The Effects of Latitudinal Shear on Equatorial Waves.  
Part I: Theory and Methods

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ABSTRACT

By using the method of multiple scales in height and a variety of methods in latitude, analytic solutions for equatorial waves in combined vertical and horizontal shear are derived. In contrast to the formulation of Andrews and McIntyre (1976b), latitudinal shear is incorporated at lowest order in the vertical shear expansion, showing that it is unnecessary to carry the calculation to first explicit order in \( R_i^{-1} \), where \( R_i \) is the Richardson number. The multiple-scales approximation implies that, with the exception of the overall amplitude factor and arbitrary overall constant phase factor, all properties of the wave at the height \( z = z_0 \) are determined by the local wind profile, \( V(y) = U(y, z_0) \). In consequence, understanding waves in two-dimensional shear reduces to the much simpler problem of solving the one-dimensional eigenvalue equation in latitude which is derived by assuming that the mean wind is \( V(y) \), a function of latitude only. This is done using ordinary perturbation theory, a non-perturbative analytic procedure and the Hermite spectral method for various classes of waves. Once its solutions are known, the overall amplitude factor may be found by using the wave action equation as shown in the text. When the method of multiple scales is invalid, as appears true of the tropical ocean, it is shown that Hermite spectral methods in latitude are much more accurate (at least in the absence of coastal boundaries) than the finite-difference methods used in the past. The techniques discussed here are applied to several classes of observed atmospheric equatorial waves in Part II (Boyd, 1978a).

1. Introduction

The thread that underlies both this work and its companion [Part II (Boyd, 1978a)] is the method of multiple scales in height, which allows one to study the effects of latitudinal shear on equatorial waves using a one-dimensional model in latitude only. The formal machinery of multi-scaling, which is developed in Sections 2 and 8 below, is also useful as a calculational tool, replacing the cumbersome numerical methods of Holton (1970) and Lindzen (1970) by accurate analytical solutions. The greatest significance of this approximation, however, is that it shows that at least in the lower stratosphere, local wave structure is determined by the local wind profile where "local" means "at a given height." In other words, if \( V(y) = U(y, z_0) \), where \( U \) is the mean zonal wind at the height \( z = z_0 \), then \( V(y) \) alone is sufficient to calculate all the wave properties at the height \( z = z_0 \) except for the overall amplitude and phase factors. This simplification is an enormous help in understanding the effects of latitudinal shear on equatorial waves both qualitatively and quantitatively.

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is a technique ideally suited to equatorial waves since the basis functions are also the exact eigenfunctions of a resting atmosphere. When the method of multiple scales is not applicable, the wave equations must be solved numerically, but Hermite functions in latitude can be combined with finite differences in height to obtain an algorithm which is much more efficient than that used previously by Holton (1970) and Lindzen (1970).

All this takes a good deal of space, and Part II on applications to the atmosphere was written so that it could be read independently because it was recognized that many nonspecialists will not have the patience to read through a paper as long as this. (Paper II also reviews previous work on equatorial waves in shear.) It should be emphasized, however, that a general reader without a strong background in analytical or numerical methods can probably follow the ideas of Part I without great difficulty. It is the multiplicity of equatorial wave classes that is responsible for the length of this paper, not the complexity or the subtlety of the methods applied to them.

2. The method of multiple scales. Part I: Formalism

In the absence of horizontal shear, the equatorial beta-plane is an approximation valid in the limit of short vertical wavelengths since both the meridional and vertical length scales for the wave decrease as the nondimensional eigenvalue $\epsilon^*$ increases. As first recognized by Lindzen (1971), this implies that the effects of vertical shear on equatorial waves can be handled by applying the method of multiple scales to the vertical coordinate since the wave scale (one wavelength/2$\pi$) will be small compared to the vertical scale of variations in the mean wind. Since the unperturbed problem is separable, the obvious way to proceed is to insert the local Doppler-shifted frequency into the eigenvalue relation for the meridional structure equation, thus defining a height-dependent eigenvalue, and then solve the vertical structure equation (3.5) (Section 3) by the well-known WKB method. This is basically what Lindzen (1971, 1972) did, but there is a complication. Since the meridional scale of the wave is dependent on the Doppler-shifted frequency, it, too, must change with height if vertical shear is present, and therefore the usual slowly varying WKB amplitude factor must be modified by an additional factor to allow for this lateral expansion and contraction of the wave with height.

The first step in the formalism is to define a "slow" vertical variable $\tau$ by

$$\tau = \sigma z,$$  

(2.1)

where $\sigma$, the vertical shear expansion parameter, is chosen so that $d\bar{U}/d\tau$ is $O(\bar{U})$ and a "fast" variable $\xi$ by

$$\xi = \int f(\tau) d\tau,'$$  

(2.2)

where $f(\tau)$, which will be determined in the course of the solution, is the local vertical wavenumber so that the scale of $\xi$ is that of vertical variations in the wave’s phase. The second step is, to treat the wave variables as functions of both $\xi$ and $\tau$ so that for any field $q$

$$\frac{dq}{d\tau} = \sigma^2 + f(\tau) \frac{dq}{d\xi},$$  

(2.3)

while the mean zonal wind is a function of $\tau$ only, implying

$$\frac{d\omega}{dz} = \sigma^2.$$  

(2.4)

Since all derivatives with respect to $\tau$ are multiplied by an explicit factor of $\sigma$, the zeroth-order solutions, with or without horizontal shear, will be identical to those for no vertical shear except for a parametric dependence on the slow variable $\tau$ (and with $\xi$ replacing $\tau$) Thus, one has immediately

$$\phi = A(\tau) e^{-\sqrt{\epsilon^*} \xi} e^{i\omega \tau} P(\xi) e^{i\epsilon^* z + i\omega t},$$  

(2.5)

and $P(\xi)$ is a polynomial in $\xi$ which depends on the wave type and the horizontal shear which will be calculated in later sections. Eq. (2.6) gives the same vertical phase variations factor as applying the usual WKB approximation directly to the vertical structure equation [Eq. (3.5) of Section 3].

In following sections, I give expressions for $\xi$ and $\epsilon^*$ in terms of a model in which the mean wind is a function of $\gamma$ only. However, when there is both vertical and horizontal shear, the definitions of $\gamma$ and $\epsilon^*$ as needed for (2.5) and the other wave fields are identical except that $\omega$ and the horizontal shear parameters $\gamma$ and $\delta$ are all new functions of $\tau$. For example, for the Kelvin wave to first order in the strength of the horizontal shear

$$\xi(y,\tau) = -[k\beta/\omega_0(\tau)]^{\frac{1}{2}},$$  

(2.7)

$$\epsilon(\tau) = [k^2/\omega_0^2(\tau)](1 + \delta_0(\tau)/2\beta).$$  

(2.8)

The shear and frequency parameters will be defined precisely later; what is important is that one recognize the full significance of (2.7) and (2.8) when they are
derived later in the narrower context of purely horizontal shear. Similarly, \( P(\xi) \) is a polynomial in \( \xi \) which is generated by the same \( U(y \) only) model; for the Kelvin wave in parabolic shear, \( P(\xi) \) is a quadratic polynomial; for gravity waves in the "gamma plane" approximation, it is a linear combination of three Hermite polynomials.

