 2\Omega a b_1 = -2, \quad 2\Omega a b_2 = -4, \]

\[ 2\Omega a c_1 = 30, \quad 2\Omega a c_2 = 6, \]

where the constants in (2.2) are in m sec⁻¹. That this is a close approximation to the observed winds can be seen from the values of Table 1. The zonally averaged geopotential fields are taken to be in geostrophic balance with the prescribed zonal winds.

The effect of a mean meridional circulation in the basic state on the asymmetric motion will be disregarded. The reasons for this simplification are that the present model is too crude to incorporate these effects and the actual meridional circulation is not well determined.

The basic features of the two-layer model tropics are summarized in Fig. 1. The basic state parameters are:

\[ \tilde{u} = \frac{1}{2} (\tilde{u}_1 + \tilde{u}_2) \]

\[ \Lambda = \frac{1}{2} (\tilde{u}_1 - \tilde{u}_2) \]

\[ \tilde{e} = \left( \frac{C_g}{2\Omega a} \right)^2 = 4.16 \times 10^{-6}. \]  

(2.3)

\[ \frac{d \tilde{u}_1}{dy} = -y \tilde{u}_1, \quad \frac{d \tilde{u}_2}{dy} = -y \tilde{u}_2 \]

The two-level linearized version of (2.1) in which \( u_1, v_1, \phi_1, u_2, v_2 \) and \( \phi_2 \) now denote the perturbation variables is as follows:

\[ \frac{\partial u_1}{\partial t} + (\tilde{u}_1 + \Lambda) \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + \Lambda \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = -\frac{\partial \phi_1}{\partial x} + y v_1 + \beta (u_2 - u_1) \]

\[ \frac{\partial v_1}{\partial t} + (\tilde{u}_1 + \Lambda) \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} - \Lambda \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = -\frac{\partial \phi_1}{\partial y} - y u_1 + \beta (v_2 - v_1) \]

\[ \frac{\partial u_2}{\partial t} + (\tilde{u}_2 - \Lambda) \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} - \Lambda \left( \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} \right) = -\frac{\partial \phi_2}{\partial x} + y v_2 + \beta (u_2 - u_1) \]

\[ \frac{\partial v_2}{\partial t} + (\tilde{u}_2 - \Lambda) \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} - \Lambda \left( \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} \right) = -\frac{\partial \phi_2}{\partial y} - y u_2 + \beta (v_1 - v_2) \]  

(2.4)

\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial y} \]

\[ \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial y} \]

\[ \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial y} \]

The frictional and heating effects have been parameterized by the linear laws

\[ F_1 = -\beta (V_1 - V_2), \]

\[ F_2 = \beta (V_1 - V_2) - (\alpha - \beta) V_3, \]

\[ Q = -\gamma (\phi_1 - \phi_2), \]

where \( \beta \) represents a small-scale vertical exchange of horizontal momentum between levels 1 and 2, \( (\alpha - \beta) \) a surface drag coefficient, and \( \gamma \) can be thought of as a very crude representation of radiative cooling. Their corresponding numerical values, from Charney (1959),
are taken as $0.343 \times 10^{-2}$, $2.74 \times 10^{-2}$ and $0.206 \times 10^{-2}$, respectively. They correspond to decay times of 23, 6 and 39 days, respectively.

3. Formulation of an analysis for stochastic perturbations

Let us imagine an infinite number of models as the one described in Section 2, and suppose the fluid in each of these models is subject to lateral forcings of the same amplitude but randomly different phase. The circulation in each model can then be viewed as a realization of an ensemble. Because of the randomly different phase in the forcings, each realization is then naturally different from others in detail. But one may suspect that since the amplitude of the forcings in the ensemble are the same, there may be some properties common to all realizations. Our problem is to deduce these ensemble-average properties if they exist.

a. Perturbation spectral equations

The dependent variables in (2.4) are random functions, and the derivatives are therefore meaningful only if they are interpreted in the sense of mean-square convergence (Lin, 1967). The $x$ dependence in (2.4) can be eliminated by using a Fourier series expansion for each dependent variable. Thus, if the subscript $l=1$ or 2 represents the pressure level, we write

\[
\varphi_l(x,y,l) = \sum_{n=-\infty}^{\infty} \left[ C_l(y,l;n) \cos nx + S_l(y,l;n) \sin nx \right]
\]

\[
\varphi_l = \text{Re}\left\{ \sum_{n=-\infty}^{\infty} \varphi_l e^{inx} \right\},
\]

where

\[
\varphi_l = C_l - iS_l,
\]

and $n$ is the zonal wavenumber. In the same way we have

\[
\psi_l = \text{Re}\left\{ \sum_{n=-\infty}^{\infty} \psi_l e^{inx} \right\}
\]

\[
\psi_l = \text{Re}\left\{ \sum_{n=-\infty}^{\infty} \phi_l e^{inx} \right\}.
\]

In order to formulate our statistical boundary value problem, we must seek a spectral representation for the time variability of the six random functions, $\varphi_l$, $\psi_l$ and $\psi_l$, and their derivatives. A thorough exposition of the appropriate mathematical theory can be found in Yaglom (1962). It is sometimes referred to as correlation theory, because it is developed only for examining the second moments of stationary random functions. According to this theory, if each of the six functions has a unique “non-negative definite” autocorrelation function which is at least twice differentiable with respect to the lag at zero lag, then each of them and their time derivatives has a spectral representation. The spectral representation is in the form of Fourier-Stieltjes integrals and their inverse integrals. We postulate that this comparatively mild condition is valid for the random functions in our model. In Yaglom’s terminology, we define a random point function $Z(\sigma)$ for each of the six stationary random functions $\xi(l)$, i.e.,

\[
\xi(l) = \int_{-\infty}^{\infty} e^{itl} dZ
\]

\[
Z(\sigma) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{\xi(l)}{l^\sigma} dt
\]

\[
\frac{d\xi}{dl} = \int_{-\infty}^{\infty} i\sigma e^{itl} dZ
\]

\[
dZ = Z(\sigma + \Delta\sigma) - Z(\sigma)
\]

### Table 1. Average zonal velocities (m sec$^{-1}$) for the Northern (N) and Southern (S) Hemispheres.

<table>
<thead>
<tr>
<th>Pressure level</th>
<th>200 mb (observations)</th>
<th>250 mb (model)</th>
<th>850 mb (observations)</th>
<th>750 mb (model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latitude</td>
<td>(N)</td>
<td>(S)</td>
<td>(N)</td>
<td>(S)</td>
</tr>
<tr>
<td>30</td>
<td>20.0 29.6</td>
<td>28.0</td>
<td>2.0 2.2</td>
<td>2.0</td>
</tr>
<tr>
<td>20</td>
<td>10.0 17.8</td>
<td>10.7</td>
<td>-1.0 -1.9</td>
<td>-1.5</td>
</tr>
<tr>
<td>10</td>
<td>0.0 4.5</td>
<td>1.1</td>
<td>-4.0 -3.5</td>
<td>-3.4</td>
</tr>
<tr>
<td>0</td>
<td>-4.0 0.4</td>
<td>-2.0</td>
<td>-4.0 -3.3</td>
<td>-4.0</td>
</tr>
</tbody>
</table>
The identification is
\[ \xi: U_t, V_t, \varphi_t, \psi_t, S_t \]
\[ Z: U_t, V_t, P_t, C_t, S_t. \]

The spectral form of the definition (3.1) is
\[ dV_1 = dC_1 - dS_1. \]

Since \( dC_1(-\sigma) = dC_1^*(\sigma) \) and \( dS_1(-\sigma) = dS_1^*(\sigma) \) (where the asterisk denotes complex conjugate), we obtain
\[ dC_1(\sigma) = \frac{1}{2}[dV_1(\sigma) + dV_1^*(\sigma)] \]
\[ dS_1(\sigma) = \frac{1}{2}[dV_1(\sigma) - dV_1^*(\sigma)]. \] (3.3)

We substitute the Fourier series expansion (3.1) into (2.4) and then take the Fourier-Stieltjes integral of the resulting equations according to (3.2). The following equations result, governing the increment \( dZ \) of these six random point functions:
\[
\begin{align*}
fdU_1 + (f_3 + f_4)fdV_1 + ibdU_2 &= -ndP_1 \\
fdV_1 + (f_5 + f_6)dV_1 + ibdV_2 &= -ndP_1 \\
2\pi frdU_2 - (g_5 + g_6)dV_2 &= -DdP_2 \\
2\pi frdU_2 + (g_5 + g_6)dV_2 &= -DdP_2 \\
2\pi frdU_2 - iddV_2 &= 0 \\
2\pi frdU_2 - iddV_2 &= 0 \\
\end{align*}
\] (3.4)

where
\[ f_4 = \sigma + \nu(\bar{\alpha} + 2\alpha) - i\beta, \quad g_1 = \sigma + \nu(\bar{\alpha} - 2\alpha) - i\alpha \]
\[ f_5 = -i[D(\bar{\alpha} + \alpha) - y], \quad g_2 = -i[D(\bar{\alpha} - \alpha) - y] \]
\[ f_3 = -i\beta, \quad g_3 = i\beta \]
\[ f_4 = i[\sigma + \nu(\bar{\alpha} + \alpha) - i\beta], \quad g_4 = i[\sigma + \nu(\bar{\alpha} - \alpha) - i\alpha] \]
\[ h_1 = \sigma + \nu(\bar{\alpha} + \alpha) - i\gamma, \quad h_2 = i\gamma. \]

\[ D = \frac{d}{dy} \]

