Symmetry Relationships for Scattering of Polarized Light in a Slab of Randomly Oriented Particles

J. W. Hovenier

Leiden Observatory, Leiden, The Netherlands

(Manuscript received 1 November 1968)

ABSTRACT

In this paper we discuss symmetry properties of the matrices describing the reflection and transmission of polarized radiation by a slab of randomly oriented particles. A complete set of such symmetry relations valid in the common case in which there is no birefringence or dichroism is given. The derivation proceeds via 1) the symmetry properties of the phase matrix describing the scattering in a volume element and 2) the symmetry properties of the reflection and transmission matrices based on single scattering only. Birefringence and dichroism may occur if the particles do not have a plane of symmetry. The study of symmetry relations for this case is not carried beyond the stage of the phase matrix.

Possible applications and some errors in the literature are pointed out.

1. Introduction

The study of multiple scattering of polarized light in a plane-parallel slab presents many complications. This is illustrated by the classical treatment of Rayleigh scattering by isotropic particles in a homogeneous layer (Chandrasekhar, 1960). Yet this is still the simplest case because of the form of the scattering matrix and its dependence on directions. Chandrasekhar uses some of the symmetries displayed by the phase matrix, the reflection matrix and the transmission matrix. For other cases hardly any symmetry relationships have been found apart from reciprocity relationships (cf. Section 5).

In this paper we derive, under very wide assumptions, many more symmetry relationships and group these into sets. Each set is shown to be complete and all relationships are explained in terms of symmetries in space and time.

2. The phase matrix

Consider a plane-parallel atmosphere containing particles which scatter light independently and without change of frequency. One side is called the top and the other the bottom. We shall specify directions by means of an azimuth angle and a direction cosine. At a general point in space we use $u$ as the cosine of the angle with downward normal ($-1 \leq u \leq +1$); at the top and bottom of the layer under consideration we use $\mu = |u|$ with an indication as to whether the light is incident or emergent (see Fig. 1). The azimuth angle $\phi$ is measured clockwise when looking from bottom to top.

We use the Stokes parameters $I$, $Q$, $U$ and $V$ to describe the state of polarization of a beam. These are defined such that the local meridian plane acts as a plane of reference (see Chandrasekhar, Chap. 1).

a. Particles with a plane of symmetry

We shall first consider a volume element containing either an assembly of identical particles that have a plane of symmetry and are randomly oriented, or a mixture of such assemblies. A mixture of assemblies may be due, for example, to a distribution in size or refractive index of the particles. Our assumptions include as a very special case scattering by homogeneous spherical particles of different sizes or refractive indices.

On grouping the Stokes parameters into a column vector we can say that when light is incident on the volume element from a direction $(u', \phi')$ and scattered once in a direction $(u, \phi)$, this corresponds to a transformation of the Stokes parameters by the $4 \times 4$ phase matrix $Z(u, u', \phi - \phi')$ apart from some scalars which are not directionally dependent. As shown by Chandrasekhar (1960) this phase matrix can be written as a product of three $4 \times 4$ matrices as

$$Z(u, u', \phi - \phi') = L(\pi - i_2) F(\theta) L(-i_1). \quad (1)$$

The scattering angle $\theta$ and the rotation angles $i_1$ and $i_2$ are defined in Fig. 2.

Owing to our special choice of Stokes parameters the rotation matrices $L$ have very simple forms, i.e.,

$$L(\pi - i_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2)$$

The scattering matrix $F(\theta)$ describes the scattering when the plane of scattering serves as the plane of reference for the Stokes parameters. Since we deal with randomly oriented particles, it depends only on $\cos \theta$;
and because each assembly consists of particles with a plane of symmetry, it can be expressed as follows in six real functions (van de Hulst, 1957):

\[
F(\theta) = \begin{bmatrix}
  a_1(\theta) & b_1(\theta) & 0 & 0 \\
  b_1(\theta) & a_2(\theta) & 0 & 0 \\
  0 & 0 & a_3(\theta) & b_2(\theta) \\
  0 & 0 & -b_2(\theta) & a_4(\theta)
\end{bmatrix}.
\]

(3)

Introducing the diagonal matrices

\[
P = P^{-1} = \ddot{P} = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix},
\]

and

\[
Q = Q^{-1} = \ddot{Q} = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1
\end{bmatrix},
\]

where the superscript \(-1\) stands for the inverse matrix and the tilde for the transposed matrix, we see from (3) that the scattering matrix has the symmetry properties

\[
F(\theta) = PF(\theta)P,
\]

(4)

\[
F(\theta) = PQF(\theta)QP.
\]

(5)

Pre- and postmultiplication with \(P\) amounts to changing the signs of the non-diagonal elements of the third row and column. Similarly, pre- and postmultiplication with \(Q\) brings about an inversion of the signs of the non-diagonal elements of the fourth row and column.

Relations (4) and (5) can be explained in terms of symmetry in time and space, respectively, as follows.

The first is a genuine reciprocity relationship because it corresponds to a reversal of time. The multiplications by the matrix \(P\) cause a switch of the sign of the Stokes parameters \(U\). That this should happen is easily understood when it is realized that \(U/Q\) is a measure of the angle between the long axis of the polarization ellipse and a direction parallel to the scattering plane measured in a sense that is linked to the direction of propagation.

Relation (5) reveals the symmetry with respect to the scattering plane. To obtain the Stokes parameters of a beam which is the mirror image of a second beam with respect to the scattering plane, we must only invert the sign of \(U\) of the second beam but also of \(V\), because left-handed polarized light becomes right-handed and vice versa. Further, since each assembly contains particles with a plane of symmetry, the scattering plane is a plane of symmetry for the scattering process. Hence, if an incident beam \(\{I_0, Q_0, U_0, V_0\}\) gives a scattered beam \(\{I, Q, U, V\}\), then an incident beam \(\{I_0, Q_0, -U_0, -V_0\}\) must yield a scattered beam \(\{I, Q, -U, -V\}\) in the same direction.