Thus, the following symbolic equation is true:

\[
\text{(analytic solutions for equatorial waves in pure horizontal shear)} + \text{(method of multiple scales)} = \text{(analytic solutions for equatorial waves in combined horizontal and vertical shear)}.
\] (2.9)

Understanding (2.9) is crucial to appreciating the rest of the paper because until I return to the method of multiple scales in Section 8, I will be discussing equatorial waves under the assumption that the mean wind is a function of \( y \) only, describing ways and means for computing the first unknown on the left-hand side of (2.9).

The only part of (2.5) which is not determined explicitly in terms of the solutions of the separable model is \( A(\tau) \). For the inviscid case, this can be calculated by requiring that the \( \gamma \) average of the equation of nondivergence of wave action flux equal zero, i.e.,

\[
\frac{d}{ds} e^{\gamma} \langle 0 \rangle = 0,
\] (2.10)

where the overbar denotes a zonal average and

\[
\langle \rangle = \int_{-\infty}^{+\infty} dy.
\] (2.11)

Eqs. (2.2) and (3.3) imply

\[
w = \omega(\tau)f(\tau)\phi/N^2.
\] (2.12)

Expanding \( d/ds \) in terms of \( \xi, \tau \) and \( \xi \) to first order and remembering that dependence on \( \xi \) is removed by zonal averaging, Eq. (2.10) becomes

\[
\left\langle \frac{\partial}{\partial \tau} e^{\gamma} \langle 0 \rangle \right\rangle = 0,
\] (2.13)

which is a first-order linear differential equation in \( \tau \) for \( A^2(\tau) \).

As can be seen from the simplicity of (2.13), it is possible to use the method of multiple scales to bypass large, expensive, two-dimensional finite-difference models as were used in the past by Lindzen (1970) and Holton (1970, 1971) even if the analytic solutions for \( f(\tau), \xi(\tau), P(\tau,\xi) \) and \( e(\tau) \) are too cumbersome to write out in closed form. In Section 8, I will return to the method of multiple scales to extend it to damped waves and present a complete example of an analytic solution for an equatorial wave in two-dimensional shear.

3. Derivation of the equations for the north-south velocity and geopotential

The basic assumptions are the following:

(i) The waves are hydrostatic.

(ii) The waves are of sufficiently small amplitude so that the wave equations can be linearized about the zonally-averaged (mean) flow.

(iii) The wave equation terms involving the mean meridional and vertical winds are neglected by scaling.

(iv) The static stability is a function of pressure only.

(v) The Doppler-shifted frequency (which depends on the mean zonal wind) is a function of latitude only.

(vi) The waves are confined sufficiently close to the equator so that the equatorial beta-plane approximation, i.e., replacing the sine of latitude by latitude and the cosine by 1 in the primitive equations, is valid.

I will discuss the last restriction at the end of this section. Of the rest, the fifth assumption is the most restrictive if taken literally: in the real atmosphere, vertical shear is usually important when meridional shear is. However, as pointed out in the previous section, vertical shear is incorporated into the equations via the method of multiple scales merely by allowing the Doppler-shifted frequency to be a parametric function of \( \tau \), the slow vertical variable. Under such circumstances, the assumption of a mean wind which is a function only of latitude is no restriction at all as long as one can justify the method of multiple scales as the example of Section 8 will make clearer. Linearity is a good approximation in the lower stratosphere, but it is more questionable in the troposphere. In both regions, contrary to what I have implicitly assumed in (ii), the basic state wind varies with longitude, but this will not affect the qualitative significance of my results.

With these restrictions, the primitive wave equations on the equatorial beta-plane become (symbols are defined in Appendix A)

\[
i\omega u - i\epsilon \phi = 0,
\] (3.1)

\[
\beta y u + i\omega v = -1\frac{\partial}{\partial \tau} \phi,
\] (3.2)

\[
wN^2 = -i\epsilon \phi,
\] (3.3)

\[
\frac{1}{N^2} \nabla^2 \phi = 0.
\] (3.4)

[Note the sign convention on the frequency: \( \omega \) is positive for easterly waves such as the Rossby. Longuet-Higgins (1968) and Holton (1975) use the opposite convention.] When assumptions (i)-(v) are satisfied, the wave equations are separable and can be reduced to a single equation for either the geopotential or the north-south velocity, even on a sphere. To take the route of solving for \( u \), one needs the following identity which is derived by routine separa-
tion of variables in the geopotential equation:

\[ \epsilon^2 (e^{-i \phi_N / \sqrt{2}})_z = - \epsilon \phi, \]  

(3.5)

where \( \epsilon \), the separation constant, is usually written

\[ \epsilon = 1 / gh, \]  

(3.6)

where \( g = 10 \text{ m s}^{-2} \), the gravitational constant, and \( h \) is the so-called “equivalent depth.” All of the wave variables, \( u, v, w \) and \( \phi \) have a common vertical structure which is determined by (3.5) for a given \( \epsilon \), but each has its own horizontal structure, which is what I will chiefly discuss. If for simplicity, one takes the static stability as constant, then solutions to (3.5) can be written

\[ \phi = Y(y)e^{i2t} \exp[\pm i(N^2 \epsilon - \frac{1}{2})t]. \]  

(3.7)

Thus, modes with large positive \( \epsilon \) are vertically propagating waves and modes with negative \( \epsilon \) are vertically “trapped” or “evanescent” and cannot propagate energy up or down. The factor of \( e^{i2t} \) is a mass density growth factor. If the vertical energy flux is constant with height, as is true in the absence of dissipation, then since the flux is proportional to the product of the mass density with \( u^2 + v^2 \), etc., and since the mass density is decreasing as \( e^{-z} \) with log-pressure height, each wave field must increase as \( e^{i2t} \) in compensation. For free modes, \( \epsilon \) or \( h \) is given and we must find \( \omega \); for forced modes, \( \omega \) is given and we must find \( \epsilon \) for each meridional mode by computing the eigenvalues of the horizontal structure equation. There is an exact correspondence between the meridional structure of each wave variable of a mode in a compressible atmosphere with that of the corresponding variables of a mode in a barotropic ocean if the equivalent depth \( h \) of the atmospheric wave is equal to the depth in the shallow water wave equations—hence the name “equivalent depth.” Thus, it is only through the presence of the term involving \( \gamma \) in (3.1) and through the dependence of \( \omega \) on \( y \) that (3.1)–(3.4) differ significantly from those studied by Matsuno (1966), Longuet-Higgins (1968, his Section 8) and Lindzen (1967). The last discusses the vertical structure equation and its solutions in more detail.

To reduce (3.1)–(3.4) to a single equation for the north–south velocity, we first solve for \( u \) and \( w \) in terms of \( v \) and \( \phi \) through (3.1) and (3.3), respectively, and substitute the results in (3.4), the equation of continuity, to obtain

\[ \frac{k}{\omega} - (\beta y - \gamma)\phi + iu v - i(\epsilon^2 \phi_N / \sqrt{2})_z = 0. \]  

(3.8)

\[ \frac{k}{\omega} - (\beta y - \gamma)\phi + iu v - i(\epsilon^2 \phi_N / \sqrt{2})_z = 0. \]  

(3.8)

For upward propagating waves, the vertical energy flux \( \phi_N \) must be positive (the so-called “radiation” upper boundary condition) which implies the (+) for the sign of \( i \) in (3.7) for easterly waves and the (−) for westerly waves.

