Growth Rates and Phase Speeds of Baroclinic Waves in Multi-Level Models on a Sphere

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ABSTRACT

The complete growth rate and phase speed spectra of unstable baroclinic disturbances are studied in multi-level quasi-geostrophic spherical models and in a two-level non-geostrophic model, for two basic zonal jet-profile flows. The eigenvalue method employed has the advantages over initial value approaches of determining all the disturbance modes, being more efficient computationally and avoiding problems of slow convergence or lack of convergence when there are two modes of similar growth rates. This method is used to re-examine the spectra for Simmons and Hoskins' 30° jet and Gall's 42.1° jet. It is found that the discrepancies between our two-level and multi-level results are not as great as found by Simmons and Hoskins using primitive equation models, particularly with respect to phase speeds and momentum fluxes. Further, the short-wave maximum at zonal wavenumber 16 in their five-level primitive equation model does not occur here. Instead, our growth rates for the fastest growing modes are very similar to those of both the primitive equation and quasi-geostrophic eight-level models and in qualitative agreement with beta-plane calculations. For the 42.1° jet, our growth rate maximum occurs at wavenumber 11, in contrast with Gall's wavenumber 14 for his primitive equation model but in closer agreement with beta-plane calculations. The effects of changing the static stability in the lower troposphere and of varying the vertical and horizontal resolutions are also studied.

1. Introduction

Recently a number of studies of the effect of spherical geometry on the growth and structure of unstable baroclinic waves has appeared in the literature. A review and detailed analysis of two-level spherical models has been given by Baines and Frederiksen (1978, hereafter denoted BF). As far as multi-level models are concerned, the studies of Simmons and Hoskins (1976, 1977a) and Gall (1976a,b,c) have tended to support the qualitative results of classical quasi-geostrophic beta-plane calculations (e.g., Eady, 1949; Green, 1960; Brown, 1969), with some notable and unexpected exceptions.

Simmons and Hoskins found that, as in the corresponding spherical two-level models, many of the properties of the growing modes are similar to those of beta-plane studies, except that spherical geometry has a significant effect on the location of the disturbances and on the eddy momentum fluxes. In particular, using multi-level models with eight or more levels and with a number of different realistic basic profiles of zonal winds and temperatures, they found that the zonal wavenumber corresponding to maximum growth rate lies in the range 5-9, in agreement with quasi-geostrophic beta-plane calculations. Further, the quasi-geostrophic (QG) solutions were found to differ little from those of the primitive equations (PE). Important results of their calculations using various vertical resolutions (and which have implications for general circulation models) are as follows:

1) Vertical truncation error can give rise to erroneously large growth rates at high wavenumbers. It was found that, while for a 30° jet and with eight or more levels, the growth rate spectrum had a maximum at zonal wavenumber \( k = 8 \), for a five-level model the growth rate spectrum had a maximum near the largest zonal wavenumber examined, \( k = 16 \).

2) Two-level (primitive equation) calculations are subject to considerable errors at all wavenumbers. For their 30° jet they noted that (i) growth rates were too low; (ii) phase speeds up to wavenumber 7 were overestimated, while between wavenumbers 8 and 11 the fastest growing mode actually moved westward; (iii) at \( k = 6 \), the ratio of the largest low-level equatorward momentum transfer to the corresponding poleward transfer was about 9 in comparison with a value of 1.4 found using eight levels.

3) Again using profiles of zonal winds and temperatures consistent with climateology in multi-level PE models, Gall noted that in contrast with beta-plane instability studies, the maximum growth rate occurred at or near the highest zonal wavenumber examined, \( k = 15 \). Simmons and Hoskins (1977b) repeated the calculations of growth rates corresponding to Gall's
(1976c) 42.1° jet profile and their results showed considerable differences. In particular, their growth rate maximum occurred at \( k = 11 \) and they suggested that either Gall's results are subject to large truncation error or there is coding error in one or both of the models which manifests itself only at high wavenumbers.