The angles \(i_1, i_2\) and \(\theta\) occurring in (1) can be expressed in \(u, u'\) and \(\varphi - \varphi'\) with the help of spherical trigonometry. We find

\[
\cos \theta = uu' + (1 - u^2)(1 - u'^2) \cos (\varphi - \varphi'),
\]

(6)

\[
\cos i_1 = \frac{-u + u' \cos \theta}{\pm (1 - \cos \theta)^2(1 - u^2)^{1/2}},
\]

(7)

\[
\cos i_2 = \frac{-u' + u \cos \theta}{\pm (1 - \cos \theta)^2(1 - u'^2)^{1/2}}.
\]

(8)
where in the last two equations the plus sign is to be taken when $\pi < \phi - \phi' < 2\pi$ and the minus sign when $0 < \phi - \phi' < \pi$.

We further use

$$\sin 2\beta = 2(1 - \cos^2 \beta) \cos \beta,$$

$$\cos 2\beta = 2 \cos^2 \beta - 1,$$

where $\beta$ is $\phi_1$ or $\phi_2$. When the denominator of (7) or (8) becomes zero, one must take the limits.

Now it is clear from (6) that there are essentially three basic transpositions of variables which leave the scattering matrix unaltered, namely:

1) interchanging $\phi$ and $\phi'$,
2) interchanging $u$ and $u'$,
3) changing the sign of $u$ and $u'$ simultaneously.

Further it is clear that two or three different operations can be performed successively in arbitrary order so that we get seven different operations in total.

The behavior of the phase matrix under these operations can now be studied with the equations given above for the elements of the rotation matrices. In this way we obtain the following seven symmetry relationships for the phase matrix:

<table>
<thead>
<tr>
<th>Identification</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(Z(u', u, \phi - \phi') = QZ(u, u', \phi - \phi')Q)</td>
</tr>
<tr>
<td>B</td>
<td>(Z(-u', -u, \phi - \phi') = PZ(u, u', \phi - \phi')P)</td>
</tr>
<tr>
<td>C</td>
<td>(Z(-u, -u', \phi - \phi') = Z(u, u', \phi - \phi'))</td>
</tr>
<tr>
<td>D</td>
<td>(Z(u, u', \phi - \phi') = PQZ(u, u', \phi - \phi')QP)</td>
</tr>
<tr>
<td>E</td>
<td>(Z(u', u, \phi - \phi') = PZ(u, u', \phi - \phi')P)</td>
</tr>
<tr>
<td>F</td>
<td>(Z(-u', -u, \phi - \phi') = QZ(u, u', \phi - \phi')Q)</td>
</tr>
<tr>
<td>G</td>
<td>(Z(-u, -u', \phi - \phi') = PQZ(u, u', \phi - \phi')QP)</td>
</tr>
</tbody>
</table>

To prove these relations we start with the three basic relationships D, E and G, which correspond to the three basic transpositions.

**Proof of D.** From (7) and (8) we see that interchanging $\phi$ and $\phi'$ causes $\cos \phi_1$ and $\cos \phi_2$ to change sign. Then (9) and (10) show that $\sin 2\phi_1$ and $\sin 2\phi_2$ change sign but $\cos 2\phi_1$ and $\cos 2\phi_2$ keep their sign. Now (2) shows that the inversion of sign of $\sin 2\phi_1$ and $\sin 2\phi_2$ can be obtained by pre- and postmultiplication of the rotation matrices with $QP$ and $QP$, while according to (5) such a multiplication does not change $F(\theta)$; thus,

\[
Z(u, u', \phi - \phi') = \{PQL(-\pi - \phi_i)QP\}F(\theta)\{PQL(-\pi - \phi_i)QP\},
\]

\[
= PQ\{L(-\pi - \phi_i)L(-\phi_i)\}Q,
\]

\[
= PQZ(u, u', \phi - \phi')QP.
\]

**Proof of E.** It is clear from (7) and (8) that the interchange of $u$ and $u'$ results in an interchange of the rotation angles $\phi_1$ and $\phi_2$. Hence in view of (1), (2) and (4),

\[
Z(u', u, \phi - \phi') = L(\pi - \phi_i)F(\theta)L(-\phi_i),
\]

\[
= (PL(-\phi_i)P)\bar{F}(\theta)P(L(-\phi_i)P),
\]

\[
= P\{L(-\phi_i)\bar{F}(\theta)L(-\phi_i)\}P,
\]

\[
= P\bar{Z}(u, u', \phi - \phi')P.
\]

**Proof of G.** Eqs. (7) and (8) show that changing the signs of $u$ and $u'$ simultaneously has exactly the same effect as interchanging $\phi$ and $\phi'$, so that application of D gives directly relationship G.

The remaining relations can now be proved in a similar way or by combining relations D, E and G; for example, relation A is a combination of D and E, so that

\[
Z(u', u, \phi - \phi') = PQZ(u', u, \phi - \phi')QP,
\]

\[
= PQ(P\bar{Z}(u, u', \phi - \phi')P)QP,
\]

\[
= QZ(u, u', \phi - \phi')Q.
\]

From an algebraic point of view the simplest basis is formed by relations D, E and G. However, insight into the geometrical and time symmetries involved is probably achieved quickest by taking as a basis the relationships B, C and D, each of which has a simple explanation in terms of symmetry in space or time (Fig. 3). First we observe that the meridian planes of the incident and scattered beam can be simultaneously rotated about the vertical through an arbitrary angle without changing the phase matrix because only the difference of the azimuth angles is involved.

Relation B is a genuine reciprocity relation which should be compared with (4) and the comments made about it, while C expresses the fact that nothing changes in the scattering process when the equatorial plane, together with the incident and scattered beams, is turned upside down. Because the azimuth angle is measured clockwise when looking from bottom to top, not only the signs of $u$ and $u'$ change, but also the azimuth difference switches sign. Relation D can be explained from the symmetry with respect to the meridian plane of incidence. Its explanation is analogous to that of (5) where we used the scattering plane as a plane of symmetry. The algebraic proof of D we have given shows it to be a direct result of the symmetry property (5) of the scattering matrix $F(\theta)$. From similar proofs it can be seen that B is a direct result of (4) and that C can be proved without any symmetry relation for $F(\theta)$ apart from its dependence on $\theta$ only.