The six spectral equations in (3.4) can be reduced to two coupled equations by a relatively straightforward but algebraically involved procedure. The final perturbation spectral equations governing \( dV_1 \) and \( dV_2 \) are:
\[
\begin{align*}
L_1(dV_1) + L_4(dV_2) &= 0 \\
L_2(dV_1) + L_5(dV_2) &= 0 \\
\end{align*}
\] (3.5)

where \( L_j(j=1,2,3,4) \) are second-order operators in \( y \); and,
\[ L_j = \frac{d^2}{dy^2} + a_j + b_j + c_j, \]

where
\[ a_j = a_{1j}y^2 + a_{2j}y^4 + a_{3j}y^6 + a_{4j}, \]
\[ b_j = b_{1j}y^2 + b_{2j}y^4 + b_{3j}y^6 + b_{4j}, \]
\[ c_j = c_{1j}y^2 + c_{2j}y^4 + c_{3j}, \]
\[ a_{ij}, b_{ij} \text{ and } c_{ij} \text{ being functions of the constant parameters}. \]
\( dV_1 \) and \( dV_2 \) are related to \( dU_1, dU_2, dP_1 \) and \( dP_2 \) as follows:
\[
\begin{align*}
dU_1 &= A_1dU_1 + M_1dV_1 + M_2dV_2 \\
U_1 &= A_2dU_1 + M_1dV_1 + M_2dV_2 \\
P_1 &= A_3dU_1 + A_4dU_1 + M_1dV_1 \\
P_2 &= A_3dU_1 + A_4dU_1 + M_2dV_2 \\
\end{align*}
\] (3.6)

where \( A_j(j=1,2,3,4) \) are functions of the parameters, and \( M_m(m=1, \cdots, 6) \) are differential operators of first order in \( D \) with variable coefficients. If the basic currents in each level are independent of \( y \), then we have \( a_{1j} = a_{2j} = a_{3j} = b_{1j} = b_{2j} = c_{ij} = 0 \) (see Mak, 1968).

b. Fundamental solutions

Eqs. (3.5) and the associated relations (3.6) have as dependent variables the increments \( dZ \) of the six random point functions \( Z \). The boundary conditions at \( y = \pm Y \) are formally in terms of \( dV_{1k} \) which cannot be uniquely specified. In this sense our problem is fundamentally different from the usual boundary value problem where the dependent variables and boundary conditions are deterministic quantities. Nevertheless, it will be shown that the formal properties of (3.5) play an important role in determining the response of the model to statistically prescribed forcing.

The operators \( L_j(j=1,2,3,4) \) in (3.5) are second-order differential operators, and are even operators with respect to \( y \). Without loss of generality, the general solution for \( dV_1 \) and \( dV_2 \) may be written in terms of two even and two odd solutions of (3.5), \( \tilde{V}_1^e \) and \( \tilde{V}_2^e \) (j = 1, 2, 3, 4), as
\[
\begin{align*}
dV_1 &= \sum_{j=1}^{4} a_0j\tilde{V}_1^e(y) \\
dV_2 &= \sum_{j=1}^{4} a_0j\tilde{V}_2^e(y) \\
\end{align*}
\] (3.7)

where the values of \( \tilde{V}_1^e \) and \( \tilde{V}_2^e \) at \( y = 0 \) and \( Y \) are given in Table 2, and where the coefficients \( a_0j \) alone contain the stochastic properties of the system, i.e.,
\[
\begin{align*}
d\alpha_1 &= dV_{1+} - dV_{1-} \\
d\alpha_2 &= dV_{2+} - dV_{2-} \\
d\alpha_3 &= dV_{1+} - dV_{1-} \\
d\alpha_4 &= dV_{2+} - dV_{2-} \\
\end{align*}
\] (3.7a)

| Table 2. Values of \( \tilde{V}_1^e \) and \( \tilde{V}_2^e \). |
|---|---|---|---|
| \( j \) | \( \tilde{V}_1^e \) | \( \tilde{V}_2^e \) | \( \tilde{V}_1^e \) | \( \tilde{V}_2^e \) |
| 1 | 0.0 | 0.0 | 0.5 | 0.0 |
| 2 | 0.0 | 0.0 | 0.0 | 0.5 |
| 3 | 0.0 | 0.0 | 0.5 | 0.0 |
| 4 | 0.0 | 0.0 | 0.0 | 0.5 |
The solutions \( \tilde{V}_i^j \) and \( \tilde{P}_i^j \) can be obtained by solving the governing equations (3.5) four times using the four different sets of boundary conditions summarized in Table 2, and will be referred to as the fundamental solutions.

The general solutions for \( dU_i \) and \( dP_i \), \( l = 1, 2 \), associated with those for \( dV_i \) in (3.7) are

\[
\begin{align*}
\dot{U}_i &= \sum_{j=1}^{4} \alpha_j \dot{U}_j^i, \\
\dot{P}_i &= \sum_{j=1}^{4} \alpha_j \dot{P}_j^i
\end{align*}
\]  
(3.8)

\( \dot{U}_j^i, \dot{P}_j^i \) being related to \( \tilde{V}_i^j \) for each \( j \) in the same way that \( dU_i \) and \( dP_i \) are related to \( dV_i \) in Eq. (3.7).

c. **Response functions for second-moment statistics**

For clarity let us consider a particular correlation function; namely, the zonally averaged correlation function between \( v_1(t, x) \) and \( v_2(t, x) \). Let angular braces represent an ensemble average and an overbar an \( x \) average. By direct substitution from the Fourier series representation (3.1) of \( v_1 \) and \( v_2 \), we obtain

\[
\langle v_1 v_2 \rangle = \sum_{n=1}^{\infty} \frac{1}{\pi} \left[ \varphi_n \psi_n (\tau) + \varphi_n^* \psi_n (\tau) \right],
\]  
(3.9)

where \( \varphi_n, \psi_n \) is the correlation function of \( \xi \) and \( \eta \). We now apply the Wiener-Khintchine theorem for stationary random functions (Lumley and Panofsky, 1964, p. 16), and write (3.9) when \( \tau = 0 \) as

\[
\langle v_1 v_2 \rangle = \sum_{n=1}^{\infty} \frac{1}{2} \int_{-\infty}^{\infty} \{ \langle dc_y dc^*_x \rangle + \langle ds_y ds^*_x \rangle \}. \tag{3.10}
\]

It will now be assumed, without further specification, that all variances refer to \( x \) averaged expressions and the overbar will be omitted for convenience. By using (3.3) we can rewrite (3.9) as

\[
\begin{align*}
\langle v_1 v_2 \rangle &= \sum_{n=1}^{\infty} \frac{1}{4} \int_{-\infty}^{\infty} \langle dV_1(\sigma) dV^*_{1}(\sigma) \\
&\quad + \langle dV^*_{2}(\sigma) dV_{2}(-\sigma) \rangle \). \tag{3.11}
\end{align*}
\]

the integrand at \( -\sigma \) being equal to its complex conjugate at \( +\sigma \). Because of this symmetry, (3.11) can be written as

\[
\langle v_1 v_2 \rangle = \sum_{n=1}^{\infty} \frac{1}{2} \int_{-\infty}^{\infty} \text{Re} \{ \langle dV_1 dV^*_{2} \rangle \}. \tag{3.12}
\]

The reason for this symmetry is simple. It arises from our choice of dealing with two-sided spectra, i.e., for both positive and negative frequency. But as far as the covariance is concerned, there is no physical difference between positive and negative frequency and contributions from them are thus necessarily equal.

Substituting the general solutions for \( dV_i \) in the form of (3.7), we obtain

\[
\langle v_1 v_2 \rangle = \frac{1}{2} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \text{Re} \{ \sum_{j=1}^{4} \langle d\alpha_j \tilde{V}_j^i d\alpha^*_m \tilde{V}_m^* \rangle \}
\]

\[
= \frac{1}{2} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \text{Re} \{ \sum_{j=1}^{4} \langle \tilde{V}_j^i \tilde{V}_m^* \rangle (\alpha_j \alpha^*_m) \}. \tag{3.13}
\]

That the \( \tilde{V}_i^j \) are deterministic functions make possible the last step, which is critical in this analysis.

The \( \alpha_j \) are given by the boundary values of \( dV_i \) at \( y = \pm V \), as defined in (3.7). Again by the Wiener-Khintchine theorem, each of the 16 \( \langle d\alpha_j d\alpha^*_m \rangle \) can be expressed in the general form \( \Phi^k \sigma, k = 1, 2, \cdots, 16 \). Eq. (3.13) thereby reduces to an expression of the general form

\[
\langle v_1 v_2 \rangle = \int_{-\infty}^{\infty} \left[ \sum_{n=1}^{16} \sum_{l=1}^{18} H^k (y, \sigma; n) \Phi^k (\sigma; n) \right] d\sigma. \tag{3.14}
\]

The \( H^k (y, \sigma; n) \) play the role of the “response functions” or “system functions” used in analysis of electrical systems. The square bracketed quantity is the spectrum for the covariance of \( v_1 \) and \( v_2 \), and we see that it is a superposition of the responses associated with the 16 “input” spectra \( \Phi^k \) for each wavenumber \( n \).

This is the statement for our linear model of the general law that the “output” spectrum of any linear system is equal to the product of the system function and the input spectrum.