From the zonal and meridional momentum equations, we solve for \( v \) in terms of \( \phi \):

\[ \Delta v = -i\omega M \phi, \]  

(3.9)

\[ \Delta = \beta^2 y^2 - \beta \gamma y - \omega^2, \]  

(3.10)

\[ M (a \text{ linear operator}) = \left( \frac{\partial}{\partial y} - \frac{k}{\omega} \right). \]  

(3.11)

We now use (3.5), the separability relation, to rewrite (3.8) as

\[ \frac{k}{\omega} - (\beta y - \gamma) v + u + i(\epsilon^2 \phi_N / \sqrt{2})_z = 0. \]  

(3.12)

and define

\[ q = (\omega^2 - k^2 / \omega^2)^{-1}. \]  

(3.13)

Multiplying (3.12) by \( q \) and then operating on it with \( M \) gives an equation for \( v \) alone:

\[ v = \frac{2q k}{\epsilon^2 \phi_N / \sqrt{2}} \frac{\partial}{\partial y} - \frac{k}{\omega} - \frac{2q k}{\epsilon^2 \phi_N / \sqrt{2}} \frac{k}{\omega} (\beta y - \gamma) - \frac{k}{\omega} (\beta y - \gamma) - \frac{k}{\omega} (\beta y - \gamma) - \frac{k}{\omega} (\beta y - \gamma) - \frac{k}{\omega} (\beta y - \gamma). \]  

(3.14)

Setting \( \gamma = 0 \) reduces this equation to the one previously studied by Matsuno, Longuet-Higgins and Lindzen.

Eq. (3.14) is singular, where \( \omega = 0 \) and \( \epsilon = k^2 / \omega^2 \). In this work, I will only consider wind profiles such that \( \omega \) never vanishes: Boyd (1978c) discusses what happens when it does.

The other class of singularity is only apparent as shown in an appendix of Section 4 of Boyd (1969a); by “apparent” I mean that both linearly independent solutions of the differential equation are analytic where \( \epsilon = k^2 / \omega^2 \) (instead of one solution being non-analytic as one might expect) even though the coefficients of the differential equation are singular there. As will be discussed in the next section, it is usually possible to neglect the terms involving this singularity by scaling, but there are some important exceptions. Since (3.14) is nonlinear in the eigenvalue \( \epsilon \) when the apparently singular terms are retained, it is preferable, when these terms cannot be neglected, to solve the equation for the meridional structure of the geopotential which constitutes a linear eigenvalue problem. The \( \phi \) equation is easily derived by solving (3.1) and (3.2) for \( u \) and \( v \) in terms of \( \phi \) and (3.3) for \( w \) in terms of \( \phi \) and then substituting the results into (3.4), the equation of continuity, to give

\[ \phi_{uu} - (\ln \Delta) \phi u \]

\[ + \left\{ \frac{k}{\omega} - \left[ (\beta y \ln \Delta) - \beta \right] - \epsilon \Delta \right\} \phi = 0. \]  

(3.15)
Eq. (3.15) has apparent singularities at the inertial
latitudes, i.e., at the roots of $\Delta$, and, unfortunately, it is not possible to scale away the apparent singularities for any class of equatorial waves. Thus, (3.14) is vastly preferable to (3.15) whenever the $\nu$ equation can be reduced to a linear eigenvalue equation by scale analysis.

4. Strategy and scale analysis

The explicit eigenvalues for an atmosphere without horizontal shear are given by (Holton, 1975, pp. 63–70, 134–143)

$$\epsilon = \frac{k^2}{\omega^2} \quad \text{(Kelvin wave; all } \chi),$$

$$\epsilon = \frac{k^2}{\omega^2} (1 - \chi^{-1})^2 \quad \text{(mixed Rossby-gravity wave; } \chi < 1),$$

$$\epsilon = \frac{k^2}{(2n+1)^2} \frac{2n+1}{\omega^2} \quad \text{(gravity waves; small } \chi),$$

$$\epsilon = \frac{k^2}{\omega^2} (1 - \chi^{-1}) \quad \text{(gravity waves; large } \chi),$$

$$\epsilon = \frac{k^2}{2n+1}\frac{1}{\omega^2} \quad \text{(Rossby waves; small } \chi) \text{, for } n \geq 1,$$

where $\chi$ is the nondimensional parameter $(k\omega/\beta)$. The equatorial beta-plane is inapplicable for Rossby waves when $\chi \gg 1$. From (4.1), one can see immediately that (3.14) is highly inappropriate for studying Kelvin waves—in the absence of shear, $\nu = 0$ for this mode—so, in Section 5, I will attack the original system of equations (3.1)–(3.4) directly to obtain perturbative solutions for Kelvin waves.

The mixed Rossby-gravity wave is well-defined: for small $\chi$, it is a gravity wave and will satisfy the gravity wave scaling given below, but for $\chi \gg 1$, it is a planetary Rossby wave and the equatorial beta-plane is not appropriate for describing it. For $\chi = 0.5$, its eigenvalue is identical to that for the Kelvin wave, so one must either attack the original system of equations or the geopotential equation (3.15). Unfortunately, the observed stratrophic mixed Rossby-gravity has $\chi \approx 0.5$ over a significant height range, so this intermediate range of $\chi$ does not constitute a trivial exception.

For large $\chi$, Eq. (4.3) shows that for gravity waves, one is again forced to attack (3.15) since the equation for $\nu$ is not useful when $\epsilon = k^2/\omega^2$ because then the apparent singularities in (3.14) cannot be ignored by scaling. Fortunately, Eq. (3.15) can be easily solved by the numerical method of Section 7. For analytical purposes, however, as in Section 6, one’s primary concern must be with that range of $\chi$, i.e., $\chi$ small, for which the apparent singularities in (3.14) can be discarded, not only because this makes the equation tractable but also because the mixed Rossby-gravity wave, the principal observed gravity wave, is gravity wavelike only for small $\chi$.

For $\chi \ll 1$ (mixed Rossby-gravity) and $\chi \ll 1$ (pure gravity waves), one finds from (4.2) and (4.3) that

$$\frac{k^2}{\omega^2} = \mathcal{O} \left[ \frac{x^2}{(2n+1)^2} \right] \epsilon \ll \epsilon.$$

With the assumptions

(i) $\gamma = 0(\omega),$

(ii) $\frac{\partial \gamma}{\partial \theta} \ll \beta,$

which, as discussed in Part II, are quite modest for observed atmospheric waves, it follows from (4.5) that

$$-2\left( \frac{\omega}{\omega - k^2/\omega^2} \right) \beta \gamma \omega = \mathcal{O} \left[ \frac{2x^2}{(2n+1)^2} \right] \beta \gamma \omega,$$

$$2\left( \frac{\omega}{\omega - k^2/\omega^2} \right) \beta \gamma \omega = \mathcal{O} \left[ \frac{2x^2}{(2n+1)^2} \right] \beta \gamma \omega,$$

$$k(\beta \gamma \omega) = \mathcal{O} \left[ \frac{x}{(2n+1)} \right] \beta \gamma \omega.$$

From the Hermite function equatorial wave solutions in the absence of shear, it is straightforward to derive the estimate for the meridional scale of the wave

$$L = \frac{\omega_0}{\beta (2n+1)} \quad \text{(gravity waves)},$$

which with (i) and (4.5) implies

$$-2\gamma k \left( \frac{\omega}{\omega - k^2/\omega^2} \right) \omega_\nu = \mathcal{O} \left( \frac{2x}{(2n+1)} \right) \omega_\nu.$$

Notice that the terms in (4.6) are $O(x^2)$ while those of (4.7) and (4.8) are $O(x)$. The terms which are first order in $x$ will be discussed in Section 6, but to lowest order in $x$, Eq. (3.14) becomes simply

$$\nu_{xx} - \epsilon (\beta \gamma \omega - \gamma - \omega^2) \nu = \mathcal{O} (\text{gravity waves}).$$

For Rossby waves, the exact eigenrelation shows that $\epsilon$ is a monotonically decreasing function of $\chi$. 