A very noteworthy corollary of D is that the elements $Z_{21}, Z_{35}, Z_{45}, Z_{43}, Z_{15}, Z_{14}, Z_{23}$ and $Z_{21}$ are odd functions of $(\phi - \phi')$, and the other elements are even functions of $(\phi - \phi')$. In a Fourier series expansion the first 8 elements will therefore contain only sine terms and the remaining 8 elements only cosine terms.

Fig. 3 also shows an example of a more complicated situation corresponding to relation A. This is a configuration in which the directions of the incident and scattered beams are exchanged. Such an exchange can...
be obtained from the initial situation by applying the following three basic operations in arbitrary order: i) reverse the equatorial plane together with the incident and scattered beam, ii) mirror the scattered beam with respect to the meridian plane of incidence, iii) reverse the time. Hence the algebraic result that relation A is a combination of B, C and D is clear.

The reader will have no difficulties now in finding similar interpretations for relations E, F and G.

Consequently, we have now reached the important conclusion that all of the relations A to G can be explained by symmetry arguments only. As an illustration of the insight obtained now we return to the fact that changing the signs of \( u \) and \( u' \) simultaneously has the same effect as the interchange of \( \phi \) and \( \phi' \) (see algebraic proof of \( G \)). This is now easily understood because the incident and scattered beams may be mirrored with respect to the equatorial plane by applying the operations i) and ii) mentioned above; stated differently, relation \( G \) is a combination of \( D \) and \( C \).

b. Particles without a plane of symmetry

Consider a volume element containing an assembly of identical particles that have no plane of symmetry and are randomly oriented. The scattering matrix
still depends only on \( \cos \theta \) but it now contains 10 real functions as shown by van de Hulst (1957), i.e.,

\[
F(\theta) = \begin{bmatrix}
  a_1(\theta) & b_1(\theta) & b_3(\theta) & b_4(\theta) \\
  b_1(\theta) & a_2(\theta) & b_4(\theta) & b_5(\theta) \\
  -b_4(\theta) & -a_4(\theta) & a_3(\theta) & b_3(\theta) \\
  b_5(\theta) & b_6(\theta) & -b_1(\theta) & a_4(\theta)
\end{bmatrix}
\]  \( (11) \)

The same form will hold in general for a mixture of such assemblies. It is only in some special cases that the scattering matrix has again the form expressed by (3) in which case all considerations of Section 2a are valid. This happens, for instance, when the mixture is such that for all particles their mirror images are found in equal numbers and in random orientation. A further example is Rayleigh scattering by randomly oriented particles of any kind.

The matrix given by (11) still obeys (4) but no longer (5). This can be understood immediately from the explanations we have given for these equations. As a result we can prove now in the same way as in Section 2a that only relations B, C and E, but not A, D, F and G, hold in the case of a scattering matrix expressed by (11). This is exactly what one would expect in view of their interpretations in terms of geometrical and time symmetries as presented in Section 2a.

3. The reflection and transmission matrix

The various symmetries displayed by the phase matrix give rise to several symmetries in the light that emerges at the top and bottom of the slab shown in Fig. 1. This is supposed not to emit light by itself and to have a perfectly absorbing lower boundary. We shall confine our treatment to cases where the scattering matrix per unit volume \( F(\theta) \) is of the form (3). The full treatment of the reflection and transmission on the basis of (11), leading to birefringence and dichroism, would be time-consuming and of little practical interest. With our assumptions we have now a unique optical depth \( 0 \leq \tau \leq b \) measured from the top, and an albedo \( a \).

Further, we suppose that incident polarized light from either top or bottom is normalized in such a way that the incoming flux is \( \pi \). The reflection matrix \( R(\mu, \mu_0, \varphi - \varphi_0) \) and the transmission matrix \( T(\mu, \mu_0, \varphi - \varphi_0) \) are now defined by

\[
I_{\text{out.top}}(\mu, \varphi) = \frac{1}{\pi} \int_0^1 \mu d \mu_0 \int_0^{2\pi} d \varphi_0 
R(\mu, \mu_0, \varphi - \varphi_0) I_{\text{in.top}}(\mu_0, \varphi_0),
\]

\[
I_{\text{out.bottom}}(\mu, \varphi) = \frac{1}{\pi} \int_0^1 \mu d \mu_0 \int_0^{2\pi} d \varphi_0 
T(\mu, \mu_0, \varphi - \varphi_0) I_{\text{in.bottom}}(\mu_0, \varphi_0),
\]

where the symbol \( I \) incorporates all four Stokes parameters, each of which has the dimension of a specific intensity.

In the same way we can direct light into the slab at the bottom giving

\[
I_{\text{out.bottom}}(\mu, \varphi) = \frac{1}{\pi} \int_0^1 \mu d \mu_0 \int_0^{2\pi} d \varphi_0 
R(\mu, \mu_0, \varphi - \varphi_0) I_{\text{in.top}}(\mu_0, \varphi_0),
\]

\[
I_{\text{out.top}}(\mu, \varphi) = \frac{1}{\pi} \int_0^1 \mu d \mu_0 \int_0^{2\pi} d \varphi_0 
T(\mu, \mu_0, \varphi - \varphi_0) I_{\text{in.bottom}}(\mu_0, \varphi_0).
\]

a. Homogeneous atmospheres

In this section we shall consider a homogeneous atmosphere; that is, a plane-parallel atmosphere in which the albedo and scattering matrix do not depend on the optical depth.

If we look at the light which is scattered only once we have

\[
T(\mu, \mu_0, \varphi - \varphi_0) = c(\mu, \mu_0) Z(\mu, \mu_0, \varphi - \varphi_0),
\]

\[
R(\mu, \mu_0, \varphi - \varphi_0) = d(\mu, \mu_0) Z(-\mu, \mu_0, \varphi - \varphi_0),
\]

\[
T^*(\mu, \mu_0, \varphi - \varphi_0) = c(\mu, \mu_0) Z(-\mu, -\mu_0, \varphi - \varphi_0),
\]

\[
R^*(\mu, \mu_0, \varphi - \varphi_0) = d(\mu, \mu_0) Z(\mu, -\mu_0, \varphi - \varphi_0),
\]

where

\[
c(\mu, \mu_0) = c(\mu_0, \mu) = (\mu_0 - \mu)^{-1} \left[ \exp(-b/\mu_0) - \exp(-b/\mu) \right],
\]

\[
d(\mu, \mu_0) = d(\mu_0, \mu) = (\mu_0 + \mu)^{-1} \left[ 1 - \exp(-b/\mu_0 - b/\mu) \right].
\]

A derivation can be readily found by means of the formulae contained in the Appendix.