Eq. (3.13) can be generalized to determine other variances and covariances in the model. Let \( \xi \) and \( \eta \) denote any pair of the six variables \( v_1, w_1 \) and \( \phi_1 \) and let \( \tilde{Z}^i \) and \( \tilde{W}^j \) denote the corresponding deterministic solutions \( \tilde{V}_i^j, \tilde{U}_i^j \) and \( \tilde{P}_i^j \). Then the general form for the covariance is

\[
\langle \xi \eta \rangle = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \text{Re} \{ \sum_{j=1}^{4} \sum_{m=1}^{4} \tilde{Z}^j \tilde{W}^m (\alpha_j \alpha^*_m) \}. \tag{3.15}
\]

4. **Statistical boundary conditions**

a. **Formulation**

A closed set of statistical boundary conditions for the spectral equations (3.5) is now formulated on the basis that only \( v_1 \) and \( v_2 \) at 30N are known for a sufficiently long period of time. For the moment, however, let us assume that long records of \( v_1 \) and \( v_2 \) at both 30N and 30S are available. We first expand each of these functions into Fourier series of longitude, truncated at a certain wavenumber \( N \), i.e.,

\[
v_{1, \pm}(t, x) = \sum_{n=1}^{N} \left[ C_{1, \pm}(n; x) \cos \alpha + S_{1, \pm}(n; x) \sin \alpha \right], \tag{4.1}
\]

where \( l = 1, 2 \) and the subscript \( \pm \) stands for \( y = \pm Y \).
As shown in (3.15), any second-moment statistic in the model can be uniquely determined if the statistical boundary conditions (s.b.c.) are such that they enable 
\[
\langle d\alpha_k d\alpha_k^* \rangle, \quad j=1,2,3,4; \quad m=1,2,3,4,
\]
to be evaluated. According to the definition of \(d\alpha_k\) in (3.7a) the required s.b.c. for the governing spectral equations (3.5) are 
\[
\langle dV_{1k} dV_{1k}^* \rangle, \quad l=1,2; \quad k=1,2.
\]
They can be synthesized from 
\[
\langle dC_{1k} dC_{1k}^* \rangle, \quad \langle dS_{1k} dS_{1k}^* \rangle, \quad \langle dC_{1k} dS_{1k} \rangle, \quad \langle dS_{1k} dC_{1k} \rangle,
\]
and \(\langle dS_{1k} dS_{1k} \rangle\), since \(dV_i\) is related to \(dC_i\) and \(dS_i\) by \(dV_i = dC_i - i dS_i\). By the Wiener-Khintchine theorem, the s.b.c. for each wavenumber \(n\) can therefore be constructed from all the independent spectra

\[
\Phi_{\ell}(\ell \pm k, \ell \pm k), \quad \Phi_{\ell}(\ell \pm k, \ell \pm k), \quad \Phi_{\ell}(\ell \pm k, \ell \pm k).
\]

These spectra can be obtained if \(v_{1\ell}\) and thus \(c_{1\ell}\) and \(s_{1\ell}\) as defined in (4.1), are available. But only observations for \(v_{1k}\) are available in sufficient detail for the Fourier expansion (4.1). The simplest and perhaps the most reasonable way to overcome this handicap is to assume that \(v_{1k}\) is statistically independent of \(v_{1k}\), but is otherwise statistically similar to the latter. This is just a working assumption. The first part of the assumption is not unreasonable because there is no significant correlation between the baroclinic activities at the midlatitudes of the two hemispheres, which, after all, are primarily responsible for the flow near 30N and 30S. The second part of the assumption is reasonable only to the extent that hemispheric symmetry is a sufficiently good approximation. It cannot be rigorously justified, since topological differences do exist between the two hemispheres. Much of the interhemispheric statistical difference presumably is expressed in seasonal fluctuations. Our detailed analysis, however, will consider only periods between 3 days and about 3 months. Therefore, the assumption of symmetry is not as weak as it appears. For consistency the basic state must also be symmetric, as has already been assumed.

The assumption of statistically independent forcings at \(y=\pm Y\) can be expressed as

\[
\langle v_{1p}(x,\ell) v_{1p}(x,\ell, l, \tau) \rangle = 0 \quad (4.2)
\]

for all \(\Delta, \tau\), and for \(l=1,2; \quad \ell=1,2\). The condition that \(v_{1p}\) is statistically similar to \(v_{1p}\) can be formulated similarly as

\[
\langle v_{1p}(x,\ell) v_{1p}(x,\ell, l, \tau) \rangle = \langle v_{1p}(x,\ell) v_{1p}(x,\ell, l, \tau) \rangle. \quad (4.3)
\]

Expansion of (4.2) and (4.3) by using (4.1) leads to four relations for each wavenumber \(n\), since each coefficient of \(\sin \Delta \) and \(\cos \Delta\) in those expansions is necessarily zero. Finally, by the Wiener-Khintchine theorem those relations can then be written in the spectral form:

\[
\begin{align*}
\langle dC_{1p} dC_{1p}^* \rangle + \langle dS_{1p} dS_{1p}^* \rangle &= 0, \\
\langle dC_{1p} dS_{1p} \rangle - \langle dS_{1p} dC_{1p} \rangle &= 0, \\
\langle dC_{1p} dS_{1p}^* \rangle + \langle dS_{1p} dS_{1p} \rangle &= \langle dC_{1p} dC_{1p} \rangle + \langle dS_{1p} dS_{1p} \rangle, \\
\langle dC_{1p} dS_{1p}^* \rangle - \langle dS_{1p} dC_{1p} \rangle &= \langle dC_{1p} dC_{1p} \rangle - \langle dS_{1p} dS_{1p} \rangle. 
\end{align*}
\]

With these conditions the statistical boundary conditions (3.3) now reduce to

\[
\langle dV_{1p} dV_{1p} \rangle = 0, \quad (4.5)
\]

together with the four real functions \(F_1, F_2, F_3\) and \(F_4\):

\[
\begin{align*}
F_1 &= \langle |dV_{1p}|^2 \rangle = \langle |dV_{1p}|^2 \rangle = \langle |dV_{1p}|^2 \rangle = \langle |dV_{1p}|^2 \rangle, \\
F_2 &= \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle, \\
F_3 &= \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle, \\
F_4 &= \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle = \langle dV_{1p} dV_{1p} \rangle.
\end{align*}
\]

The \(\langle d\alpha_k d\alpha_k^* \rangle\) appearing in (3.15) can then be written as

\[
\begin{align*}
\langle |d\alpha_1|^2 \rangle &= \langle |d\alpha_1|^2 \rangle = \langle |d\alpha_1|^2 \rangle = \langle |d\alpha_1|^2 \rangle = \langle |d\alpha_1|^2 \rangle, \\
\langle |d\alpha_2|^2 \rangle &= \langle |d\alpha_2|^2 \rangle = \langle |d\alpha_2|^2 \rangle = \langle |d\alpha_2|^2 \rangle = \langle |d\alpha_2|^2 \rangle, \\
\langle |d\alpha_3|^2 \rangle &= \langle |d\alpha_3|^2 \rangle = \langle |d\alpha_3|^2 \rangle = \langle |d\alpha_3|^2 \rangle = \langle |d\alpha_3|^2 \rangle = \langle |d\alpha_3|^2 \rangle = 0, \\
\langle |d\alpha_4|^2 \rangle &= \langle |d\alpha_4|^2 \rangle = \langle |d\alpha_4|^2 \rangle = \langle |d\alpha_4|^2 \rangle = \langle |d\alpha_4|^2 \rangle = \langle |d\alpha_4|^2 \rangle = 0.
\end{align*}
\]

Finally, upon substitution from (4.7), the general covariance defined in (3.17) can be written as

\[
\langle \xi_\eta \rangle = \sum_{\eta=-1}^{12} \sum_{\tau=-1}^{4} H_{\ell \tau}^i(y; \eta; n) F_i(\sigma; n), \quad (4.8)
\]

where

\[
\begin{align*}
H_{\ell \tau}^1 &= \text{Re}\{2|\tilde{W}_{\ell \tau}|^2 + 2|\tilde{W}_{\ell \tau}|^2, \\
H_{\ell \tau}^2 &= \text{Re}\{2|\tilde{W}_{\ell \tau}|^2 + 2|\tilde{W}_{\ell \tau}|^2, \\
H_{\ell \tau}^3 &= \text{Re}\{2|\tilde{W}_{\ell \tau}|^2 + 2|\tilde{W}_{\ell \tau}|^2 + 2|\tilde{W}_{\ell \tau}|^2, \\
H_{\ell \tau}^4 &= \text{Im}\{-2|\tilde{W}_{\ell \tau}|^2 + 2|\tilde{W}_{\ell \tau}|^2 + 2|\tilde{W}_{\ell \tau}|^2, \\
\end{align*}
\]

and the integration is over the frequency \(\sigma\). [Note the \(\Delta \sigma\) appearing in (4.6).]

b. Data analysis

The data used for computing the statistical boundary conditions is the streamfunction field \(\psi\) analyzed with the “balance equation” by the National Meteorological Center (see Shuman, 1957). The data is given in a stereographic grid system which covers the Northern Hemisphere north of about 15N, twice daily at 0000 and 1200 GMT. Data for the period June 1963 through April 1965 for the 200- and 850-mb levels were made available to the writer by the National Center for Atmospheric Research.

A linear interpolation scheme was used to obtain the value of \(\psi\) at every 5° of longitude along 30N at each
observation time. These 72 values were then resolved into Fourier components in longitude. In view of the relatively sparse network over the major oceans, the Fourier series was truncated after wavenumber 12. Since $\psi'$ is nondimensionalized as $\psi' = \psi/(2\Omega^2)$, the dimensionless meridional velocity is $v_{14} = (1/\cos 30^\circ)$ $(\partial \psi'/\partial x)$. These Fourier coefficients for $\psi'$ are readily related to those for $v_{14}$ in (4.1) with $N = 12$.