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*The equatorial beta-plane is valid only for large $\epsilon$ in the absence of shear. It is shown in Longuet-Higgins (1968) that for fixed $k$ and $\beta$, $\epsilon$ is a monotonically decreasing function of $\chi$ for Rossby and mixed Rossby-gravity waves and that $\epsilon = 0$ for $x = x^* \left[ (n+2) (n+s+1) \right]$. Thus, the limiting value of $\chi$ for which the beta-plane is useful is always less than 1 for all Rossby waves and for the mixed Rossby-gravity wave.
so (4.4) is an upper bound on $\epsilon$. Thus, for all Rossby modes,

$$\epsilon \ll k^2/\omega^2$$  \hspace{1cm} (4.10)

is a very good approximation, implying

$$\epsilon/(\epsilon-k^2/\omega^2) = -\epsilon/(k^2/\omega^2),$$ \hspace{1cm} (4.11)

$$L = (\omega_0/k\beta)^4$$  (Rossby waves),

and further implying under the other assumptions above

$$-2\gamma k \left( \frac{\epsilon}{\omega - k^2/\omega^2} \right) v_\nu/v_\eta = O \left[ \frac{2 \sqrt{\chi}}{(2n+1)^2} \right] \ll 1$$ \hspace{1cm} (4.12)

and reducing (3.14) to

$$v_\nu + \left\{ k(\beta-\partial_\gamma/\partial y) \omega \epsilon - k^2 - \epsilon(\beta \gamma - 2\gamma)(\beta \gamma - \gamma) \right\} v = 0$$ \hspace{1cm} (Rossby waves). \hspace{1cm} (4.13)

Note that (4.13), the Rossby wave equation, differs from (4.9), the gravity wave equation, only through the presence of the additional underlined terms in (4.13).

Given an equation or system of equations, there are three principal means for solving it: perturbation theory; Taylor expansion of the frequency and shear terms up to second order in $\gamma$ followed by transformation to the harmonic oscillator equation; and direct spectral solution via Hermite functions. For the Kelvin wave, perturbation theory is useful because, as the perturbative solutions indicate, shear-induced changes in this mode are small, for the observed parameters. For gravity waves with $x$ small, shear may produce $O(1)$ changes in the wave fields even when the relative change in frequency between the turning points is small. Ordinary perturbation theory cannot handle such large changes from the unperturbed fields, but a procedure based on transformation to the harmonic oscillator equation (whose solutions are known analytically) is quite simple and accurate as will be discussed in Section 6. For the $n=1$ Rossby wave in the troposphere and the mixed Rossby-gravity wave in the stratosphere, neither of the two previous approaches is entirely satisfactory although perturbation theory for both is discussed in Section 9, but solving either (4.13) or (3.15) spectrally through expansions in Hermite functions is extremely efficient and easy because these polynomials multiplied by a Gaussian are the exact eigenfunctions of the meridional structure equation when there is no shear. The Hermite spectral method is the topic of Section 7.

5. The Kelvin wave in shear

As noted in Section 2, Eq. (3.14) is useless for the Kelvin wave because the apparent singularities are important for this one mode. Consequently, using conventional perturbation theory, I will solve the system

$$i\omega u + (-\beta y + \gamma) v = -ik\phi,$$ \hspace{1cm} (5.1)

$$\beta u v + i\omega v = -\phi u,$$ \hspace{1cm} (5.2)

$$ik u + v = -i\omega \phi.$$ \hspace{1cm} (5.3)

The unperturbed solutions for the Kelvin wave are given explicitly by

$$\phi^0 = \exp(k\beta y^2/2\omega_0),$$ \hspace{1cm} (5.4)

$$\psi^0 = (k/\omega_0)\phi^0,$$ \hspace{1cm} (5.5)

$$v^0 = 0,$$ \hspace{1cm} (5.6)

$$\epsilon = k^2/\omega_0.$$ \hspace{1cm} (5.7)

We define a perturbation parameter $\zeta$ by setting

$$\zeta = [\omega(y) - \omega_0]/\omega_0,$$ \hspace{1cm} (5.8)

where $y_1$ is the northern turning latitude for the unperturbed wave $[=(\omega_0/k\beta)^4], and assume

$$\omega = \omega_0 + \gamma k\phi^0 + \delta k^2 y^2/2,$$ \hspace{1cm} (5.9)

$$\epsilon = \epsilon^0 + \epsilon^1 + \epsilon^2,$$ \hspace{1cm} (5.10)

$$\phi = \phi^0 + \phi^1 + \phi^2,$$ \hspace{1cm} (5.11)

and similar expansions for $u$ and $v$ where, for any quantity $q, q^2 = O(\zeta^2) \phi^2 = O(\zeta^2) q^2$.

For sufficiently large $y$, a parabolic wind profile as assumed in (5.9) is never a small perturbation and the scaling of (5.8) breaks down. Fortunately, all the Kelvin wave fields decay like Gaussians for large $y$, so this scaling breakdown occurs only where the wave amplitude is exponentially small. A similar situation arises for the quantum mechanical anharmonic oscillator where a potential of the form $y^2$ is perturbed by a term $\lambda y^4$. As explained in Boyd (1978b), for sufficiently large $y$, no matter how small $\lambda$, the quartic term will dominate the quadratic. However, the eigenfunctions—the unperturbed ones are the Hermite functions of Section 4—decay exponentially for large $|y|$ so for small $\lambda$, the perturbative assumption breaks down only where the eigenfunction has negligible amplitude. The result is that the Taylor series for the perturbed eigenvalue about $\lambda = 0$ converges rapidly for sufficiently small $\lambda$ and gives many decimal places of accuracy—but it has been rigorously proved that the series is asymptotic. Thus, it is likely that the perturbation series whose first terms are derived here is asymptotic, too, but as discussed in Morse and Feshbach (1953), the formal divergence of an asymptotic series is no real problem as long as the first few terms decrease rapidly in the parameter range of interest—as seems generally the case for the Kelvin wave.

The first-order perturbation equations are

$$ik[\gamma y + \delta y^2/2] \phi^1 = -\beta y v^1 - i\omega u^1 - ik\phi^1,$$ \hspace{1cm} (5.12)

$$i\omega u^1 = -\beta u v^1 - \phi^1,$$ \hspace{1cm} (5.13)

$$v^1 + ik[\gamma y + \delta y^2/2] \phi^1 = -i\omega \phi^1 - ik u^1.$$ \hspace{1cm} (5.14)
It can be shown through a modest generalization of the methods of Section 9 that if the wind is approximated by a polynomial in $y$, then the order-by-order perturbative solutions for all fields are the products of polynomials in $y$ with a Gaussian in $\gamma$. (This statement also applies to the higher modes as shown in Section 9.) Consequently, Eqs. (5.12)–(5.14) can be solved by simply matching powers of $y$. One finds\(^4\)

\[ e^i = (k^2/\omega^2)(\delta_0/2\beta) = e^i(\delta_0/2\beta), \tag{5.15} \]
\[ \nu^i = [i(k(\gamma_0+\delta_0y/2)/\beta)]u^i, \tag{5.16} \]
\[ u^i = -(k^2/\beta)[\gamma_0y+\delta_0y^2/4]u^i, \tag{5.17} \]
\[ \phi^i = (k\omega_0/\beta)[\gamma_0y^2+\delta_0y^3/4]u^i. \tag{5.18} \]

Note that $e^i$ depends only on $\delta_0$. A simple symmetry argument explains this. By definition $e^i$ must be a linear function of $\gamma_0$, so changing the sign of $\gamma_0$ must change the part of $e^i$ which depends on $\gamma_0$. However, a change of sign in $\gamma_0$ does not alter the physical situation—it merely transposes the roles of the North and South Poles—so $e^i$ must be unaffected. Thus, $e^i$ must be independent of $\gamma_0$.