Now we can easily derive for first-order scattering the following relations:

**Identification**

<table>
<thead>
<tr>
<th>Relation</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^*(\mu, \mu_0, \varphi - \varphi_0) = R(\mu, \mu_0, \varphi - \varphi_0) )</td>
<td>C</td>
</tr>
<tr>
<td>( T^*(\mu, \mu_0, \varphi - \varphi_0) = T(\mu, \mu_0, \varphi - \varphi_0) )</td>
<td>C</td>
</tr>
<tr>
<td>( R(\mu_0, \mu, \varphi - \varphi_0) = QR(\mu_0, \mu, \varphi - \varphi_0)Q )</td>
<td>F</td>
</tr>
<tr>
<td>( T(\mu_0, \mu, \varphi - \varphi_0) = PT(\mu_0, \mu, \varphi - \varphi_0)P )</td>
<td>E</td>
</tr>
<tr>
<td>( R(\mu, \mu_0, \varphi - \varphi_0) = PQR(\mu_0, \mu_0, \varphi - \varphi_0)Q )</td>
<td>D</td>
</tr>
<tr>
<td>( T(\mu, \mu_0, \varphi - \varphi_0) = POT(\mu_0, \mu, \varphi - \varphi_0)P )</td>
<td>D</td>
</tr>
</tbody>
</table>

By combining these basic relations (or using directly...
relations A to G) we also find these further relations:

<table>
<thead>
<tr>
<th>Identification</th>
<th>Relation</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>( \mathbf{R}(\mu_0, \varphi, \varphi - \varphi) = \mathbf{PR}(\mu, \mu_0, \varphi - \varphi_0) \mathbf{P} )</td>
<td>c, e</td>
</tr>
<tr>
<td>h</td>
<td>( \mathbf{T}(\mu_0, \varphi, \varphi - \varphi_0) = \mathbf{QT}(\mu, \mu_0, \varphi - \varphi_0) \mathbf{Q} )</td>
<td>d, f</td>
</tr>
<tr>
<td>i</td>
<td>( \mathbf{R}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{QR}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{Q} )</td>
<td>a, c</td>
</tr>
<tr>
<td>j</td>
<td>( \mathbf{T}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{PT}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{P} )</td>
<td>b, d</td>
</tr>
<tr>
<td>k</td>
<td>( \mathbf{R}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{PQR}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{QP} )</td>
<td>a, e</td>
</tr>
<tr>
<td>l</td>
<td>( \mathbf{T}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{PQT}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{QP} )</td>
<td>b, f</td>
</tr>
<tr>
<td>m</td>
<td>( \mathbf{R}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{PR}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{P} )</td>
<td>a, c, e</td>
</tr>
<tr>
<td>n</td>
<td>( \mathbf{T}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{QT}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{Q} )</td>
<td>b, d, f</td>
</tr>
<tr>
<td>o</td>
<td>( \mathbf{R}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{QR}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{Q} )</td>
<td>a, c</td>
</tr>
<tr>
<td>p</td>
<td>( \mathbf{T}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{PT}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{P} )</td>
<td>b, d</td>
</tr>
<tr>
<td>q</td>
<td>( \mathbf{R}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{PQR}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{QP} )</td>
<td>a, e</td>
</tr>
<tr>
<td>r</td>
<td>( \mathbf{T}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{PQT}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{QP} )</td>
<td>b, f</td>
</tr>
<tr>
<td>s</td>
<td>( \mathbf{R}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{PR}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{P} )</td>
<td>a, c, e</td>
</tr>
<tr>
<td>t</td>
<td>( \mathbf{T}^<em>(\mu_0, \mu, \varphi - \varphi_0) = \mathbf{QT}^</em>(\mu, \mu_0, \varphi - \varphi_0) \mathbf{Q} )</td>
<td>b, d, f</td>
</tr>
</tbody>
</table>

We shall prove in the Appendix that all the relations a to t are also valid for every order of scattering and thus for the total reflected and transmitted light of the layer.

We can divide these relations into three subsets:

1. Relations with no asterisks (c to h).
2. Relations with two asterisks (i to n).
3. Relations with one asterisk (a, b, and o to t).

In subsets 1) and 2) we have the transpositions i) \( \mu \leftrightarrow \mu_0 \), ii) \( \varphi \leftrightarrow \varphi_0 \), iii) \( \mu \leftrightarrow \mu_0 \) and \( \varphi \leftrightarrow \varphi_0 \); in subset 3) the additional possibility of performing no transposition exists. Thus, the total number of different relationships is \((3 \times 2) + (3 \times 2) + (4 \times 2) = 20\).

Again instead of the algebraically attractive basis a to f we can use a “space-time” basis like, for instance, a, b, c, f, g and p. The symmetries expressed by these relations are sketched in Fig. 4 and are quite analogous to those of the phase matrix. Relations a and b express the fact, when polarization is involved, that light incident at the top of a homogeneous atmosphere is equivalent to light incident at the bottom only when the azimuth is counted in the reversed sense.

Thus, the statement, for a homogeneous atmosphere, that \( \mathbf{R} = \mathbf{R}^\ast \) and \( \mathbf{T} = \mathbf{T}^\ast \), sometimes found in the literature, is incorrect and should be replaced by relations q and r. This fact has sometimes been overlooked. In particular, we wish to point out that the integral equations for the reflection and transmission matrix in case of Rayleigh scattering by isotropic particles as formulated by Chandrasekhar are not correct. The source of this error is the fact that in the mathematical expressions of the second and third principle of in-

variance for polarized light the incorrect statement mentioned above was implicitly used.