Each of the time series $c_{14}$ and $s_{14}$ was then modified by having its time mean value removed. The spectra and cross-spectra of the modified time series $\Phi_{c_{14}s_{14}}$, $\Phi_{s_{14}c_{14}}$, and $\Phi_{c_{14}s_{14}}$ were then computed. The elementary frequency interval $\delta$ used in the boundary spectra is $(270)^{-1}$ cycles per 12 hr and the resolution is 28. The frequency range we shall consider is $|\sigma| \leq 1/6$ cycles per 12 hr, corresponding to a minimum period of 3 days. This limitation on $\sigma$ is imposed by the nature of the boundary data, since this consisted of stream-function analyses which are essentially based on the quasi-geostrophic theory. The details of the spectral analysis as well as a complete listing of $F_1$, $F_2$, $F_3$, and $F_4$ for this frequency range for each of the 12 wavenumbers can be found in Mak (1963). As a sample, $F_1$ and $F_2$ for wavenumbers 2, 4, 5, 7, and 11 are shown in Figs. 2 and 3. Here we find, as one would expect, that much more energy is associated with the motions at the 200-mb level than at 850 mb. Most of the energy in $F_1$ and $F_2$ belongs to the intermediate wavenumbers 5–8. Furthermore, the spectra for the low wavenumbers 1–4 have a larger magnitude at positive frequency than at negative frequency, while the opposite is true for the higher wavenumbers. This can be interpreted as meaning that the waves of low zonal wavenumbers at 30°N are moving relatively more westward than those of higher wavenumbers. This is a well-known characteristic of Rossby waves.

5. Determination of the fundamental solutions

We now return to the problem of determining the four pairs of fundamental solutions which are to satisfy
the same differential equations (3.5) but have the four
different sets of boundary conditions summarized in
Table 2. The differential operators $L_4$ in (3.5) have
variable coefficients and have no simple relationships
among themselves. It is virtually impossible to obtain
the fundamental solutions in analytic form. We shall
therefore solve them numerically. But in order to es-
blish a complete picture of the fundamental solutions,
we must numerically integrate (3.5) for a sufficiently
dense combination of wavenumbers and frequencies.
Otherwise, certain combinations of $n$ and $\sigma$ which give
strong response might be overlooked. It is therefore a
very practical matter that we obtain at least some broad
notion about how the fundamental solutions vary in the
wavenumber-frequency domain. We might gain such
information if we could find a special case in which ana-
lytic solutions for (3.5) are possible. Not only could they
serve as a guide for making an adequate and efficient
scanning over the frequency scale, but they also would
shed considerable light on the intrinsic characteristics
of the system. Fortunately, there exists such a special
case. We will, therefore, first show the analysis for this
special case, which reveals the conditions under which
resonance could occur. A description of the numerical
analysis for the general case is then given.

a. Special case

The four differential operators $L_4$ reduce to remark-
ably simple forms when all the dissipative coeffi-
cients and the basic shear vanish. Hence, the special
case under consideration is a two-layer nondissipa-
tive model with a barotropic basic state. When $a = \beta = \gamma = 0,$
$\Lambda = 0,$ and $\sigma =$ constant, we find

$$L_4 = -L_4 = \left[ \frac{s^3 - 2\xi}{n^2} \right] \frac{d^2}{dy^2} + s^2 \frac{d^2}{n dy^2}$$

$$= \left[ \frac{s^3}{n} + \frac{1}{n} - \frac{2}{s} \right] \frac{s^2}{n^2} + s^2 \frac{dy^2}{n^2} \right\}, \quad (5.1)$$

$$L_4 = -L_4 = \frac{s^3}{n^2} \frac{d^2}{dy^2} + \frac{s^2}{n^2} \frac{d^2}{n dy^2} + \frac{s^2}{n^2} \frac{dy^2}{n^2} \right\}, \quad (5.1)$$

where $s = \sigma + n\eta.$ The symmetry among the operators
now enables us to combine the two coupled equations
in (3.10) into the two uncoupled equations

$$\left( \frac{d^2}{dy^2} + \frac{s^2}{n^2} \right) \left\{ \tilde{V}_i \pm \tilde{V}_j \right\} = 0, \quad (5.2)$$

$$\left[ \frac{d^2}{dy^2} + \left( \frac{s^2}{n^2} \right) \right] \left\{ \tilde{V}_i \mp \tilde{V}_j \right\} = 0. \quad (5.3)$$

Provided $s^2 - n^2 \xi \neq 0,$ Eq. (5.3) is the counterpart of Eq.
(6) in Matsuno (1966). In this special case ($\tilde{V}_i \pm \tilde{V}_j$)
and ($\tilde{V}_i \mp \tilde{V}_j$) can clearly be identified as the baro-
tropic and the baroclinic components, respectively.

BAROTROPIC COMPONENT. The barotropic component
is governed by (5.2). Its solutions which satisfy the
boundary conditions in Table 2 are

$$\tilde{V}_i + \tilde{V}_j = \frac{\cos(n^2 - n^2 \xi)}{2 \cos(n^2 - n^2 \xi)} \right\}, \quad j = 1, 2$$

$$\tilde{V}_i + \tilde{V}_j = \frac{\sin(n^2 - n^2 \xi)}{2 \sin(n^2 - n^2 \xi)} \right\}, \quad j = 3, 4$$

(5.4)

The barotropic component therefore becomes infinite
whenever

$$\left( \frac{n}{\sqrt{m^2 + n^2}} \right) \frac{m}{2} \frac{n}{\pi} = \frac{\sigma}{2Y}, \quad m = \pm 1, \pm 2, \cdots \quad (5.5)$$

In terms of dimensional parameters, (5.5) is equivalent
to

$$\sigma_{\text{resonance}} = \frac{2\Omega n}{\frac{1}{2Y}} \frac{\tilde{U}_n}{a}. \quad (5.6)$$

These frequencies can therefore be called resonant fre-
quencies associated with the barotropic mode. They
Correspond to barotropic Rossby waves in a channel
$|\gamma| \leq Y$. For each wavenumber $n$, there are an infinite
number of barotropic resonant frequencies, and $\sigma_{\text{resonance}}$
is a monotonically decreasing function of $|m|$. The upper
and lower bounds are $\sigma(m = \pm 1)$ and $\sigma(m = \pm \infty)$,
respectively. The distribution of these barotropic resonant
modes is shown by the dashed curves in Fig. 4. (The con-
stant value of $\tilde{U}_n$ used here corresponded to $+3$ m
sec$^{-1}$.)

BAROCLINIC COMPONENT. The baroclinic component
is governed by (5.3). Its solutions are known to be para-
bolec cylinder functions. Let us first transform (5.3)
into a standard form by using a new independent variable
$\varepsilon = (4/\tilde{\varepsilon}) y$; then

$$\left[ \frac{d^2}{dy^2} + \left( \frac{1}{2} s^2 + \Gamma \right) \right] \left\{ \tilde{V}_i \mp \tilde{V}_j \right\} = 0, \quad (5.8)$$

where

$$\Gamma = \frac{-1}{2(\tilde{\varepsilon})^2} \left( \frac{\eta n}{s} \right). \quad (5.8a)$$

The two independent solutions of (5.8) are well-known
and can be given in terms of the confluent hyperge-
ometric function $\text{JF}_1(a, b, x)$. The even and odd solutions
are

$$M^I(z; \Gamma) = e^{-\tilde{\varepsilon}^2} \text{JF}_1 \left( \frac{1}{2} z \Gamma + \frac{1}{2}, \frac{1}{2}, \frac{1}{2} z^2 \right)$$. \quad (5.9a)

$$M^{II}(z; \Gamma) = e^{-\tilde{\varepsilon}^2} \text{JF}_1 \left( \frac{1}{2} z \Gamma + \frac{1}{2}, \frac{1}{2}, \frac{1}{2} z^2 \right)$$. \quad (5.9b)

When the boundary conditions are incorporated, we
obtain

\[ \tilde{\psi}_1 - \tilde{\psi}_2 = - (\tilde{\psi}_1^2 - \tilde{\psi}_2^2) = \frac{1}{2} M'(z_+; \Gamma) \]

(5.10)

\[ \tilde{\psi}_1^2 - \tilde{\psi}_2^2 = - (\tilde{\psi}_1^2 - \tilde{\psi}_2^2) = \frac{1}{2} M''(z_+; \Gamma) \]

Hence the baroclinic component becomes infinite whenever

\[ M'(z_+; \Gamma) = 0 \quad \text{or} \quad M''(z_+; \Gamma) = 0. \]

Since \( e^{-i z s} \) and \( z_+ \) are positive quantities, this is equivalent to either of the two conditions

\[ \mathbf{1}_F \left( \frac{1}{2} \Gamma + \frac{1}{2}, \frac{1}{2}, \frac{1}{2} z_+ s \right) = 0 \]

\[ \mathbf{1}_F \left( \frac{3}{2} \Gamma + \frac{1}{2}, \frac{3}{2}, \frac{3}{2} z_+ s \right) = 0 \]

Since \( z_+ \) only depends on the static stability and the latitude of the northern boundary, its numerical value is known. The baroclinic resonant frequencies for each wavenumber \( n \) are obtained by first finding all values of \( \Gamma \) which satisfy (5.11), and then solving for the three roots of \( \sigma \) (or \( z \)) associated with each of those values of \( \Gamma \) according to (5.8a).