Eqs. (5.17) and (5.18) imply

\[ u^i = -e^i(k/\omega_0)\phi^i, \tag{5.19} \]

which is identical to (5.5) except that the zeroth-order zonal wind and geopotential have been replaced by the corresponding first-order quantities. Eqs. (5.19) and (5.7) show that the right-hand sides of (5.12) and (5.14) are both zero. Since the perturbed mean wind appears only in the these latter two equations, this implies that the effects of shear are balanced entirely by the first-order meridional wind and first-order eigenvalue. The only role of the first-order geopotential and zonal wind is to balance the $\nu^i$ term in the meridional momentum equation (5.13).

The most important characteristic of the explicit solutions is that to first order

\[ (\sigma/u) = O(\omega_0/\beta). \tag{5.20} \]

Most observational studies have relied heavily on the smallness of this ratio to detect Kelvin waves via spectral analysis of the horizontal wind fields and other methods, and (5.20) shows that for long-period Kelvin waves ($\omega \ll 20$), this ratio will always be small unless $\omega$ is changing on a meridional scale much smaller than the radius of the earth.

Eq. (5.15) can be rewritten as

\[ e^i = e^i[U_{uv}/(2\beta)]. \tag{5.21} \]

The Rayleigh–Kuo criterion for barotropic instability (Kuo, 1949) is $U_{uv} > \beta$ somewhere in the flow. Thus, for westerly jets which are stable according to the Rayleigh–Kuo criterion, the relative first order change is at most 50% in $e^i$, 25% in vertical wavelength (proportional to $e^{-1}$) and 12% in meridional scale (proportional to $e^{-1}$).

Since $u^i$ and $\nu^i$ are proportional to $k^2$ and $\nu^i$ only to $\beta$, it follows that (i) $u^i$ and $\phi^i$ may be more significant than $\nu^i$ although $\nu^i > u^i$ for the example given in Part II and (ii) the perturbation series will break down as $k \rightarrow \infty$ even if $\omega(y)$ is held fixed. This zonal wavenumber dependence is crucial and is most easily understood by noting that for the special cases of pure linear and pure quadratic shear, respectively, one can rewrite (5.17) as

\[ u^i = -x\left[\frac{\omega(y) - \omega_0}{\omega_0}\right]u^i \quad \text{(linear shear)}, \tag{5.22} \]
\[ u^i = -x\left[\frac{\omega(y) - \omega_0}{2\omega_0}\right]u^i \quad \text{(quadratic shear)}, \tag{5.23} \]

where

\[ x = k\omega_0/\beta, \tag{5.24} \]

the nondimensional parameter that also plays a crucial role in the theory of the “gamma-plane” approximation given in the next section.

Eqs. (5.22) and (5.23) show that for large enough $|y|$, $u^i \approx O(u^i)$ no matter how small $\gamma_0$ and $\delta_0$ are, which is why the perturbation series is asymptotic instead of convergent. If the condition $|u^i| < |u^i|$ is violated only for $|y|$ large enough so that both $u^i$ and $\nu^i$ are exponentially small, however, then the perturbation series will give an accurate approximation to the eigenvalue and, near the equator, to the winds and geopotential as well.

For large $x$, it is clear that $u^i \approx O(u^i)$ close to the equator and the perturbation expansion is useless. For small $x$, however, the perturbation series will be accurate even in such strong shear that $\omega(y)$ changes by a factor of 2 in the region where the Kelvin wave has most of its amplitude. For the example of Part II (Boyd, 1978a) $x = 0.07$, so even very strong shear has an almost ludicrous small effect on the Kelvin waves observed in the atmosphere. When the zonal wave-number or frequency are higher, however, as may be the case in the oceans, wind shear may have drastic effects on Kelvin waves—but then the perturbation series cannot help us.

As a final remark, however, note that the explicit forms of the first-order fields are all independent of $\omega_0$. This implies that as $\omega_0 \rightarrow 0$, i.e., a critical level, the perturbation expansion does not break down if the other parameters are fixed. Thus, for the particular observed Kelvin waves discussed in Part II, the horizontal shear perturbation theory is always useful even when the wind profile vanishes at a critical level.

The first-order zonal and meridional velocities for linear shear only and parabolic shear only are presented in Figs. 1–4. In each case, $[\omega_0] = 60$ m s$^{-1}$ and the

---

\(^4\) The second-order solutions were included in an earlier draft of this paper and are available from the author.
zonal wavenumber is 1 [corresponding to the 15-day modes reviewed in Holton (1975) and Wallace (1973)], the zeroth-order geopotential is normalized to a maximum value of 1 in MKS units, and the shears are normalized so that the values for $u'$ and $v'$ on the graphs are those for $[U(a) - U(0)] = 2$ m s$^{-1}$, where $a$ is the earth's radius. The shapes of the first-order solutions are invariant, so the graphs shown apply to any Kelvin wave with appropriate rescaling of the axes. It is important to note, however, that for both cases (which give the general parabolic shear solution by superposition), $v' > u'$ for this set of zeroth-order parameters. For high zonal wavenumber and frequency, the opposite would be true as explained above.

6. The equatorial gamma-plane approximation for gravity and mixed Rossby-gravity waves

As shown in Appendix B, any equation of the form

$$v_{yy} + (E+Fy)v_y + (A+By-Cy^2)v = 0,$$

(6.1)

where $A$, $B$, ..., $F$ are constants, can be transformed into Schroedinger's equation for the harmonic oscillator by simple, explicit changes of both the independent and dependent variables. Explicit solutions for the eigenvalues and eigenfunctions of the transformed equation can then be looked up in any elementary quantum mechanics text. The equatorial beta-plane approximation itself is merely a Taylor expansion in $y$. If one similarly expands $\omega(y)$ in $y$
and keeps only the linear terms in the coefficient of $v_y$ and the quadratic terms in the coefficient of $v_x$, then the transformations of Appendix B and the Hermite function solutions of Schrödinger's equation provide analytic solutions for equatorial waves in shear.