Relations e and f describe the symmetries with respect to the meridian plane of incidence and give a separation into a set of eight elements that are even functions of \( \varphi - \varphi_0 \) and another set of eight elements that are odd functions of \( \varphi - \varphi_0 \).

Relations g and p are reciprocity relationships.

Relations o and h correspond to an exchange of the incident and outgoing beams; o can be interpreted as a combination of a, e and g, and h as a combination of b, f and p.

For the same reason as mentioned above, Chandrasekhar incorrectly considered both the fundamentally different relations g and h to be the result of the principle of reciprocity.

We could also have drawn initial situations with incident light from below. Then we would have seen, for example, that the reversal of time, with light from below, gives relation m, for reflection; for transmission the reversal gives the relation p as in the case of light from above. Similarly, the exchange of directions with light from below gives relation o for reflection as in the light-from-above case, while relation n results for transmission. Hence all the relations a to t can be explained by symmetry arguments only.

b. Inhomogeneous atmospheres

We consider a plane-parallel atmosphere in which the albedo and scattering matrix are not both independent of optical depth. From symmetry arguments it is directly clear that the reciprocity relationships g, m and p as well as the relations e, f, k and l, which express the symmetry with respect to the meridian plane of incidence, keep their validity in this case. Algebraic proofs may be obtained along similar lines as we used for the symmetry relationships for the homogeneous atmosphere. By combining the preceding equations we also find relations c, i and t to be valid so that there exist for the inhomogeneous atmosphere under consideration only 10 of the 20 symmetry relations a to t.

4. Some applications

It is clear that the symmetry relations for the phase, reflection and transmission matrices can be very useful. In order to be more specific we mention four possible applications.

1) The light which is scattered by a cloud or a planet will have certain symmetries that can be easily derived from the given relationships. This makes it possible to make a quick comparison between the observations and theory, giving checks on the quality of the observations or on assumptions made in the theory (cf. Minnaert, 1941).
2) The addition and subtraction of layers is a beautiful device for obtaining short derivations of some complicated formulae in the theory of radiative transfer. Moreover, a numerical solution is based on these principles which has been shown to work very well; namely, the adding method of van de Hulst. In applying these principles to polarized light, the difference between light falling in at the top and at the bottom, as given by relations a and b, must be realized.

Fig. 4. Symmetry relations of the reflection matrix (left side) and the transmission matrix (right side) when in the initial situation light is incident at the top.
3) The old work of Chandrasekhar for Rayleigh scattering by isotropic particles and the work of Sekera still in progress for other types of scattering of polarized light show that symmetry relations play an important role in reducing the complicated integral equations for \( R(\mu, \mu_0, \varphi - \varphi_0) \) and \( T(\mu, \mu_0, \varphi - \varphi_0) \) to simpler forms.

4) For numerical computations symmetry relationships may be used either as a check or to reduce the number of computations considerably. For example, if \( Z(u, u', \varphi - \varphi') \) is required for a fixed value of \( (\varphi - \varphi') \) [or alternatively for a specific Fourier component] and a number of discrete values of \( -1 \leq u \leq 1 \) and \( -1 \leq u' \leq +1 \), it suffices to take \( 0 \leq u \leq 1 \) and \( |u'| \leq 1 \).

5. Related work of others

Chandrasekhar (1960), in considering Rayleigh scattering by isotropic particles, which is a very special case, mentions only relations A and B for the phase matrix explicitly. The remaining relations are easily verified with his formulae. He also relates relation g for the reflection matrix and relation h for the transmission matrix and regards them both as reciprocity relations, although, as we have seen above, the term is not quite correct for g.

General reciprocity relations are discussed, for example, by van de Hulst in a forthcoming book.

Apart from this we have found in the literature only relation A, although for much more restricted cases than ours (Sekera, 1966). His derivations are different from ours and he also discusses other representations of polarized light that the Stokes parameters we have chosen. We have not done this because a transformation to another representation may always be made if desired.

A'Hearn (1966) finds for a scattering matrix of the type (3) the same division of the phase matrix in even and odd functions of the azimuth difference as we do. However, he does not put this result in the framework of symmetry relationships and his proof is somewhat more elaborate than ours. He also extended this result to the reflection and transmission matrix of a homogeneous atmosphere by using the same technique that Chandrasekhar used for proving the reciprocity relationships in the case of scattering without polarization. In this proof the incorrect integral equations are used for the reflection and transmission matrix. However, due to a cancellation of errors, his result is correct and agrees with our relations e and f. Both authors do not explain their results in terms of symmetries in space and time.

Busbridge (1960) does not consider polarization, so that instead of a phase matrix she has a phase function. With this simplification she mentions lemma 1 (see Appendix) for \( b = \infty \) and no azimuth dependence. Then she gives for the case \( v = 1 \) only an explicit proof, which is similar to ours under part A of lemma 1.

Acknowledgments. I wish to express my gratitude to Prof. H. C. van de Hulst for much advice and discussion. A stimulating discussion with Prof. Z. Sekera is also acknowledged.

APPENDIX

Higher-order Scattering for a Homogeneous Atmosphere

In Section 3a we have proved a set of symmetry relations for the reflection and transmission matrices of a homogeneous atmosphere, with a scattering matrix per unit volume of the form (3), which refer only to light which is scattered once. The explanation of these relationships in terms of symmetry arguments makes it hard to believe that higher order scattering can destroy them. In other words, we expect them to be also valid for every order of scattering and thus for the total reflected and transmitted light of the layer. We shall now show that this is indeed true by proving it by induction for the basic relations e, f, g, h, n, q and r.

Let light be incident on the top only from one direction \( (\mu_0, \varphi_0) \). The light which is not scattered at all at a depth \( r \) measured from the top may be represented by the 4\( \times \)4 matrix

\[
I_0(\tau, \mu, \mu_0, \varphi - \varphi_0) = \pi \mu_0^{-1} \delta(\mu - \mu_0) \delta(\varphi - \varphi_0) \exp(-\tau/\mu) J,
\]

where J is the 4\( \times \)4 unit matrix and \( \delta \) is Dirac's delta function. The Stokes parameters of this light are obtained by multiplication with a column vector which corresponds to the incident light and consists of four constants the first of which is 1.