A simple graphical method to determine the most important values of \( \Gamma \) with sufficient accuracy is given in Mak (1968). This method is based upon two general properties of confluent hypergeometric function in regard to its zeros, and also upon the relation between it and the error function. The resulting baroclinic resonant frequencies are shown in Fig. 4 by the solid curves. [Matsumo's (1966) study was effectively to solve for the free solutions of (5.8) under the restriction that they are to vanish when \( s \to \infty \). It is equivalent to finding the eigen values of \( \Gamma \) in (5.11) for \( z_+ = \infty \). Exact results are known for this limiting case, and can also be obtained with the graphical method mentioned above.] One distinct feature of Fig. 4 is that the resonant modes in this special case fall into three groups. Only one group of them is bounded within an upper and a lower value, and are characterized by their small frequency. They include all the barotropic modes and one-third of the baroclinic modes. These are barotropic and baroclinic Rossby waves moving slowly westward with respect to the basic current. The other two groups of resonant modes are characterized by large positive and negative frequency. These are internal gravity-inertia waves which can travel both eastward and westward at relatively high speed.

b. General case

We now consider the numerical analysis of the fundamental solutions when the friction and cooling coeffi-
ciests, and the basic shear are incorporated. Fifty-one grid points, \( k = 1, 2, \ldots, 51 \), were used to cover the latitude zone from equator to 30°N. The variables \( U_i^k \) and \( \bar{P}_i^k \) are defined at the same grid points, whereas \( \bar{V}_i^k \) are defined at points midway between the points for the former. For clarity we omit temporarily the symbol \( \cdot \) for the dependent variables as well as the superscript \( j \) which distinguishes the four different sets of fundamental solutions, and instead use a new superscript \( k \) to denote the grid points. (The difference equations for the four sets of fundamental solutions are identical; it is only the four sets of boundary values in Table 2 which distinguish them.) The finite difference notation is then

\[
\begin{align*}
\bar{U}_k^k &= \frac{1}{\Delta} f_k \bar{U}_k^{k+1} + \frac{1}{2} f_k \bar{V}_k^{k+1} + V_k^k + \frac{1}{\Delta} f_k \bar{V}_k^{k-1} - \bar{V}_k^k - i\beta \bar{V}_k^k - n \bar{P}_k^k \\
\bar{V}_k^k &= \frac{1}{\Delta} g_k \bar{V}_k^{k+1} + \frac{1}{2} g_k \bar{U}_k^{k+1} + V_k^k + \frac{1}{\Delta} g_k \bar{U}_k^{k-1} - \bar{V}_k^k - i\beta \bar{U}_k^k - n \bar{P}_k^k \\
\bar{P}_k^k &= \frac{1}{\Delta} \bar{P}_k^k - \bar{P}_k^{k-1} \\
\bar{V}_k^k &= \frac{1}{\Delta} \bar{V}_k^k - \bar{V}_k^{k+1} - \bar{V}_k^{k-1} + 2 \bar{V}_k^k \\
\bar{P}_k^k &= \frac{1}{\Delta} \bar{P}_k^k - \bar{P}_k^{k+1} - \bar{P}_k^{k-1} + 2 \bar{P}_k^k \\
\end{align*}
\]

(5.12)

The terms \( U_k^k \) and \( P_k^k \) are then eliminated from (5.12). The resulting second order difference equations are the counterpart of (3.5) and apply at \( k = 1-50 \); thus

\[
\begin{align*}
\bar{V}_k^k &= \frac{1}{\Delta} f_k \bar{V}_k^{k+1} + \frac{1}{2} f_k \bar{V}_k^{k+1} + \frac{1}{2} f_k \bar{V}_k^{k-1} + \frac{1}{2} f_k \bar{V}_k^{k-1} + \frac{1}{\Delta} f_k \bar{V}_k^{k-1} - \bar{V}_k^k - i\beta \bar{V}_k^k - n \bar{P}_k^k \\
\bar{V}_k^k &= \frac{1}{\Delta} g_k \bar{V}_k^{k+1} + \frac{1}{2} g_k \bar{V}_k^{k+1} + \frac{1}{2} g_k \bar{V}_k^{k-1} + \frac{1}{\Delta} g_k \bar{V}_k^{k-1} - \bar{V}_k^k - i\beta \bar{V}_k^k - n \bar{P}_k^k \\
\bar{P}_k^k &= \frac{1}{\Delta} \bar{P}_k^k - \bar{P}_k^{k-1} - \bar{P}_k^{k-1} + 2 \bar{P}_k^k \\
\bar{V}_k^k &= \frac{1}{\Delta} \bar{V}_k^k - \bar{V}_k^{k+1} - \bar{V}_k^{k-1} + 2 \bar{V}_k^k \\
\bar{P}_k^k &= \frac{1}{\Delta} \bar{P}_k^k - \bar{P}_k^{k+1} - \bar{P}_k^{k-1} + 2 \bar{P}_k^k \\
\end{align*}
\]

(5.13)

The wavenumber range is \( n = 1-12 \) and the frequency range is from \( \sigma = -\frac{1}{2} \) to \( \sigma = +\frac{1}{2} \) cycles (12 hr)\(^{-1} \), which is to be scanned through adequately and efficiently using Fig. 4 as a guide. A very small frequency interval, (2700)\(^{-1} \) cycles (12 hr)\(^{-1} \), was used in the neighborhood of the resonant frequencies of the special case, and larger frequency intervals [as large as (14)\(^{-1} \) cycles (12 hr)\(^{-1} \)] were used elsewhere. The small fre-

Table 2 which distinguish them.) The finite difference notation is then

\[
\begin{align*}
y^k &= (k-1)\Delta = (k-1)\frac{1}{50} \\
V^k &= V \at \ y = (k-1)\Delta, \\
U^k, P^k &= u, p \at \ y = (k-\frac{1}{2})\Delta.
\end{align*}
\]

We begin with the unreduced spectral equations (3.4) whose centered difference form is as follows:

\[
\begin{align*}
f_k V_{k+1}^k + \frac{1}{2} f_k V_{k-1}^k - \beta V_{k+1}^k &= \frac{1}{\Delta} \left( P_{k+1}^k - P_{k-1}^k \right) \\
g_k V_{k}^k + \frac{1}{2} g_k V_{k-1}^k - \beta V_{k-1}^k &= \frac{1}{\Delta} \left( P_{k-1}^k - P_{k-2}^k \right) \\
n \left( V_{k+1}^k + V_{k-1}^k \right) &= \left( V_{k+1}^k + V_{k-1}^k \right) \\
\frac{2}{\Delta} \left( V_{k+1}^k - V_{k-1}^k \right) &= \bar{h}_{k+1} \left( P_{k+1}^k P_{k+2}^k \right)
\end{align*}
\]

The terms \( U_k^k \) and \( P_k^k \) are then eliminated from (5.12). The resulting second order difference equations are the counterpart of (3.5) and apply at \( k = 1-50 \); thus

\[
\begin{align*}
p_k V_{k+1}^k + \frac{1}{2} p_k V_{k-1}^k + p_k V_{k-1}^k + p_k V_{k-1}^k + p_k V_{k-1}^k &= 0 \\
q_k V_{k}^k + q_k V_{k}^k + q_k V_{k}^k + q_k V_{k}^k + q_k V_{k}^k &= 0
\end{align*}
\]

(5.13)

The boundary conditions giving \( V_k^k \) at \( k = 0 \) and \( k = 51 \) correspond to those in Table 2 and are stated in Table 3.

Eq. (5.13) together with the boundary values in Table 3 can be put into a vector form

\[
A_{m1} = \begin{cases} m = 1, \bar{V}_k^k, \bar{V}_k^k & \text{odd solutions} \\
m = 2, \bar{V}_k^k, \bar{V}_k^k & \text{even solutions}
\end{cases}
\]

(5.14)

where \( A_{m1} \) is a 100×100 matrix whose elements consist of \( p_k, q_k, V \) is a 100×2 matrix containing even (odd) fundamental solutions \( V_k^k \), and \( F_{m1} \) is a 100×2 matrix containing the even (odd) boundary conditions. Such a system can be easily solved with the Gauss elimination method; the subroutine "GELB" at the MIT Computation Center was used.

| Table 3. Boundary values of \( V_k^k \) at \( k = 0 \) and \( k = 51 \). |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( V_1^0 \)    | \( V_1^0 \)    | \( V_1^0 \)    | \( -V_1^0 \)    | \( -V_1^0 \)    |
| \( V_2^0 \)    | \( V_2^0 \)    | \( V_2^0 \)    | \( -V_2^0 \)    | \( -V_2^0 \)    |
| \( V_3^0 \)    | 0.5             | 0.5             | 0               | 0               |
| \( V_4^0 \)    | 0               | 0.5             | 0               | 0               |
frequency interval is 1/20 of the frequency resolution in the input spectra. It should be noted that the resonant frequencies in the special case treated earlier have an accumulation point at \( \sigma = -n\bar{u} = -n(0.00322) \) as the north-south wavenumber increases in the Rossby modes. The small frequency interval \( \delta_0 = (2700)^{-1} \) was small enough to show conclusively that this fine structure was smeared out in the general case by friction and basic shear. For purpose of comparison, the fundamental solutions were also computed numerically for the special case for wavenumber \( n = 4 \) by simply setting \( \alpha, \beta, \gamma \) and \( \Lambda \) equal to zero, and \( \bar{u} = 3 \) m sec\(^{-1}\) in the computation of \( \rho^h \) and \( q^h \) in (5.13).

c. Properties of the general fundamental solutions

In view of the complicated dependence of the response functions of the second-moment statistics upon the fundamental solutions [see (3.17)], little would be gained from a close examination of the detailed structure of each of the \( \hat{V}_j^f \). Therefore, only one broad aspect of them will be discussed; namely, the latitudinal sum of \( |\hat{V}_j^f| \). These quantities give a crude indication of the intensity of the integrated responses as a function of wavenumber and frequency.