4) However, Gall and Blakeslee (1977) proposed that growth rate spectra of baroclinic waves are very sensitive to the details of the zonal mean flow profiles, especially in the lower troposphere, such that even small differences due to truncation errors may result in considerably different growth rate spectra. They supported this proposal by reference to the study made by Staley and Gall (1977) using a four-level quasi-geostrophic betaplane model. Gall and Blakeslee also pointed out that if these spectra are as sensitive to small changes in the vertical profiles of zonal mean winds and temperatures as proposed, then linear growth rate spectra may be rather unimportant in determining the energy spectrum of the atmosphere.

The above studies have been largely or exclusively concerned with only the fastest growing wave at a given zonal wavenumber, since the initial value method employed does not facilitate the determination of other modes. In this paper, we generalize the eigenvalue approach of BF to multi-level QG models on a sphere and study the growth rates and phase speeds of all the growing modes at each zonal wavenumber. In particular, we examine whether propositions 1)–4) above apply to multi-level QG models (and the non-geostrophic L model of BF), using this eigenvalue formulation which does not suffer from slow or lack of convergence when there are two or more modes of similar growth rate. While the eigenvalue method is in general more efficient computationally than the initial value approach, it requires considerable storage. Thus, to maximize the horizontal resolution that may be attained, we shall in the present analysis restrict our consideration to multi-level QG models (and the non-geostrophic L-model). Our results are therefore not directly applicable to PE models, although we note that Simmons and Hoskins' multi-level QG and PE results were found to be very similar. In fact, we find that, for the fastest growing modes, our QG model results for the 30° and 42.1° jets are very similar to the PE results of Simmons and Hoskins, with eight or more levels.

In Section 2, we summarize the equations defining the multi-level spherical QG model and in Section 3 we obtain the complete growth rate and phase speed spectra for eight-, five- and two-level models with a 30° jet basic profile [as used by Simmons and Hoskins (1976)] and we study propositions 1) and 2). The complete growth rate and phase speed spectra for a 9-level QG model with a 42.1° jet basic profile [as used by Gall (1976c)] are obtained in Section 4 and we also study propositions 3) and 4). Our conclusions, summarizing the facts that some of the propositions 1)–4) are not valid for the corresponding QG and L models, and emphasizing the desirability of obtaining the complete growth rate spectra are given in Section 5. Appendices A and B summarize the eigenvalue equations and the characteristics of the basic profiles.

2. Model details

The QG spherical multi-level equations with a continuous vertical representation may be written in pressure coordinates in the form (Lorenz, 1960)

\[
\frac{\partial \nabla \psi}{\partial t} = -J(\psi, \nabla \psi + f) - \nabla \cdot (f \nabla \chi), \tag{2.1}
\]

\[
\frac{\partial T}{\partial t} = -J(\psi, T) + \Sigma w, \tag{2.2}
\]

\[
\frac{\partial \Phi}{\partial \rho} = \frac{R T}{\rho}, \tag{2.3}
\]

\[
\nabla^2 \Phi = \nabla \cdot (f \nabla \psi), \tag{2.4}
\]

\[
\frac{\partial \omega}{\partial \rho} = \nabla^2 \chi, \tag{2.5}
\]

\[
f = 2 \Omega \mu, \tag{2.6}
\]

\[
\Sigma = \left( \frac{\partial \nabla T}{\partial \rho} \right). \tag{2.7}
\]

Here \( \psi \) is the streamfunction, \( \chi \) the velocity potential, \( T \) the temperature, \( \overline{T} \) the horizontally averaged temperature, \( \Sigma \) the static stability parameter, \( w = \frac{\partial \rho}{\partial t} \), the "vertical" velocity, \( \Phi \) the geopotential height, \( R \) the gas constant for air, \( f \) the Coriolis parameter, \( \kappa \) the ratio of the gas constant to the specific heat of air at constant pressure, \( \mu = \sin \) (latitude) and \( \rho = \) pressure. Details of finite differencing the above continuous equations in the vertical to obtain a multi-level model and of linearizing the equations about a basic state of zonal flow and formulating the eigenvalue problem, are given in Appendix A. We note here, however, that in order to optimize the possible horizontal resolution attainable with the model, streamfunctions and temperatures are specified at alternate full and half levels; in this respect, the model differs from Simmons and Hoskins' (1976) QG model where the temperatures are also specified at the full levels.