Calling \( n \geq 1 \) the order of scattering we have for the \( n \)th order 4\( \times \)4 source matrix

\[
N_n(\tau, \tau, \mu, \mu_0, \varphi - \varphi_0) = \int_{-1}^{1} \int_{0}^{2\pi} Z(u, u', \varphi - \varphi') \times I_{n-1}(\tau', \mu_0, \varphi' - \varphi_0) du' d\varphi',
\]

and for the \( n \)th order 4\( \times \)4 intensity matrix

\[
I_n(\tau, \mu, \mu_0, \varphi - \varphi_0) = \begin{cases} 
\int^{\tau'}_{0} N_n(\tau', \mu, \mu_0, \varphi - \varphi_0) \exp\left(-\frac{\tau - \tau'}{\mu}\right) d\tau'/\mu, & \text{if } u > 0 \\
\int_{\tau}^{b} N_n(\tau', \mu, \mu_0, \varphi - \varphi_0) \exp\left(-\frac{(\tau' - \tau)}{(-u)}\right) d\tau'/(-u), & \text{if } u < 0 \\
N_n(\tau, 0, \mu_0, \varphi - \varphi_0), & \text{if } u = 0
\end{cases}
\]

where \( Z(u, u', \varphi - \varphi') \) is given by (1) and \( F(\theta) \) is normalized such that \( \int_{0}^{2\pi} F(\theta) d\omega = 1 \).
The $n$th order reflection and transmission matrices are defined by
\[ R_n(\mu, \mu_0, \varphi - \varphi_0) = I_n(0, -\mu, \mu_0, \varphi - \varphi_0), \quad T_n(\mu, \mu_0, \varphi - \varphi_0) = I_n(b, \mu, \mu_0, \varphi - \varphi_0). \]
The total reflection and transmission matrices of the diffuse light, i.e., light scattered at least one time, are obtained from these by multiplication with $a^n$ and summation over $n$. For the directly transmitted light $I_0(b, \mu, \mu_0, \varphi - \varphi_0)$, the above mentioned symmetry relations are obvious.

1. Relations e and f

By elimination of the source matrix we find
\[ I_n(\tau, \mu, \mu_0, \varphi - \varphi_0) = \int_{-1}^{1} \int_{0}^{\pi} \int_{0}^{\pi} Z(u, u', \varphi - \varphi') I_{n-1}(\tau', u', \mu_0, \varphi' - \varphi_0) \exp\left(-\frac{(\tau - \tau')}{u}\right) du' \, d\varphi' \, d\tau' / u, \quad \text{if } u > 0, \]
\[ I_n(\tau, \mu, \mu_0, \varphi - \varphi_0) = \int_{-1}^{1} \int_{0}^{\pi} \int_{0}^{\pi} Z(u, u', \varphi - \varphi') I_{n-1}(\tau', u', \mu_0, \varphi' - \varphi_0) \times \exp\left(-\frac{(\tau - \tau')}{(-u)}\right) du' \, d\varphi' \, d\tau' / (-u), \quad \text{if } u < 0. \]

Application of these relations for $n = 1$ gives
\[ I_1(\tau, \mu, \mu_0, \varphi - \varphi_0) = PQI_1(\tau, \mu, \mu_0, \varphi - \varphi_0) \]
in view of symmetry relation D for the phase matrix. Now if such a relation is true for $n - 1$ then it is also true for $n$, since for $u > 0$
\[ I_n(\tau, \mu, \mu_0, \varphi - \varphi_0) = \int_{-1}^{1} \int_{0}^{\pi} \int_{0}^{\pi} Z(u, u', \varphi - \varphi') I_{n-1}(\tau', u', \mu_0, \varphi' - \varphi_0) \exp\left(-\frac{(\tau - \tau')}{u}\right) du' \, d\varphi' \, d\tau' / u, \]
\[ = \int_{-1}^{1} \int_{0}^{\pi} \int_{0}^{\pi} PQZ(u, u', \varphi - \varphi_0) QPQI_{n-1}(\tau', u', \mu_0, \varphi' - \varphi_0) \exp\left(-\frac{(\tau - \tau')}{u}\right) du' \, d\varphi' \, d\tau' / u, \]
and this is equal to $PQI_n(\tau, \mu, \mu_0, \varphi - \varphi_0)Q$ as is clear by the substitution $\varphi' = \varphi + \varphi_0 - \psi$ and the periodicity in azimuth.

The same proof can be used for $u < 0$.

Hence, we have proved that for every $n$
\[ I_n(\tau, \mu, \mu_0, \varphi - \varphi_0) = PQI_n(\tau, \mu, \mu_0, \varphi - \varphi_0)Q. \]

By substitution of $\tau = 0, u = -\mu$ and $\tau = b, u = \mu$, we find in particular relations e and f to hold with subscript $n$.

2. Relations g and h

By elimination of the intensity matrix we have
\[ N_n(\tau, \mu, \mu_0, \varphi - \varphi_0) = \int_{-1}^{1} \int_{0}^{\pi} \int_{0}^{\pi} Z(u, u', \varphi - \varphi') \int_{0}^{\pi} \exp\left(-\frac{(\tau' - \tau) / (-u')}{u}\right) N_{n-1}(\tau', u', \mu_0, \varphi' - \varphi_0) du' \, d\varphi' \, d\tau' / (-u') \]
\[ + \int_{0}^{1} \int_{0}^{\pi} \int_{0}^{\pi} Z(u, u', \varphi - \varphi') \int_{0}^{\pi} \exp\left(-\frac{(\tau - \tau') / u'}{u}\right) N_{n-1}(\tau', u', \mu_0, \varphi' - \varphi_0) du' \, d\varphi' \, d\tau' / u', \]
\[ = \int_{-1}^{1} \int_{0}^{\pi} \int_{0}^{\pi} Z(u, u', \varphi' - \varphi_0) du' \, d\varphi' \int_{0}^{\pi} N_{n-1}(\tau', u', \mu_0, \varphi' - \varphi_0) k(\tau', u') d\tau', \]
where
\[ k(t, u) = \begin{cases} |u|^{-1} \exp(-t/u), & (t/u) > 0 \\ 0, & (t/u) < 0 \end{cases} \]
so that
\[ k(t, u) = k(-t, -u). \]
This can be written even shorter as
\[ N_n(\tau, \mu, \mu_0, \varphi - \varphi_0) = \Delta_{\tau, \mu, \varphi}(N_{n-1}(\tau', \mu, \mu_0, \varphi' - \varphi_0)). \]
Further
\[ N_1(\tau, u, \mu_0, \varphi - \varphi_0) = Z(u, \mu_0, \varphi - \varphi_0) \pi \mu_0^{-1} \exp(-\tau/\mu_0). \]