The even and odd "quasi-barotropic" integrated response components, \( \sum_j |\hat{V}_j^f| \), for \( n = 4 \), are shown separately in two plots in Fig. 5. The solid curves are for the special case and the broken curves are for the general case. The short vertical arrows along the frequency axes locate the resonant barotropic frequencies determined analytically for the special case. Fig. 6 does likewise for the "quasi-baroclinic" components \( \sum_j |\hat{V}_j^f - \hat{V}_j^s| \). [The numerical results in the special case were identical for \( j = 1, 2 \) and for \( j = 3, 4 \) in agreement with the analytic solutions (5.4) and (5.10). Only one set of solid curves is therefore necessary on each of the four diagrams in Figs. 5 and 6.]

We first note that the theoretically deduced arrows agree well with the location of the corresponding numerically computed peaks of the solid curves on both Figs. 5 and 6. All solid peaks shown correspond to Rossby waves, except for the peak at \( \sigma = 0.259 \) in Fig. 6, which represents the internal gravity wave of smallest positive \( \sigma \) at this wavenumber. The theoretical Rossby wave accumulation point at \( \sigma = -4\bar{u} = -0.0129 \) also checks. The quantitative agreements just described provide a welcome verification of the numerical method. (The behavior with \( y \) in the special case of the numerical barotropic values also agrees well with the corresponding theoretical solutions.)

The broken curves have only two broad but well resolved peaks associated with the Rossby waves, and one peak associated with the above-mentioned internal gravity wave. Damping due to dissipation is clearly dominant for those Rossby waves of high north-south wavenumbers located at very low frequencies. It is noted that the Rossby-wave peaks are shifted to smaller frequencies relative to their counterparts in the special
case. Furthermore, while the barotropic and baroclinic peaks of the latter are located at different frequencies, those (broken-line) peaks of the "quasi-barotropic" and "quasi-baroclinic" coincide. This indicates that the basic shear and the dissipative processes cause the previously separated barotropic and baroclinic components to interact with one another. As a result the sum and difference of the velocities at the upper and lower levels no longer represent the actual barotropic and baroclinic components. There is also a shift of the broken-line peak of the internal gravity wave in Fig. 6 relative to the solid peak, even though it has no corresponding barotropic peak to interact with. This suggests that the apparent interaction is between the basic currents and the internal gravity wave; as a result, a fairly large secondary maximum appears in Fig. 5 near that frequency.

Finally, it is of interest to examine how the fundamental solutions vary in the wavenumber-frequency space. As an example, we show one even solution of the "quasi-baroclinic" component, \( \Sigma_\nu |V_\nu^1 - V_\nu^2| \), in Fig. 7. The maximum has the same general pattern as the distribution of the resonant modes shown in Fig. 4. The most striking feature, however, is the location of maximum response around \( n = 3 \) at frequencies corresponding to the lowest latitudinal wavenumber of the Rossby waves and internal gravity waves. Even a small boundary forcing at such values of \( n \) and \( \sigma \) could then excite considerable response.

6. The predicted statistics of the model

With the statistical boundary conditions obtained in Section 4 and the fundamental solutions in Section 5, we now can compute any variance or covariance by (4.8). Each of these statistics is either an even or an odd function of latitude \( \gamma \), because the basic state and the boundary forcings in the model are both symmetric with respect to the equator. As shown in (3.17) each statistic is equal to an integral over frequency \( \sigma \) and a sum over wavenumber \( n \) of a quantity, say \( X \), which depends on \( n \), \( \sigma \), \( \gamma \), the boundary spectra and the basic state parameters. The model statistics will generally be presented simply as functions of \( \gamma \), although in several cases the dependence of the integrand-summand \( X \) on \( n \) and \( \sigma \) will also be displayed. The integration over \( \sigma \) was performed by a trapezoidal sum, in which the frequency interval employed was small enough (generally equal to \( \delta \), in fact) to adequately sample the detailed behavior with \( \sigma \) of the fundamental solutions and of the boundary spectra. The contribution from those individual frequency intervals were then combined into 15 frequency bands of width 1/90 = 0.011 cycles (12 hr)\(^{-1}\). These bands are centered at \( \sigma = 1/180, 3/180, \ldots, 29/180 \) cycles (12 hr)\(^{-1}\). They are wide enough to insure meaningful spectral resolution and yet narrow enough to show the distinct frequency dependence of the statistics.

In Mak (1968) computations were made for a basic current having no \( \gamma \) dependence at each level, with
\( \bar{u}_1 = 8 \text{ m sec}^{-1} \) and \( \bar{u}_2 = -2 \text{ m sec}^{-1} \). [These values are equal to the latitudinal averages of the \( \bar{u}_1 \) and \( \bar{u}_2 \) defined in (2.2).] The results of this earlier computation having only vertical shear in \( \bar{u} \) will be referred to as Case A and shown by \textit{dashed curves} in the figures. The more general case containing lateral and vertical shear, with \( \bar{u} \) from (2.2), will be referred to as Case B and shown by \textit{solid curves}.

**Fig. 7.** Variation of \( \sum |V_1 - V_2| \) in frequency-wavenumber domain.

**Fig. 8.** Predicted standard deviation of \( \bar{v} \) at 250 and 750 mb, and observed values by Kidson, using crosses for 250 mb and dots for 850 mb: dashed lines, Case A; solid lines, Case B.
a. Horizontal velocity statistics

Fig. 8 shows the square root of \( \langle v_1^2 \rangle \) and \( \langle v_2^2 \rangle \) as a function of latitude, as well as the observed values at four latitudes for the pressure levels 250 and 850 mb. The latter were recently obtained from five years of data by John Kidson of the Planetary Circulations Project at Massachusetts Institute of Technology. In Case A, the theoretical values predict too large \( \langle v_2^2 \rangle \) and too small \( \langle v_2^2 \rangle \) in the equatorial region. The theoretical value of \( \langle v_2^2 \rangle \) is greatly reduced in Case B, whereas \( \langle v_2^2 \rangle \) changes only slightly. They both decrease monotonically with latitude and agree at least qualitatively with the observed values. The difference between the observed data and Case B is primarily a consistent underestimate of about 2 m sec\(^{-1}\). Since this occurs also at 30°N, where \( \langle v_2^2 \rangle \) is prescribed in the model, this discrepancy may be due primarily to an underestimate of streamfunction amplitudes in the streamfunction analyses.
Fig. 9 shows the same plot for \( \langle u^2 \rangle \), together with the observed Southern Hemisphere values which seem to be similar to those of the Northern Hemisphere (Obasi, 1963). Agreement in Case B is excellent. The introduction of lateral basic shear (especially pronounced in the upper layer) in going from Case A to Case B affects a marked reduction in \( \langle u^2 \rangle \), while producing little change in \( \langle v^2 \rangle \). The presence of horizontal shear is evidently important in determining the latitudinal distribution of eddy kinetic energy, even though the zonal current does not have a maximum in absolute vorticity and is therefore barotropically stable. We recall that \( \bar{u}_1 \) becomes easterly equatorward of 8° and \( \bar{u}_2 \) equatorward of 25°. To this extent the present results are consistent with the remark by Charney (WMO, 1967) that "stationary or westerly wave-disturbances cannot propagate far into an easterly regime." A similar suggestion for stationary waves was made by Eliassen and Palm (1960). In any event, the model demonstrates that lateral coupling can account for much of the observed large-scale eddy kinetic energy in low latitudes.

A plot showing the contribution to the latitudinal sum of the variance from each wavenumber \( n \) and frequency band will indicate the dominant modes of the flow averaged over the region. Such statistics for Case B are shown in Figs. 10 and 11 for \( \langle v^2 \rangle \) and \( \langle u^2 \rangle \), respectively. (The corresponding plots for \( \langle v^2 \rangle \) and \( \langle u^2 \rangle \) are not presented because their small magnitude make any physical interpretation of detailed structure rather irrelevant.) Fig. 10 shows that most of the areal-integrated variance of \( \langle v^2 \rangle \) is associated with wavenumbers 3–7 and in frequency bands corresponding to

---

**Fig. 11.** Observed values of \( \sigma^2 \) in wavenumber-frequency domain.

**Fig. 12.** Observed values of the ratio of \( \langle v^2 \rangle \) at the equator to the latitudinal average of \( \langle v^2 \rangle \) in wavenumber-frequency domain.
a period of 10–40 days. In Fig. 11 we also find overwhelming dominance by the low frequency modes, although it should be noted that the low wavenumbers (1–5) are most important. This difference in $n$ dependence already suggests the horizontal nondivergent character of the flow, since $v$ is proportional to $n$ in nondivergent motion. These statistics for case A are given in Mak (1968). They are qualitatively the same as in Case B, but their magnitudes are generally larger. Since the statistics in Figs. 10 and 11 require a Fourier decomposition in longitude, existing low-latitude wind data is insufficient to allow a comparison.