3. The 30° jet

The first basic profile that we consider is the 30° jet of Simmons and Hoskins (1976); it is described in detail in Appendix B.

a. Eight-level results

The eight-level vertical structure is defined as in
Simmons and Hoskins (1976) by "half levels,"

$$\sigma^{2l} = \frac{\rho^{2l}}{1000} = \frac{l}{8} \left[ 1 + \frac{56}{96} \left( \frac{1 - l}{8} \right) \right], \quad l = 0, 1, \ldots, 8, \quad (3.1a)$$

and by "full levels"

$$\rho^{2l+1} = \frac{1}{2} (\rho^{2l} + \rho^{2l+2}), \quad l = 0, 1, \ldots, 7, \quad (3.1b)$$

with streamfunctions defined at the full levels but with temperatures specified at half levels. Throughout this article we shall be concerned only with even disturbance modes (antisymmetric streamfunctions); with the perturbation streamfunction given by the general form in Eq. (A2) of BF, the total wavenumber $n$ of the antisymmetric streamfunction then takes the values
phase speeds are, however, slightly larger in the five-level model. For the slower growing modes, the differences between the growth rates in the eight- and five-level models increase with decreasing growth rates at a fixed $k$. For modes B and C the growth rates are in quite good agreement near $k=7$. Moreover, the five-level modes stabilize earlier as $k$ increases than do the eight-level modes, as expected from beta-plane studies.

It is not clear why the growth rates of the fastest growing modes for the five- and eight-level QG models should be in such good agreement here compared with the corresponding PE results of Simmons and Hoskins. We note that the modes A and B for the five-level model have growth rates which are well separated even at large wavenumbers and one would therefore expect that had we used an initial value approach we should have had no difficulty in attaining convergence; this was indeed found by Simmons and Hoskins in their model. Discarding the possibility of error in either model, possible explanations for the discrepancy are (a) the spurious (from the point of view of higher vertical resolution studies) short-wave growth rate maximum only occurs in the PE model (it is not clear whether Simmons and Hoskins also studied the five-level QG model) or (b) our different evaluation of the temperature fields has avoided the erroneous short wave growth. In any event, our results in Fig. 2 are in close qualitative agreement with the classical beta-plane findings.

c. Two-level results

For their two-level PE model with the 30° jet, Simmons and Hoskins found that their calculations were subject to considerable errors at all wavenumbers, as mentioned in 2) in the Introduction. Growth rates were underestimated and phase speeds were overestimated up to $k=7$ beyond which the fastest growing modes appeared quite spurious with large negative phase speeds corresponding to westward movement and most amplitude at the upper level. At $k=6$, the disturbance maximum was positioned to within 3° using two layers, but the largest low-level equatorward momentum flux was found to be nine times larger than the corresponding poleward transfer in comparison with a value of 1.4 using eight levels. We suggested on general grounds in BF that the negative phase speeds and momentum fluxes may be erroneous. In view of the theoretical importance of two-level models, it seems of interest to study the findings 2) in more detail using the same 30° jet profile but employing an eigenvalue approach.

For this study we use the nongeostrophic L-model with two equally spaced levels. The L-model, like the PE model, has no short wave cutoff. In fact, it was con-

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1 After this work was first submitted for publication, I was informed by A. J. Simmons that they did not examine the corresponding QG model.
cluded in BF that the absence of the short wave cutoff in two-layer models is directly attributable to the advection of the density field by the divergent wind field even in models as complete as the PE models. Nevertheless, as mentioned in the Introduction, L-model and PE model results may differ.