Although this approach is fairly general, in this work I shall explicitly apply it only to gravity waves and to the mixed Rossby-gravity wave in its "gravity wave-like" regime and only for small $x$, where $X = k_x / \beta$. For Rossby waves, the dominant shear term in Eq. (4.13) comes from the expansion of $k_x / \omega$, not from the terms explicitly involving $\gamma$. Thus, the effect of shear on Rossby waves is directly proportional to the change in $\omega$ between the turning points of the wave. Because of the relatively broad meridional extent of long-period Rossby waves, such changes in $\omega$ may be of the same magnitude as the equatorial frequency $\omega_0$ itself, but in this situation the Taylor expansion of $\omega^2$ breaks down. Thus, the convert-to-the-harmonic-oscillator-equation technique is equivalent to the ordinary perturbation expansion derived in Section 9, and more powerful techniques for Rossby waves will be given in the next section. In contrast, the equation-transformation approach is a nonperturbative method for gravity waves of small $X$ in the sense that it gives accurate solutions even when shear induces large changes in the wave fields as will be seen below. There are two reasons why this special range of small $X$ is particularly interesting. First, for large $X$ the gamma-effect which I shall discuss below is not important and the equation-transformation approach becomes merely an ordinary perturbation method. Second, the only gravity wave that has been observationally isolated in the atmosphere is the mixed Rossby-gravity wave, which is gravity wave-like only when $X$ is small compared to 1 (e.g., near a critical level).

It is easy to show from (5.19) that the change in Doppler-shifted frequency between the turning points of $O(X^2)$ Part I, i.e., the region where the wave has most of its amplitude is $O(X^{2}(\gamma_0 + \omega_0(\delta_0/2\beta)))$. It follows that when $X$ is small, $\omega(y)$ can be replaced by the constant $\omega_0$, its equatorial value, with little error over the range of latitude in which the wave is confined. However, it would be quite wrong to conclude that all effects of shear vanish as $X \to 0$ because $\gamma(y) = \partial U / \partial y$ appears explicitly in the zonal momentum equation and this one term, as shall be seen below, can produce order-of-magnitude changes in the wave fields.

These twin facts suggest the approximation for small $X$ of treating $\omega(y)$ in the same way as the midlatitude beta-plane treats the Coriolis parameter: replacing it by a constant mean value except where it is differentiated. Because of the analogy with the midlatitude beta-plane, I have dubbed this approximation "the equatorial gamma-plane approximation."

The basic gamma-plane approximation and first order in $X$ corrections for the special case of the mixed Rossby-gravity wave are considered in the next two subsections, respectively.

\subsection{The equatorial gamma-plane approximation}

After eliminating $w$ and using the separability condition the gamma plane approximation gives the set

$$i\omega u + \left[ -\beta y + \gamma(y) \right] u = -ik\phi, \quad (6.2)$$

$$\beta y u + i\omega \phi = -\phi_y, \quad (6.3)$$

$$iku + v_y = -i\omega_0 \phi. \quad (6.4)$$

Routine elimination of $u$ and $\phi$ and neglect of $(k\beta/\omega - k^2)$ and $v_y$ by scaling as in (4.6) and (4.7) give

$$v_{yy} - \epsilon(\beta y[\beta y - \gamma(y)] - \omega_0^2)v = 0, \quad (6.5)$$

which could be alternatively derived by applying the gamma-plane approximation or small $X$ scale analysis directly to (4.9). As in the previous section, it is convenient to represent $\gamma(y)$ as linear, i.e.,

$$\gamma(y) = \gamma_0 + \delta_0 y$$

and to examine the effects of nonzero $\gamma_0$ and $\delta_0$ separately.

When $\gamma_0 = 0$ (quadratic shear) the wave fields are constrained to be either symmetric or antisymmetric about the equator, so a nonzero value of $\delta_0$ will not produce large changes in wave shape. Its principal effect is to replace $\beta^2$ by $\beta(\beta - \delta_0)$ in the usual gravity wave eigenvalue formula (4.3) which can be written as

$$\epsilon = \frac{\beta(\beta - \delta_0)(2n+1)^2}{\omega_0^4}. \quad (6.6)$$

For the value of $\delta_0$ associated with the quasi-biennial oscillation at its level of maximum amplitude, this is enough to increase $\epsilon$ by about 50% and double the net acceleration of the mean by the wave as is shown in Part II, Section 4. Because symmetric shear does not alter the shape of the wave except for small changes in latitudinal scale, I will take $\delta_0 = 0$ in the rest of this section.

The situation for linear shear is very different. With or without shear, the wave is latitudinally trapped between the turning latitudes $\gamma_1$, where the coefficient of $e$ in (6.5) is zero, i.e.,

$$\gamma_1 = \gamma_0 / 2\beta \pm (\gamma_0^2 + 4\omega_0^2)^{1/2} / 2\beta. \quad (6.7)$$

Observations show $\gamma = 0(\omega_0)$ is not an unreasonably large value for the shear. One finds from (6.7)

$$\gamma_1 = \frac{\omega_0}{\beta} \begin{cases} (\pm 1, 1), & \gamma_0 = 0 \\
(- \frac{62}{100}, \frac{62}{100}), & \gamma_0 = \omega_0 \end{cases} \quad (6.8)$$

independent of $X$ for $X$ small. Thus, even though $\omega(y)$ is approximately constant over the region between
the turning points, the shear has drastically altered the location of the turning points and shifted the region in which the wave has significant amplitude. Expressed another way, the shear has significantly changed the effective location of the equator. Such a drastic change would ruin a conventional perturbation expansion such as that of Andrews and McIntyre (1976b), but it is precisely these cases where horizontal shear induces large changes in the wave that are of the greatest interest. The reduction to the harmonic oscillator equation, however, effectively incorporates this shear term into the lowest order solution, so it constitutes a nonperturbative technique.

The transformations of Appendix B give explicit solutions for \( v \) and \( e \). To obtain expressions for \( u \) and \( \phi \), one solves Eqs. (6.3)–(6.5) for \( u \) and \( \phi \) in terms of \( v \), changes variable from \( y \) to \( \xi \), and exploits these identities of the Hermite polynomials:

\[
\frac{dH_n(\xi)}{d\xi} = 2nH_{n-1}(\xi),
\]

\[
\xi H_n(\xi) = nH_{n-1}(\xi) + H_{n+1}(\xi)/2
\]

neglecting \( k/\omega_0 \) in comparison to \( \epsilon^4 \) as one goes in accordance with (4.5). The results are

\[
e^4 = \frac{2n+1}{\gamma_0^2 + \gamma_0^2/4},
\]

\[
\xi = (\gamma_0^2/2\delta)\delta^4 e^4,
\]

\[
v = e^{-\epsilon^4/2} \frac{H_n(\xi)}{\gamma_0},
\]

\[
u = -\frac{1}{\omega_0} \frac{\gamma_0}{\gamma_0^2 + \gamma_0^2/4} \left( \frac{\gamma_0^2 + \gamma_0^2/4}{2n+1} \right) \xi,
\]

\[
\phi = \frac{i\epsilon^4}{\omega_0} \frac{e^{-\epsilon^4/2}}{\left[ nH_{n-1}(\xi) - H_{n+1}(\xi)/2 \right]}
\]

Figs. 5–7 compare the exact (dotted line) and gamma-plane (solid line) approximate geopotential, meridional wind and zonal wind, respectively, for a case with zonal wavenumber 1, \( \omega_0 = 10^{-5} \text{ s}^{-1} \), \( \gamma_0 = 2\omega_0 \) and \( \chi = 0.07 \). As expected from (6.12) and (6.14), \( v \) and \( \phi \) are shifted almost entirely into the hemisphere of larger frequency with little change of shape. The zonal wind, however, which in the absence of shear would be similar in shape to the geopotential, i.e., two peaks of equal amplitude and an equatorial node, has been distorted almost beyond recognition. A single large peak, centered almost at the equator itself, dominates the response; the secondary peak is only 20% of the height of the first. Although this example (and its unrealistically strong shear) were chosen somewhat arbitrarily to illustrate the accuracy of the gamma-plane approximation, these same quali-

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**Fig. 5.** Gamma-plane (solid line) and numerical (dashed line) geopotential for the mixed Rossby-gravity wave, Case III (arbitrary units).