**Lemma 1:**
\[ \int_0^b \psi(\tau) \Lambda_{\tau-\mu, \varphi} Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau') d\tau = \int_0^b \chi(\tau) P[\Lambda_{\tau-\mu, \varphi} Z(u', \mu, \varphi' - \varphi) \psi(\tau') \psi(\tau')] P d\tau \]
for every natural number \( \nu \). Here the superscript \( t \) denotes the transposed matrix and
\[ \Lambda_{\tau-\mu, \varphi} = \Lambda_{\tau-\mu, \varphi} \Lambda_{\tau-\mu', \varphi'} \cdots \Lambda_{\tau-(\nu+1)u', \varphi'-(\nu+1)}. \]

**Proof:**

A) Consider \( \nu = 1 \). Then
\[ \int_0^b \psi(\tau) \Lambda_{\tau-\mu, \varphi} Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau') d\tau = \int_0^b \psi(\tau) d\tau \int_0^{b+1} \int_0^{\tau'} Z(-\mu, u', \varphi - \varphi') du' d\varphi' \int_0^b Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau') k(\tau - \tau', u') d\tau', \]
which on writing \( u' = -u' \) and interchanging \( \tau \) and \( \tau' \) becomes
\[ \int_0^b \chi(\tau) d\tau \int_0^{b+1} \int_0^{\tau'} Z(-\mu, -u', \varphi - \varphi') du' d\varphi' \int_0^b Z(-u', \mu_0, \varphi' - \varphi_0) \psi(\tau') k(\tau - \tau', u') d\tau'. \]

Using relation B we can write for this
\[ \int_0^b \chi(\tau) d\tau \int_0^{b+1} \int_0^{\tau'} Z(-\mu, -u', \varphi - \varphi') du' d\varphi' \int_0^b Z(-u', \mu_0, \varphi' - \varphi_0) \psi(\tau') k(\tau - \tau', u') d\tau' = \int_0^b \chi(\tau) P[\Lambda_{\tau-\mu, \varphi} Z(u', \mu, \varphi' - \varphi) \psi(\tau')] P d\tau. \]

Hence, the lemma holds for \( \nu = 1 \).

B) Suppose the lemma to be true for a certain \( \nu \). Then
\[ \int_0^b \psi(\tau) \Lambda_{\tau+1, \mu, \varphi} Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau') d\tau = \int_0^b \psi(\tau) \Lambda_{\tau+1, \mu, \varphi} \left( \int_0^{b+1} \int_0^{\tau'} Z(u', u', \varphi' - \varphi') du' d\varphi' \int_0^b Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau') k(\tau' - \tau, u') d\tau' \right) d\tau, \]
\[ = \int_0^b d\tau' \int_0^{b+1} du' \int_0^{\tau'} d\varphi' \left[ \int_0^b \psi(\tau) \Lambda_{\tau+1, \mu, \varphi} Z(u', u', \varphi' - \varphi') k(\tau' - \tau, u') d\tau' \right] Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau'), \]
\[ = \int_0^b d\tau' \int_0^{b+1} du' \int_0^{\tau'} d\varphi' \left[ \int_0^b k(\tau - \tau', u') P(\Lambda_{\tau+1, \mu, \varphi} Z(u', \mu, \varphi' - \varphi) \psi(\tau')) P d\tau \right] Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau'). \]

On interchanging \( \tau \) and \( \tau' \), substituting \( u' = -u' \) and using B we get
\[ \int_0^b d\tau \int_0^{b+1} du' \int_0^{\tau'} d\varphi' \left[ \int_0^b k(\tau - \tau', u') P(\Lambda_{\tau+1, \mu, \varphi} Z(u', \mu, \varphi' - \varphi) \psi(\tau')) P d\tau \right] Z(-\mu, -u', \varphi_0 - \varphi_0) P \chi(\tau), \]
\[ = \int_0^b \chi(\tau) d\tau P \left( \int_0^{b+1} \int_0^{\tau'} Z(-\mu, -u', \varphi_0 - \varphi_0) du' d\varphi' \int_0^b \Lambda_{\tau+1, \mu, \varphi} Z(u', \mu, \varphi' - \varphi) \psi(\tau') k(\tau - \tau', u') d\tau' \right) P, \]
\[ = \int_0^b \chi(\tau) d\tau P[\Lambda_{\tau+1, \mu, \varphi} Z(u', \mu, \varphi' - \varphi) \psi(\tau')] P d\tau. \]

Hence, the lemma holds also for \( \nu + 1 \) and thus for every \( \nu \). This completes the proof.
Lemma 2:
\[
\int_0^b \psi(b-\tau) A_{\tau,\mu, \varphi}(Z(u', \mu_0, \varphi' - \varphi_0) \chi(\nu')) \, d\tau = \int_0^b \chi(b-\tau) Q A_{\tau,\mu, \varphi}(Z(u', \mu_0, \varphi' - \varphi) \psi(\nu')) \, d\tau.
\]

Proof:
The proof is similar to that of lemma 1. The only difference is that we leave \(u'\) unaltered, first substitute \(\nu'=b-\tau\) and \(\nu=b-\nu'\) and then make use of relation A. This completes the proof.