Shown in Fig. 3 are the growth rates and phase speeds of the six fastest growing modes in the L-model and as well the growth rates and moduli of the phase speeds of the fastest growing mode in Simmons and Hoskins' PE model; note that for $8 \leq k \leq 11$ the latter phase speeds are in fact negative. We notice that the maximum growth rates are in reasonable agreement, except that the L-model growth rates are slightly smaller. This is to be expected since in the L-model, unlike the PE model, the static stability at 750 mb (and 250 mb) is set equal to that at 500 mb and in general the growth rate decreases with increasing static stability in the

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**Fig. 2.** Growth rates (a) and phase speeds (b) of the seven fastest growing disturbance modes in the five-level QG model (A-G) for the 30° jet and of the fastest growing mode in Simmons and Hoskins' five-level PE model (X).
lower troposphere. For \( k \leq 9 \), the L-model has a number of modes with similar growth rates; these may have been difficult to separate had we used an initial value approach. We note that at \( k = 12 \), Simmons and Hoskins did not attain convergence in their PE model. For \( k \leq 7 \), the phase speeds of the modes D, C and A correspond closely with Simmons and Hoskins' results, while the growth rates are somewhat smaller as might be expected on the basis of the above discussion. For \( k \geq 8 \) our fastest growth rates are comparable with those of Simmons and Hoskins but the phase speeds bear little resemblance.\(^2\)

For the corresponding quasi-geostrophic two-level P-model, discussed in BF, modes similar to A, C and D occur, but modes B, E and F are absent; thus the model has the usual short wave cutoff characteristic of two-level quasi-geostrophic models.

At \( k = 6 \), the disturbance maximum for the L-model is positioned within 3° or 4° of that of our eight-level QG model, as found by Simmons and Hoskins for the corresponding PE models. However, in contrast to their value of 9 for the ratio of the largest low-level equatorial momentum transfer to the corresponding poleward transfer in their two-level model, we find it to be only 1.6 which compares much more favorably with the value of 1.4 obtained for both their PE and our QG models.

We see, however, that there remain considerable differences between two-level (L-model) and multi-level results. The crudity of two-level models and their dependence on the method of finite differencing in the vertical, particularly with respect to obtaining the static stability parameter, have previously been discussed by Charney and Phillips (1953; see also Hollingworth, 1975 and BF). However, two-level models may still provide useful insights if for suitable realistic static stability parameters they are able to describe the

\(^2\) After this work was first submitted for publication, I was informed by A. J. Simmons that their calculation of phase speed is subject to a possible error of \( N (360/k) \) degrees of longitude per day, where \( k \) is the zonal wavenumber of the mode and \( N \) an integer. He suggests taking \( N = 1 \), i.e., adding an amount \( (360/k) \) deg day\(^{-1} \) for \( k \geq 8 \). The phase speeds are then all eastward but somewhat larger than for mode B and in fact are closer to modes F and G.
gross features characteristic of multi-level models. Since it is known that the growth rates of multi-level models are quite sensitive to the static stability in the lower troposphere, it seems more appropriate to use, in the L-model, a value characteristic of the lower troposphere than at 500 mb (such an approach was followed by Hollingworth and in BF). For this reason, we have recalculated the growth rate and phase speed spectra in the L-model with one-half of the static stability used for Fig. 3; the results are shown in Fig. 4. The growth rates of the fastest growing waves now compare more favorably with those for the eight-level QG model, while the phase speeds are still somewhat too large, particularly for the shorter waves. At \( k = 6 \), the disturbance maximum is now positioned to within 1° of that for the eight-level model, while the ratio of the low-level equatorward momentum transfer to the poleward transfer is 0.82.