**Fig. 6.** As in Fig. 5 except for the meridional wind.

**Fig. 7.** As in Fig. 5 except for the zonal wind.
Table I. Comparison of the gamma-plane [Eq. (6.15)] and exact nondimensional eigenvalues. Case III is illustrated in Figs. 5–7. All cases are identical except for the differences indicated below.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Gamma-plane</th>
<th>Exact</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I ((\gamma_0=0))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4.91E04</td>
<td>4.98E04</td>
<td>1.4%</td>
</tr>
<tr>
<td>2</td>
<td>4.99E05</td>
<td>5.05E05</td>
<td>1.1%</td>
</tr>
<tr>
<td>3</td>
<td>1.40E06</td>
<td>1.41E06</td>
<td>1.1%</td>
</tr>
<tr>
<td>Case II ((\gamma_0=\omega_0))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.02E04</td>
<td>2.74E04</td>
<td>10.2%</td>
</tr>
<tr>
<td>2</td>
<td>3.18E05</td>
<td>2.94E05</td>
<td>8.2%</td>
</tr>
<tr>
<td>3</td>
<td>8.93E05</td>
<td>8.25E05</td>
<td>8.3%</td>
</tr>
<tr>
<td>Case III ((\gamma_0=2\omega_0))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.03E04</td>
<td>9.33E03</td>
<td>10.8%</td>
</tr>
<tr>
<td>2</td>
<td>1.23E05</td>
<td>1.02E05</td>
<td>20.1%</td>
</tr>
<tr>
<td>3</td>
<td>3.48E05</td>
<td>2.88E05</td>
<td>20.8%</td>
</tr>
<tr>
<td>Case IV ((\gamma_0=2\omega_0; \omega_0&lt;0))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.77E04</td>
<td>2.52E04</td>
<td>29.8%</td>
</tr>
<tr>
<td>2</td>
<td>1.30E05</td>
<td>1.77E05</td>
<td>26.6%</td>
</tr>
<tr>
<td>3</td>
<td>3.55E05</td>
<td>4.80E05</td>
<td>26.0%</td>
</tr>
</tbody>
</table>

It is similarly straightforward to compute \(u, v\) and \(\phi\) as done above, but when first-order \(X\) effects are included, the algebra becomes messy and the final results are complicated. For this reason, and because the techniques are the same as described above, I omit a detailed discussion and simply write down the answers for the mixed Rossby gravity wave:

\[
v = \exp \left[ -\frac{\beta}{2\omega^2} (1-x) y + \frac{\gamma_0 \beta}{2\omega^2} (1+3x)y \right],
\]

\[
\phi = \frac{i}{2\beta} \left[ \frac{\omega_0 \gamma_0}{2\beta} (1+3x) - \omega_0 y + k \gamma_0 y^2 \right],
\]

\[
u = \frac{i}{2\beta} \left[ \frac{-\gamma_0}{\omega_0} (1-x/2) - \frac{\beta}{\omega_0} (1-x) y - \frac{k \gamma_0 \beta}{\omega_0^2} y^2 \right],
\]

to first order in \(\gamma_0, X\) and \(\gamma_0 X\). [For simplicity, terms of \(O(\gamma_0^2)\) were neglected in (6.17)–(6.19) so that they correspond to a regular perturbation procedure.] It can be shown that the quadratic terms in the curly brackets in (6.18) and (6.19) are \(O(X)\) in comparison to the other terms. It is then easy to show that the latitude where \(v\) is a maximum and \(u\) and \(\phi\) have nodes are given by

\[
y(v\text{ maximum}) = \frac{-\gamma_0}{2\beta} (1+4x),
\]

\[
y(\phi\text{ node}) = \frac{-\gamma_0}{2\beta} (1+3x),
\]

\[
y(u\text{ node}) = \frac{-\gamma_0}{\beta} (1+x/2).
\]

To zeroth order in \(X\), this implies that \(\phi\) is antisymmetric about the latitude where \(v\) is a maximum. For nonzero \(X\), however, the node in \(\phi\) is a little closer to the equator than the peak in \(v\), which implies that the geopotential peak in the hemisphere of large \(\omega\) will be slightly larger in absolute value than the other peak. This is clearly evident in the numerical solution shown in Fig. 5. For the zonal wind, the reverse is true, even for zero \(X\). The node in \(u\) is about twice as far from the equator as the maximum in \(v\), which as noted earlier can lead to a marked asymmetry in the magnitude of its two peaks. Non-zero \(X\) brings the peak of \(v\) and the node of \(u\) into closer accord, thus reducing this asymmetry. Simmons (1978) noted in his numerical experiments a tendency for the node of \(u\) to occur where the mean absolute vorticity vanishes, i.e., at \(y = \gamma_0/\beta\). Although the formulas of this section can be applied to the observed mixed Rossby gravity wave only heuristically (away from a critical surface) because \(x=0.5\), a rather large value, it is, nonetheless, noteworthy that (6.22) confirms Simmon's observation.
7. Hermite spectral methods

Unfortunately, the analytic techniques of earlier sections are inadequate to study all equatorial waves or even all equatorial waves observed in the stratosphere, so one must sometimes resort to numerical methods. This section describes an algorithm that is ideally suited to equatorial waves because it employs expansion functions which are the exact eigenfunctions of the horizontal structure equation in the absence of lateral shear. Combined with finite differences in height, Hermite spectral methods can also be used to attack the nonseparable, two-dimensional wave equation directly, as is necessary when the multiscaling approximation is invalid as is the case for tropospheric and oceanic equatorial waves. Because of its broad usefulness, therefore, the Hermite algorithm and its numerical efficiency will be analyzed in detail.

To apply Galerkin’s spectral method, the variant I will use here, to the ordinary differential equation eigenvalue problem

\[ Qu = \mu r(y)u, \quad (7.1) \]

where \( \mu \) is the eigenvalue and

\[ Q = a_1(y) \frac{d^2}{dy^2} + a_1(y) \frac{d}{dy} + a_0(y), \quad (7.2) \]

one assumes that \( u(y) \) can be represented to acceptable accuracy by a truncated series of functions

\[ u(y) = \sum_{n=0}^{N-1} u_n \phi_n(y), \quad (7.3) \]

where in the present case

\[ \phi_n(y) = e^{-y/2} H_n(y), \quad (7.4) \]

where \( H_n(y) \) is the \( n \)th Hermite polynomial (Abramowitz and Stegun, 1965). The coefficients \( u_n \) and the eigenvalue \( \mu \) for each mode are given by the solutions of the algebraic problem

\[ Qu = \mu Ru, \quad (7.5) \]

where \( Q \) and \( R \) are \( N \times N \) square matrices whose elements are given by

\[ Q_{ij} = (\phi_i | Q \phi_j), \quad (7.6) \]

\[ R_{ij} = (\phi_i | R \phi_j), \quad (7.7) \]

with the inner product in (7.6) and (7.7) defined by

\[ (\phi(y) | q(y)) = \int_{-\infty}^{\infty} dy \phi(y)q(y). \quad (7.8) \]

If \( R \) is the identity matrix, Eq. (7.5) is the ordinary algebraic eigenvalue problem; if \( R \) is not the identity matrix but is nonsingular (as is the case in applications to equatorial waves), Eq. (7.5) can be converted into the ordinary eigenvalue problem by multiplying through by \( R^{-1} \). For fixed \( N \), only the lowest eigen-modes of (7.5), i.e., those for whom the coefficients \( u_n \) decrease sufficiently fast where “sufficiently” will be made more precise below, correspond to eigensolutions of the original differential equation; higher order algebraic modes whose coefficients are all roughly the same magnitude are strictly numerical. To accurately compute more modes of the differential equation, one must increase \( N \). I will not attempt to justify (7.5) here but instead refer the reader to Finlayson (1972).