Proceeding with the proof of g and h we now have
\[
R_n(\mu_0, \mu_0, \varphi - \varphi_0) = I_n(0, -\mu_0, \mu_0, \varphi - \varphi_0) = \int_0^b \exp(-\nu/\mu) N_n(\nu_0, -\mu_0, \mu_0, \varphi - \varphi_0) \, d\nu/\mu,
\]

where lemma 1 is applied with
\[
\psi(\nu) = \exp(-\nu/\mu) \quad \text{and} \quad \chi(\nu') = \exp(-\nu'/\mu_0).
\]

Further,
\[
T_n(\mu_0, \mu_0, \varphi - \varphi_0) = I_n(b, \mu_0, \mu_0, \varphi - \varphi_0) = \int_0^b \exp(-(b-\nu)/\mu) N_n(\nu_0, \mu_0, \mu_0, \varphi - \varphi_0) \, d\nu/\mu,
\]

where lemma 2 is applied with
\[
\psi(b-\tau) = \exp(-(b-\tau)/\mu) \quad \text{and} \quad \chi(\nu') = \exp(-\nu'/\mu_0).
\]

Hence, relations g and h hold for every \(n\).

3. Relations q and r

When light is incident from the bottom the only difference with the foregoing is the first-order source function and the way
\[
R^q_n(\mu_0, \mu_0, \varphi - \varphi_0) \quad \text{and} \quad T_n^q(\mu_0, \mu_0, \varphi - \varphi_0) \quad \text{depend on} \quad N_n(\nu_0, \mu_0, \mu_0, \varphi - \varphi_0).
\]

We have
\[
R^q_n(\mu_0, \mu_0, \varphi - \varphi_0) = \int_0^b \exp(-(b-\nu)/\mu) A_{\nu_0, \mu, \varphi}(Z(u', -\mu_0, \nu_0 - \varphi_0) \exp(-(b-\nu_0)/\mu_0) \exp(0)) \, d\nu/\mu,
\]

\[
T_n^q(\mu_0, \mu_0, \varphi - \varphi_0) = \int_0^b \exp(-\nu/\mu) A_{\nu_0, \mu, \varphi}(Z(u', \mu_0, \nu_0 - \varphi_0) \exp(-\nu/\mu) \chi(b-\nu)) \, d\nu/\mu.
\]

Lemma 3:
\[
\int_0^b \psi(b-\tau) A_{\tau,\mu, \varphi}(Z(u', -\mu_0, \varphi' - \varphi_0) \chi(b-\tau)) \, d\tau = \int_0^b \psi(\nu) PQ A_{\nu_0, \mu, \varphi}(Z(u', \mu_0, \varphi' - \varphi_0) \chi(\nu')) \, d\tau.
\]

Proof:
A) The lemma is true for \(v = 1\) as is clear from the transitions \(b-\nu\) to \(\nu\), \(\nu'-\tau\) to \(\nu'\), \(u'-\mu\) to \(-u'\) and relationship G.

B) Suppose the lemma to hold for a certain \(v\). Then
\[
\int_0^b \psi(b-\tau) A_{\nu_0, \mu, \varphi}(Z(u', -\mu_0, \varphi' - \varphi_0) \chi(b-\tau)) \, d\tau
\]

\[
= \int_0^b \int_0^{\nu_0} \psi(b-\tau) A_{\nu_0, \mu, \varphi}(Z(u', \mu_0, \nu_0 - \varphi_0) \chi(b-\tau)) \, d\tau \int_0^{\tau_0} A_{\nu_0, \mu, \varphi}(Z(u', \mu_0, \nu_0 - \varphi_0) \chi(b-\tau)) \, d\tau
\]

\[
= \int_0^b \int_0^{\nu_0} \psi(b-\tau) A_{\nu_0, \mu, \varphi}(Z(u', \mu_0, \nu_0 - \varphi_0) \chi(b-\tau)) \, d\tau \int_0^{\tau_0} A_{\nu_0, \mu, \varphi}(Z(u', \mu_0, \nu_0 - \varphi_0) \chi(b-\tau)) \, d\tau
\]

\[
= \int_0^b \int_0^{\nu_0} \psi(b-\tau) \chi(b-\tau) \, d\tau \int_0^{\tau_0} A_{\nu_0, \mu, \varphi}(Z(u', \mu_0, \nu_0 - \varphi_0) \chi(b-\tau)) \, d\tau
\]

\[
= \int_0^b \psi(b-\tau) A_{\nu_0, \mu, \varphi}(Z(u', -\mu_0, \varphi' - \varphi_0) \chi(b-\tau)) \, d\tau.
\]
Substituting \( \tau' = b - \tau', \ u' = -u' \) and using \( G \) gives
\[
\int_0^b d\tau \int_{-1}^{1} d\mu' \int_0^{2\pi} d\varphi' \int_0^b d\tau' \psi(\tau) PQ \Lambda^*_{\tau, -\mu, \varphi} \{Z(u', u', \varphi', \varphi') k(\tau' - \tau', u')\} QPPQZ(u', \mu_0, \varphi' - \varphi_0) QPX(\tau')
\]
\[
= \int_0^b \psi(\tau) PQ \Lambda^{r+1}_{\tau, -\mu, \varphi} \{Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau')\} QP d\tau,
\]
so that the lemma holds also for \( r + 1 \). This completes the proof.

Lemma 4:
\[
\int_0^b \psi(\tau) \Lambda^*_{\tau, -\mu, \varphi} \{Z(u', -\mu_0, \varphi' - \varphi_0) \chi(b - \tau')\} d\tau = \int_0^b \psi(b - \tau) PQ \Lambda^*_{\tau, -\mu, \varphi} \{Z(u', \mu_0, \varphi' - \varphi_0) \chi(\tau')\} QP d\tau.
\]

Proof:

Apply lemma 3 with \( \mu = -\mu \) and the proof is complete.

On employing lemma 3 and 4 we find
\[
R_n^*(\mu, \mu_0, \varphi - \varphi_0) = PQ R_n(\mu, \mu_0, \varphi - \varphi_0) QP,
\]
\[
T_n^*(\mu, \mu_0, \varphi - \varphi_0) = PQ T_n(\mu, \mu_0, \varphi - \varphi_0) QP.
\]

Hence, \( q \) and \( r \) hold for any \( n \).

REFERENCES