4. The 42.1° jet

Next we study the growth rate and phase speed spectra in the QG model for the 42.1° jet analyzed by Gall (1976c), and subsequently by Simmons and Hoskins (1977b), in PE models; a summary of the properties of this profile is given in Appendix B. The same nine levels as given in Table 1 of Gall (1976c) are used, with the streamfunction calculated at full levels and the temperatures at half levels. The winds are those used by Gall, but for this QG model, the temperatures are determined geostrophically as described in Appendix B. In all other respects (apart from truncation error)
our basic state is identical with that of Gall between latitudes 23.2 and 61°N. However, while Gall's calculations were carried out with walls placed at these latitudes, for both Simmons and Hoskins' and our models, the zonal mean flow is assumed to be zero outside this region. As discussed by Simmons and Hoskins, this is unlikely to affect their subsequent differences at high wavenumbers.

The growth rates and phase speeds of the five fastest growing modes in the QG model, shown in Fig. 5, were calculated using $J=25$ and a 12th-degree even polynomial fit in $\mu$ to the latitudinal variation of the basic streamfunction $\partial\psi/\partial\mu$ as described in Appendix B. Also shown are the spectra for the fastest growing modes in Gall's PE model and in Simmons and Hoskins' 16-level PE model. Both Simmons and Hoskins' and


Table 1. Growth rates and speeds of the fastest growing mode in the QG model for the 42.1° jet and with various horizontal resolutions \(J\), vertical resolutions \(L\) and zonal wavenumbers \(k\), and as well for various degree polynomial fits to the derivative of the basic streamfunction \(\frac{\partial \phi}{\partial u}\).

<table>
<thead>
<tr>
<th>(J)</th>
<th>(L)</th>
<th>Degree of polynomial</th>
<th>(k)</th>
<th>Growth rate (day(^{-1}))</th>
<th>Phase speed (deg day(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>9</td>
<td>18</td>
<td>15</td>
<td>0.7663</td>
<td>11.968</td>
</tr>
<tr>
<td>25</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>0.7693</td>
<td>12.033</td>
</tr>
<tr>
<td>25</td>
<td>9</td>
<td>10</td>
<td>15</td>
<td>0.7762</td>
<td>12.083</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>12</td>
<td>15</td>
<td>0.8087</td>
<td>11.259</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>10</td>
<td>15</td>
<td>0.8030</td>
<td>11.404</td>
</tr>
<tr>
<td>25</td>
<td>9</td>
<td>12</td>
<td>7</td>
<td>0.7246</td>
<td>12.757</td>
</tr>
<tr>
<td>29</td>
<td>8</td>
<td>10</td>
<td>7</td>
<td>0.7381</td>
<td>12.716</td>
</tr>
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<td>10</td>
<td>7</td>
<td>0.7665</td>
<td>12.585</td>
</tr>
</tbody>
</table>

Our growth rates have a maximum at \(k = 11\), in contrast to Gall’s maximum at \(k = 14\); moreover, our maxima are appreciably smaller than Gall’s.

We also wish to examine whether Gall and Blakeslee’s proposition 4) applies for our QG model. That is, we examine whether small changes in the flow profile are able to produce changes in the growth rate spectra comparable with the differences between Gall’s and Simmons and Hoskins’ results. Shown in Fig. 5 are also the largest growth rates in the QG model using again the 42.1° jet profile but with the static stability below 700 mb decreased by 10%. Such a change should increase the short-wave growth rate by an amount that would far outweigh any change due to possible truncation errors. However, the position of the new growth rate maximum only changes to \(k = 12\). Some experimentation with different vertical and horizontal resolutions has also been carried out and representative results are shown in Table 1. For the nine-level models \((L = 9)\) the vertical structure is that of Gall, while for the eight-level models, it is as given in Eq. (3.1). It was found that the 12th-degree polynomial fit to \(\frac{\partial \phi}{\partial u}\) produced very little difference from the 18th-degree fit; even the relatively poorer 10th-degree polynomial fit, corresponding to a somewhat broader jet, produced results which are not substantially different.