The secondary maximum in Fig. 10 at $n=3$, $\sigma=22/180$ is associated with motions of wavelengths about 10,000 km and periods about 5 days. Yanai and Maruyama (1966) found westward-propagating disturbances of this wavelength and period to be very noticeable around 20 km over the central equatorial Pacific during March–July 1958. Although there is no stratospheric region as such in our model, its upper level might reflect some of the lower stratospheric features of the real atmosphere. It is, therefore, of special interest to determine the extent to which the motions associated with this maximum in Fig. 10 are concentrated over the equatorial region. Fig. 12 contains the ratio $R=(\langle v^2 \rangle)$ at equator)/\langle (\bar{v}^2) \rangle$, on a wavenumber-frequency plot. The values of $R$ reach a maximum of 2.4 at $n=3$ and $\sigma=21/180$ cycles (12 hr)$^{-1}$. Detailed examination of the latitudinal distribution of $\langle v^2 \rangle$ in this range of $\sigma$ and $n$ verifies that these motions are confined near the equator, with $\langle v^2 \rangle$ dropping by as much as a factor of 5 from the equator to 20°. This agrees with the latest finding of Maruyama (1967). Furthermore, we may deduce that these motions have a westward propagation, since they evidently arise from the large values of the fundamental solutions associated with the Rossby wave of smallest latitudinal wavenumber (see Figs. 5 and 6). In view of all these common features, the dominant response in $\langle v^2 \rangle$ found at the equator in our model tropics appears to be the counterpart of the real phenomenon reported by Yanai and Maruyama. Since such waves in the model tropics

**Table 4. Spectral analysis of $\langle v^2 \rangle$ for various latitudes.**

<table>
<thead>
<tr>
<th>Center of frequency band</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>[cycles (12 hr)$^{-1}$]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/180</td>
<td>0.0</td>
<td>0.5</td>
<td>2.6</td>
<td>7.6</td>
</tr>
<tr>
<td>3/180</td>
<td>0.4</td>
<td>1.3</td>
<td>5.0</td>
<td>14.5</td>
</tr>
<tr>
<td>5/180</td>
<td>1.1</td>
<td>2.7</td>
<td>5.5</td>
<td>12.8</td>
</tr>
<tr>
<td>7/180</td>
<td>0.4</td>
<td>5.1</td>
<td>4.5</td>
<td>9.4</td>
</tr>
<tr>
<td>9/180</td>
<td>0.3</td>
<td>5.2</td>
<td>4.5</td>
<td>7.9</td>
</tr>
<tr>
<td>11/180</td>
<td>0.2</td>
<td>0.5</td>
<td>1.3</td>
<td>6.5</td>
</tr>
<tr>
<td>13/180</td>
<td>0.3</td>
<td>0.3</td>
<td>0.5</td>
<td>4.3</td>
</tr>
<tr>
<td>15/180</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>4.4</td>
</tr>
<tr>
<td>17/180</td>
<td>0.7</td>
<td>0.4</td>
<td>0.4</td>
<td>3.6</td>
</tr>
<tr>
<td>19/180</td>
<td>1.4</td>
<td>0.8</td>
<td>0.4</td>
<td>3.4</td>
</tr>
<tr>
<td>21/180</td>
<td>2.6</td>
<td>1.7</td>
<td>0.5</td>
<td>3.4</td>
</tr>
<tr>
<td>23/180</td>
<td>3.9</td>
<td>1.2</td>
<td>0.5</td>
<td>3.3</td>
</tr>
<tr>
<td>25/180</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
<td>2.0</td>
</tr>
<tr>
<td>27/180</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>2.0</td>
</tr>
<tr>
<td>29/180</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>2.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Wavenumber</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.6</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>3.9</td>
<td>2.6</td>
<td>1.5</td>
<td>1.8</td>
</tr>
<tr>
<td>3</td>
<td>4.1</td>
<td>6.0</td>
<td>3.9</td>
<td>3.8</td>
</tr>
<tr>
<td>4</td>
<td>1.8</td>
<td>5.8</td>
<td>4.8</td>
<td>6.1</td>
</tr>
<tr>
<td>5</td>
<td>1.6</td>
<td>3.4</td>
<td>4.3</td>
<td>9.9</td>
</tr>
<tr>
<td>6</td>
<td>0.8</td>
<td>1.4</td>
<td>5.0</td>
<td>17.1</td>
</tr>
<tr>
<td>7</td>
<td>0.4</td>
<td>0.7</td>
<td>3.0</td>
<td>14.8</td>
</tr>
<tr>
<td>8</td>
<td>0.1</td>
<td>0.2</td>
<td>1.7</td>
<td>10.9</td>
</tr>
<tr>
<td>9</td>
<td>0.0</td>
<td>0.1</td>
<td>1.1</td>
<td>9.9</td>
</tr>
<tr>
<td>10</td>
<td>0.0</td>
<td>0.0</td>
<td>0.5</td>
<td>5.5</td>
</tr>
<tr>
<td>11</td>
<td>0.0</td>
<td>0.0</td>
<td>0.3</td>
<td>4.3</td>
</tr>
<tr>
<td>12</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
<td>2.4</td>
</tr>
</tbody>
</table>
derive their energy from lateral forcing, their counterparts in the atmosphere are presumably maintained in a similar manner.

The $R$-plot for $\langle u^2 \rangle$ is shown in Fig. 13. Here there is no corresponding maximum at the 5-day period, but instead there is a maximum at frequency corresponding to periods around 12 days and wavelength around 10,000 km. This feature has not been observed and hence remains to be confirmed or rejected. Further theoretical investigation about this is being made.

Table 4a gives, at four latitudes, the total contribution to $\langle v^2 \rangle$ from the 12 wavenumbers in each frequency band (i.e. the prediction of the power spectrum of $\langle v^2 \rangle$), and Table 4b the wavenumber variation of the

---

**Fig. 14.** Predicted standard deviation of $u$ velocity, $\omega$, at 500 mb:
dashed line, Case A; solid line, Case B.

**Fig. 15.** Predicted standard deviation of temperature at 500 mb, and observed values by Peixoto:
dashed line, Case A; solid line, Case B.
total contribution from the 15 frequency bands for the same latitudes.

As far as the frequency dependence is concerned, \( \langle v_f^2 \rangle \) in the northern half of the model tropics has a maximum at a period of about 20 days and decreases monotonically with increasing frequency; whereas in the southern half of the model tropics it has a secondary maximum at a period of about 5 days. The latter can be identified with the Yanai-Maruyama phenomenon. As far as the wavenumber dependence is concerned, \( \langle v_f^2 \rangle \) at 30N is mainly associated with wavenumbers from 5-9. But at decreasing latitudes, the maximum of \( \langle v_f^2 \rangle \) shifts systematically towards the smaller wavenumbers.

The corresponding results for \( \langle u_f^2 \rangle \) are not shown. Most of the contributions are associated with periods from 2 weeks to 40 days and with wavenumbers from 2-5. No systematic dependence on latitude is present. Verification of these predicted characteristics would be interesting. The frequency dependence is simpler since only single station data may suffice. For checking the wavenumber dependence, on the other hand, good data coverage in longitude is necessary.

b. Variance of \( \omega \) and temperature at the 500-mb level

Fig. 14 shows the theoretical value of the square root of \( \langle \omega^2 \rangle \) at the 500-mb level of the model as a function of latitude. In Case B it drops from about \( 10 \times 10^{-4} \) mb sec\(^{-1} \) at 30N to about \( 1.5 \times 10^{-4} \) mb sec\(^{-1} \) at 20N and then decreases gradually to \( 0.4 \times 10^{-4} \) mb sec\(^{-1} \) at the equator. This variation corresponds to an rms value of horizontal divergence at the two layers of \( 2 \times 10^{-6} \), \( 0.3 \times 10^{-6} \), and \( 0.15 \times 10^{-6} \) sec\(^{-1} \). The value at 30N in the model is comparable to the typical values in middle latitudes. There are no observed values of this quantity for the tropical region. Nevertheless, the theoretical prediction of a decrease of horizontal divergence by a factor of 7 from the sub-tropics to the tropics is again consistent with Charney’s (1963) scale argument. [Put briefly, his argument is that the decrease of the Coriolis parameter equatorward requires smaller horizontal pressure and temperature gradients if the horizontal velocity and length scales remain similar to those of higher latitudes. The approximately constant static stability is consistent with this then only if \( \omega \) decreases as well. The present paper extends this argument by not assuming anything about the horizontal velocity and length scales.]

A small \( \langle \omega^2 \rangle \) implies a small \( \langle T^2 \rangle \), in the absence of intense local heating. Fig. 15 shows \( \langle T^2 \rangle \) as a function of latitude, together with the observed values collected by Peixoto (1960). The theoretical result does show a substantial decrease with latitude, and is therefore compatible with the result for \( \omega^2 \) mentioned above. The agreement between the observed value of \( \langle T^2 \rangle \) and the theoretical values is very good in both cases, and thus leads one to believe that the theoretical value of \( \omega^2 \) should also be reasonably realistic. Since the variance of temperature is proportional to the eddy available potential energy, the theoretical result in Fig. 15 can then be interpreted as a rapid equatorward decrease of eddy available potential energy.

c. Cross-latitude eddy fluxes of sensible heat, wave energy and momentum

Fig. 16 shows \( 2R \ln 2^{-1} (v_1 + v_2)/(\phi_1 - \phi_2) \), where \( R \) is the gas constant for dry air, as a function of latitude. This is the eddy contribution to \( \langle T \rangle_{\text{TOTAL}} \) at 500 mb, by virtue of the hydrostatic approximation. The pre-
dicted value for both cases is negative everywhere and its magnitude is small. Except near 30N, the observed values at 500 mb obtained by Kidson at M.I.T. and Peixoto (1960) are also negative and agree well with the theoretical values. A negative value of \( \langle \gamma T \rangle \) represents a countergradient heat flux since the basic temperature gradient is poleward. Such a feature has also been reproduced in a numerical experiment by Smagorinsky et al. (1965). Because this corresponds to a net conversion from eddy available potential energy to zonal available potential energy, it implies a severe constraint upon the energetics of the asymmetric motions in the model. The parametrized radiation cooling destroys eddy available potential energy; it follows that the latter must be main-

![Diagram](http://journals.ametsoc.org/jas/article-pdf/26/1/41/3416712/1520-0469(1969)026_0041_ldsmit_2_0_co_2.pdf)

**Fig. 17.** Predicted poleward wave energy flux at 250 and 750 mb, and observed values by Kidson, using crosses for 250 mb and dots for 850 mb: dashed lines, Case A; solid lines, Case B.