Thus, it seems that our QG model is not so sensitive to small truncation errors as to be able to produce differences in growth rates as large as those between Simmons and Hoskins’ and Gall’s PE results.

5. Conclusions

The complete growth rate and phase speed spectra of unstable baroclinic disturbances in spherical multi-level models have been obtained for two basic jet-profile flows. The eigenvalue method employed has the advantages over initial value approaches of determining all the disturbance modes, being considerably more efficient computationally and avoiding problems of slow or lack of convergence when there are two or more modes of similar growth rates.

The findings 1) and 2) of Simmons and Hoskins (1976) and the findings 3) and 4) of Gall (1976c) in PE models (mentioned in the Introduction) have been examined using their respective basic profiles in a multi-level QG model. Our five-level QG model does not produce the short-wave growth rate maximum at wavenumber 16 discovered in Simmons and Hoskins’ five-level PE model. Instead, the growth rate spectrum for the fastest growing wave in our five-level model is found to be very similar to that of the eight-level model, with a maximum at intermediate wavenumbers and a decrease at large wavenumbers. Our two-level \(L\)-model calculations agree with 2a) in that the growth rates are too low compared with eight-level calculations. However, the phase speeds are found to be positive everywhere, including between wavenumbers 8 and 11 where Simmons and Hoskins’ PE phase speeds are negative.\(^3\) Moreover, in contrast to their two-level PE model findings, the ratio of our largest low-level equatorward momentum transfer to the corresponding poleward transfer in the \(L\)-model agrees quite well with the value obtained in both PE and QG eight-level models.

With Gall’s 42.1° basic jet profile, our growth rate maximum in a nine-level QG model occurs at wavenumber 11, in agreement with Simmons and Hoskins’ (1977b) PE calculations for the same jet but in contrast to Gall’s maximum at wavenumber 14. Our QG model is not so sensitive to small truncation errors as to be able to produce differences in growth rate as large as those between Simmons and Hoskins’ and Gall’s PE results.

APPENDIX A

The Eigenvalue Equations for General Zonal Flow

The eigenvalue equations for the multi-level QG model may be obtained in a similar way to that described in Appendix A of BF for the two-level models. First, Eqs. (2.1)–(2.7) are scaled with \(a\) (earth’s radius) and \(\Omega^{-1}\) (earth’s angular velocity)\(^{-1}\), as length and time scales, and \(a^2 g / R\) as a temperature scale. Second, Eqs. (2.1)–(2.5) are linearized by replacing each of the fields by a zonal average (denoted by a bar) and a perturbation field and retaining in the equations for the perturbation fields only quantities which are first order in the perturbation fields. The continuous vertical representation is then approximated by a number of discrete levels with streamfunctions and geopotentials calculated at the full levels and temperatures and “vertical” velocities at half levels. The streamfunctions are interpolated to the half levels by using half of the sum of the values at the two surrounding full levels. We use simple vertical finite differences for the vertical derivatives and the boundary conditions for the “vertical” velocities \(w^0 = 0 = w^L\), where \(w(\rho = 1000) = w^0, w(\rho = 0) = w^L\).

\(^3\) As discussed in Section 3, A. J. Simmons has since pointed out an ambiguity in their determination of phase speeds.
The streamfunctions and geopotentials at the full levels are denoted by superscripts $2l+1$, $l=0,1,\ldots,L-1$ and the temperatures and vertical velocities at the half levels are denoted by superscripts $2l$, $l=0,1,\ldots,L$, where $L$ is the number of levels. For the basic zonally averaged streamfunctions we take general representations of the form

$$\tilde{\varphi}_{2l+1}^{\ast} = -A_{0N}^{2l+1}P_{N}^{l}(\mu),$$

(A1a)

where summation over positive odd $N$ up to an arbitrary odd $N_{\text{max}}$ is implied. The scaled equations corresponding to Eqs. (2.1)–(2.7) are then satisfied by the basic fields provided we take

$$\tilde{T}_{2l} = -\frac{2\rho^{2l}}{p^{2l+1}-p^{2l-1}}C_{2l+1}^{N+1}P_{N+1}^{l}(\mu) + \text{constant},$$

(A1b)

where

$$C_{2l+1}^{N+1} = R(0,N+1)[A_{0N}^{2l-1}-A_{0N}^{2l+1}] + S(0,N+1)[A_{0N}^{2l-1}-A_{0N}^{2l+1}],$$

(A1c)

and $R(k,n)$ and $S(k,n)$ are given in Eq. (A.3d) of BF. Further, we take

$$\chi = 0.$$  

(A1d)

Each of the perturbation fields $\varphi_{2l+1}^{\ast}$, $\Phi_{2l+1}^{\ast}$, $T_{2l}^{\ast}$, $\chi^{\ast}$ is now expanded as in Eq. (A2) of BF with the replacements $\theta \to T$, $r \to \Phi$. Substituting these expressions together with Eq. (A1) into the linearized multi-level equations, multiplying by $P_{m}^{l}(\mu)e^{-in\mu}$ and integrating over the surface of the sphere then yields a set of equations for the perturbation fields in the expansion for the perturbation fields. We choose to eliminate from these $\chi_{2l}^{\ast}$, $\Phi_{2l+1}^{\ast}$ and $T_{2l}^{\ast}$, leaving the following eigenvalue-eigenvector system for $\varphi_{2l+1}^{\ast}$ and the angular frequency $\omega$:

$$\begin{align*}
\omega[1+& \left(\tilde{L}_{2l+1}^{0}+\tilde{L}_{2l+1}^{1}\mu \right)A(k,n) \varphi_{2l+1}^{\ast} \\
&+\omega\left[\tilde{L}_{2l+2}^{0}+\tilde{L}_{2l+2}^{1}\mu \right]B(k,n) \varphi_{2l+2}^{\ast} \\
&+\omega\left[\tilde{L}_{2l+3}^{0}+\tilde{L}_{2l+3}^{1}\mu \right]C(k,n) \varphi_{2l+3}^{\ast} \\
+\frac{2k}{n} &- \frac{n+N}{n-1}D(k,n,q,N)A_{0N}^{2l+1} \varphi_{2l+1}^{\ast} \\
+ &\sum_{q=-N+1}^{N-1} \left[ L_{2l+2}^{1} \left\{ \frac{1}{4} \left( A_{0N}^{2l+1}+A_{0N}^{2l+3} \right) \right\} F(k,n,q,N) \\
&-\frac{1}{2} \left( A_{0N}^{2l+1}+A_{0N}^{2l+3} \right)E(k,n,q,N) \right] \varphi_{2l+2}^{\ast} \\
&-\omega L_{2l+3}^{0} A(k,n) \varphi_{2l+3}^{\ast} \\
&-\omega L_{2l+3}^{1} B(k,n) \varphi_{2l+3}^{\ast} - \omega L_{2l+3}^{1} C(k,n) \varphi_{2l+3}^{\ast} \\
- &\sum_{q=-N+1}^{N-1} \left[ L_{2l+4}^{0} \left\{ \frac{1}{4} \left( A_{0N}^{2l+1}+A_{0N}^{2l+3} \right) \right\} F(k,n,q,N) \\
&+\frac{3}{2} \left( A_{0N}^{2l+1}+A_{0N}^{2l+3} \right)E(k,n,q,N) \right] \varphi_{2l+4}^{\ast} = 0,
\end{align*}$$

where

$$\begin{align*}
L_{2m+1}^{0} &\equiv \frac{4\rho^{2l}}{\Sigma^{2l}_{m=1}(\rho^{2m+1}-\rho^{2m-1})(\rho^{2m+2}-\rho^{2m})}, \\
L_{0}^{0} &= 0 = L_{2L}^{0}.
\end{align*}$$

(A3a)

(A3b)

In (A2) the summation over $q$ is in steps of 2 and an additional summation over positive odd values of $N$ is implied. The other functions needed in Eq. (A2) are given in Appendix A of BF.