![Diagram](http://journals.ametsoc.org/jas/article-pdf/26/1/41/3416712/1520-0469(1969)026_0041_ldsmit_2_0_co_2.pdf)

**Fig. 18.** Predicted poleward zonal momentum flux by eddies at 250 and 750 mb, and observed values by Starr and White, and by Kidson: dashed lines, Case A; solid lines, Case B.
tained by a net conversion from eddy kinetic energy, which in turn has to be replenished either by conversion from zonal kinetic energy or by an inflow of wave energy into the model tropics.

Fig. 17 shows \(\langle v\phi_1 \rangle\) and \(\langle v\phi_2 \rangle\) as functions of latitude. They represent the poleward wave-energy flux at the 250- and 750-mb levels, and are negative for both levels, increasing monotonically toward zero at the equator. The magnitude of \(\langle v\phi_1 \rangle\) is much larger than that of \(\langle v\phi_2 \rangle\). Such results imply a net pressure work done on the model tropics by the lateral forcing, primarily at the upper level. The agreement between the observed and the theoretical values is good.

Fig. 18 shows the Reynolds stresses \(\langle u_1 v_1 \rangle\) and \(\langle u_2 v_2 \rangle\). They are proportional to the poleward momentum flux by eddies at the 250- and 750-mb levels, and are positive at both levels, with \(\langle u_1 v_1 \rangle\) much larger than \(\langle u_2 v_2 \rangle\) except in the equatorial region where both decrease toward zero. The observed values obtained by Kidson at M.I.T. and Starr and White (1952, 1954) are also shown in Fig. 18. The theoretical values of \(\langle u_1 v_1 \rangle\) in Case B are much more realistic than those in Case A. In fact, the former are in good agreement with the observed values. This improvement is undoubtedly related to the great reduction of \(\langle u_1^2 \rangle\) in Fig. 9 when horizontal shear is incorporated in the basic current.

In summary, the empirical lateral forcing on the model tropics results in a strong equatorward wave energy flux, a small equatorward eddy sensible heat flux and at the same time a strong poleward eddy momentum flux. All three kinds of flux are in reasonably good agreement with the observed values.

d. Energetics of the asymmetric motions in the model tropics

The energetics of the disturbances are expressed mathematically by two equations, one describing the rate of eddy kinetic energy and the other the rate of change of eddy available potential energy. They are readily derived from (3.4); thus

\[
\frac{\partial}{\partial t} (K_1 + K_2) = -\left[ (\bar{u} + \Lambda) \frac{\partial K_1}{\partial x} + (\bar{u} - \Lambda) \frac{\partial K_2}{\partial x} \right] \\
- \left[ \frac{d(\bar{u} + \Lambda)}{dy} + \frac{d(\bar{u} - \Lambda)}{dy} \right] \\
+ 2\Lambda (u_1 + u_2) \omega - 2(\phi_1 - \phi_2) \omega \\
- \left[ \begin{array}{c}
\frac{\partial}{\partial x} (u_1 \phi_1 + u_2 \phi_2) + \frac{\partial}{\partial y} (v_1 \phi_1 + v_2 \phi_2) \\
- 2[\alpha K_2 + \beta K_1 - \beta (u_1 u_2 + v_1 v_2)]
\end{array} \right] \\
= -\frac{\partial A}{\partial x} + \frac{\Lambda}{2\varepsilon} (\phi_1 - \phi_2) \\
+ 2(\phi_1 - \phi_2) \omega - 2\gamma A,
\]
where

\[ K_l = \frac{1}{2} (u_1^2 + v_1^2), \quad l = 1, 2 \]

\[ A = \frac{1}{4\xi} (\phi_1 - \phi_2)^2, \quad \omega = \frac{1}{2} \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right). \]

By taking the ensemble and \( x \) average of (6.2), we obtain

\[ 0 = 2\Delta((u_1 + u_2)\omega) - \frac{d}{dy} \left[ \frac{d}{dy} (u_1 v_1) + \frac{d}{dy} (u_2 v_2) \right] \]

\[ - 2(\phi_1 - \phi_2)\omega - \frac{d}{dy} \left[ (v_1 \phi_1) + (v_2 \phi_2) \right] \]

\[ - 2[\alpha(K_2) + \beta(K_1)] - \beta((u_1 u_2) + (v_1 v_2)) \], \quad (6.3) \]

\[ 0 = -\frac{\Delta}{2\xi} \left( (v_1 + v_2) (\phi_1 - \phi_2) + 2((\phi_1 - \phi_2)\omega) \right) \]

\[ - \frac{\gamma}{2\xi} ((\phi_1 - \phi_2)^2). \quad (6.4) \]

Writing KE for kinetic energy and APE for available potential energy, we can interpret Eqs. (6.3) and (6.4) in terms of straightforward physical processes. The first two terms in (6.3) represent the net conversion from zonal KE to eddy KE through the Reynolds stresses. The third term represents conversion from eddy APE to eddy KE. The fourth term is the convergence of the wave energy flux, sometimes known as pressure work. The last term is the frictional dissipation. The first two terms of (6.4) represent the conversion from the zonal APE and eddy KE to eddy APE. The third term is simply the destruction due to radiation.

The numerical values of the five terms in (6.3) for Case B are shown in Fig. 19. The only supply of eddy kinetic energy in the model tropics is the pressure work, from the equatorward flux of wave energy through the lateral boundaries. The other four terms are negative and represent sinks of eddy KE. (Their sum balances the source term very closely at each latitude.) The largest sink is the conversion to zonal KE through the horizontal Reynolds stresses. The frictional dissipation is also substantial. The remaining two terms are small everywhere. Hence, the asymmetric disturbances in the model tropics have no internal source of eddy KE. The inflow of wave energy is more than enough to compensate for the frictional dissipation, with most of the residue converted to zonal KE.

The numerical values of the three processes that constitute the eddy APE balance are shown in Fig. 19 with an expanded vertical scale. (The balance is not as good as that in Fig. 19; most likely, because the values in Fig. 20 are much smaller than those in Fig. 19, and a small discrepancy in the latter can lead to a sizable imbalance among the former.) The only source of eddy APE is the conversion from eddy KE; the two losses due to radiation and conversion to zonal APE are
smaller. The latter is sometimes dramatically referred to as a "refrigeration process," in the sense that the warmer part of the atmosphere is being warmed up by the advection of enthalpy by the motions. However, the values in Figs. 19 and 20 show that the amount of energy involved is small and that the refrigeration process is an inefficient one.

The energetics of the asymmetric motions in the model tropics are summarized schematically in Fig. 21. This is an incomplete description of the energetics of the model tropics because it does not include the energetics of the zonally averaged circulation. But this is all the information that one can deduce from this model as it stands.

There is apparently no comprehensive data evaluation of the energetic processes in the tropics. We may, however, compare the theoretical values with the computer-generated data based upon a presumably realistic general circulation model, such as those in Manabe and Smagorinsky (1967). They found that the energetics for low latitudes in a dry and a wet numerical model were quite different. They did not compute the wave energy flux, but in their dry model the conversion between eddy KE and eddy APE was very small and a weak equatorward flux of eddy sensible heat was also found, as in our model tropics. On the other hand, their wet model had a strong conversion from eddy APE to eddy KE, as well as from zonal KE to eddy KE. Manabe and Smagorinsky point out that this latter feature is unrealistic. If condensation is indeed the main source of eddy kinetic energy in the real tropics, any dry model such as this one would be inappropriate. It is, however, conceivable that since precipitation tends to be concentrated along narrow regions in the tropics (as suggested by satellite cloud pictures), the release of latent heat by and large mainly affects the zonally averaged motions and has only minor effects upon the large-scale asymmetric motions over the rest of the tropics. (Hurricanes are, of course, a different phenomenon.) The intensity of the large-scale tropical eddies could then, as in the present model, depend upon the baroclinic activities in higher latitudes.

As a conclusion to this presentation of the predicted energetics it is important to emphasize again that the computed wave energy flux was into the tropics from the boundaries. (This was true not only for the total $\langle \psi \phi \rangle$ but was true at all $\sigma$ for all $n$.) This result is consistent with our broad approach that the tropical eddy motions are primarily a response of the dynamically stable tropics to the unstable baroclinic processes in higher latitudes. In fact, if the model tropics had turned out to be dynamically unstable, with an outward flux of wave energy, our fundamental procedure of determining the stochastic behavior of the tropics from known stochastic boundary conditions would have been wrong in principle.

Acknowledgments. This work is an extension of the author's doctoral thesis, which was undertaken on the suggestion and under the guidance of Prof. Norman A. Phillips. The author expresses his deep gratitude to Prof. Phillips for his enthusiasm and most valuable advice throughout this research and for his help in preparing the manuscript. Thanks are due to Prof. J. Charney for many interesting discussions. The author is also indebted to Prof. C. Wunsch for his suggestions concerning the spectral analysis; to J. W. Kidson of the Planetary Circulation Project at M.I.T. for offering his unpublished results quoted in this work; and to the National Center for Atmospheric Research for providing two years' streamfunction data. Appreciation is due to Mrs. J. Thorne and Miss I. Cole for typing this manuscript and drafting the figures. The numerical computations were made at the M.I.T. Computation Center. This work was supported by the National Science Foundation under contract G-18085 with the Massachusetts Institute of Technology.

REFERENCES


——, and ——, 1954: Two years of momentum flux data for 13°N. Tellus, 6, 180-181.