**APPENDIX B**

**Summary of Zonal Flow Profiles**

a. The 30° jet

For the 30° jet of Simmons and Hoskins (1976), the zonally averaged basic flow velocity is given by

$$\begin{align*}
\vec{u}(\mu,\sigma) &= M(\mu)U_{0}(\sigma),
\end{align*}$$

(B1a)

where

$$M(\mu) = \sin^{3}\mu,$$

(B1b)

$$\mu = \sin\phi, \hspace{1em} \phi = \text{latitude}, \hspace{1em} \sigma = \rho/1000.$$

The function $U_{0}(\sigma)$ and the horizontally averaged temperature $T(\sigma)$ were chosen as follows. Fifth-degree polynomials were fitted to the values

$$\begin{align*}
U_{0}(\sigma) &= (45,35,22,12,4) \text{ m s}^{-1}, \\
T(\sigma) &= (220,230,250,267,280) \text{ K}.
\end{align*}$$

(B2a)

(B2b)

at

$$\sigma = (0.1,0.3,0.5,0.7,0.9),$$

(B2c)

respectively, with the additional constraints that

$$\begin{align*}
U_{0}'(\sigma = 1) &= U_{0}'(\sigma = 0.9), \\
T'(\sigma = 1) &= T'(\sigma = 0.9).
\end{align*}$$

(B2d)

(B2e)

In order to express the basic streamfunctions $\tilde{\varphi}_{2l+1}$ in the form given in Eq. (A1a), we use the fact that

$$\begin{align*}
\vec{u}(\mu,\sigma) &= -(1-\mu^{2})^{1/2}a\frac{\partial \vec{\varphi}}{\partial \mu},
\end{align*}$$

(B3)

and expand $\partial \vec{\varphi}/\partial \mu$ as an even polynomial in $\mu$; this is then reexpressed in terms of the derivatives of the normalized Legendre functions $\partial P_{m}^{l}(\mu)/\partial \mu$ to obtain $A_{0N}^{2l+1}$ at each level. For the eight- and five-level QG models, we use 10th-degree even least-squares poly-
nominals in $\mu$, while for the two-level $L$-model we use a sixth-degree polynomial.

b. The 42.1° jet

For the 42.1° jet of Gall (1976c), the basic zonal velocity is given by

$$
\bar{u}(\phi, \sigma) = \frac{1}{2} \bar{u}_c(\sigma) \left[ 1 - \cos \left( \frac{2\pi (\phi - \phi_0)}{\phi_L - \phi_0} \right) \right],
$$

(B4)

for $\phi_0 \leq \phi \leq \phi_L$ where $\phi_0 = 23.2^\circ$, $\phi_L = 61^\circ$ and $\bar{u}_c(\sigma)$ is given in Eqs. (4) and (5) of Gall. Outside this region we take the zonal velocity to be zero. The horizontally averaged temperature $\overline{T}(\sigma)$ is obtained geostrophically from the thermal wind equation and Eq. (2.3) with the boundary condition that the zonal mean temperature at the center of the channel, $\phi_0 \leq \phi \leq \phi_L$, is given in Eqs. (8) and (9) of Gall. The coefficients $A_{0N}^{2L+1}$ needed in Eq. (A1a) for the QG model are again obtained by least-squares polynomial fits in $\mu$ to $\partial \overline{T}/\partial \mu$.

REFERENCES


Note that some minor errors occur in Eq. (3) and Fig. 1 of Gall.