The matrix elements (7.6) and (7.7) are most easily computed by direct Gauss-Hermite quadrature:

\[ Q_{ij} = \sum_k w_k \phi_i(y_k) \left[ a_1(y_k) \frac{d^2}{dy^2} \phi_j(y_k) + a_1(y_k) \frac{d}{dy} \phi_j(y_k) + a_0(y_k) \phi_j(y_k) \right], \quad (7.9) \]

and similarly for \( R \). The weights \( w_k \) and collocation points \( y_k \) are given in Abramowitz and Stegun (1965). In analytical work, such as that leading to (7.18), it is convenient to split each matrix element into its constituent parts, e.g., \( (\phi_i | y^3 \phi_j), (\phi_i | y^4 \phi_j) \), evaluate each integral separately by Gauss-Hermite quadrature, and then assemble the final determinant by hand. For numerical work, it is easier to evaluate the \( Q_{ij} \) and \( R_{ij} \) all at once.

Computation of \( H_n(y) \) and its derivatives is trivial because of two identities of the Hermite polynomials:

\[ H_0 = 1; \quad H_1 = 2y, \quad (7.10) \]

\[ H_{n+1}(y) = 2y H_n(y) - 2n H_{n-1}(y), \quad (7.11) \]

Summing a truncated Hermite series of the form

\[ f(y) = \sum_{n=0}^{N-1} c_n H_n(y) \]

can be done via a three-term recurrence relation derived from (7.10). After the initialization \( B_1 = B_2 = 0 \), one canks through \( N \) applications of

\[ B_0 = 2[yB_1 - (N+1-k)B_k], \quad k = 1, 2, \ldots, N \]

\[ B_2 = B_1 + c_{N-k} \]

\[ B_1 = B_0 \]

Then

\[ f(y) = B_0. \quad (7.14) \]

For 1) avoiding numerical roundoff problems described in Boyd (1978d) that come from using the \( H_n \) themselves and 2) for convenience, it is advisable to
actually calculate the matrix elements in terms of the normalized Hermite polynomials $\hat{H}_n(y)$ which satisfy
\[
\hat{H}_n(y) = H_n(y) / \{\pi^{1/2} \sqrt{n!}\}, \quad (7.15)
\]
\[
\int_{-\infty}^{\infty} dy e^{-y^2} \hat{H}_n(y) \hat{H}_m(y) = 1, \quad (7.16)
\]
\[
|e^{-y^2} \hat{H}_n(y)| \leq 0.8157, \quad (7.17)
\]
for all \(n\) and all real \(y\).

The bound (7.17) is useful for two reasons. First, as explained in Boyd (1978d), with a little practice, one can distinguish between good and poor accuracy merely by inspecting the coefficients of the numerical eigenfunction. As always, however, the acid test of numerical accuracy is to repeat the calculation with a larger \(N\). Second, Eq. (7.17) can be used to estimate the importance of shear-induced changes in the equatorial wave fields since the unperturbed fields are pure Hermite functions \((u)\) or a sum of two \((u, \phi)\).

Most of the formulas and bounds above closely parallel those of other systems of orthogonal polynomials such as Chebyshev and Legendre polynomials, and like them, Hermite polynomials have the property of “infinite-order” convergence, which means that the error decreases faster than any finite power of \(N\) for the expansion of a sufficiently well-behaved function. Additional background and detail are discussed in Boyd (1978d), but it would be too far afield to give more than a cookbook description of the algorithm here. Nonetheless, enough has been presented to show that clearly, the Hermite spectral method is very simple and easy to use. A triple DO loop in \(i, j\) and \(k\) to compute the matrix elements and two three-term recurrence relations plus calls to standard, “blackbox” linear algebra routines are unlikely to tax the ability of even a novice programmer.

An important question still remains: how well do Hermite methods work when applied to equatorial waves? Because equatorial wave functions decay exponentially fast as \(|y|\) goes to infinity and because the exact constant frequency solutions are either Hermite functions or the sum of two Hermite functions, the answer is—very well. For both equatorial waves in shear and for the related problem of the quantum mechanical anharmonic oscillator (discussed in Boyd, 1978d) retaining just the lowest two Hermite functions of the appropriate parity is adequate. One usually thinks of spectral methods only in terms of multi-million dollar number-crunching computers and lengthy FORTRAN codes, but for \(N=2\), the eigenvalue is the solution of a quadratic equation and the whole problem can be solved analytically.

For the ground state of the anharmonic oscillator, for example, one finds
\[
E_0(\lambda) = 3 + 5.25\lambda - (4 + 18\lambda + 24.75\lambda^3), \quad (7.18)
\]
which is accurate to within 5% relative error for \(\lambda = 1\), which is three times larger than the range of usefulness of the perturbation series for \(E_0(\lambda)\). The principal complication in calculating and using similar analytical solutions for equatorial waves is the additional number of parameters. Some sample calculations were done for the \(n=0\) mixed Rossby-gravity wave in parabolic shear. Using the scaling of Section 9, there are three fundamental parameters: \(k\), \(\omega_0\) and \(\delta_0\), as opposed to the single parameter \(\lambda\) of the anharmonic oscillator. The result is that the coefficients of the quadratic equation for \(\epsilon\) contain no fewer than 65 terms. When \(k\) and \(\omega_0\) are fixed, one again obtains a formula with the simplicity of (7.18), and such an expression is accurate to within 4% for the strongest observed parabolic shear. Because of the complexity of the full three-parameter formulas, however, this analysis has not been included in this paper.

Nonetheless, the most powerful way to demonstrate the accuracy of a method for moderate \(N\) is to show its accuracy for very small \(N\). Thus, in the remainder of this section, \(N=2\) approximations for the other observed wave which cannot be easily studied using perturbation theory or the “gamma-plane approximation”—the \(s=10, n=1\) tropospheric Rossby wave—will be compared with the exact solution and with perturbative approximations.

The exact Hermite wave functions for a resting atmosphere are given in terms of the scaled latitudinal variable
\[
\xi = S_\theta \theta,
\]
with \(S_\theta = \xi^4 = 8.45\) for the present case, where \(\xi^4\) is the nondimensional eigenvalue when the shear is zero and \(\theta\) is latitude. It is obviously important to use the scaled variable \(\xi\) in numerical computations, but since the observed tropospheric wind shear significantly widens the wave, as shown in the last figure of Part II (Boyd, 1978a), it is no longer clear the “obvious” choice of scaling factor, \(S_\theta = \xi^4\) is the best. Thus, it is useful to examine the effect of varying \(S_\theta\) on the accuracy of the solution.
Table 3. A comparison of exact and approximate eigenfunctions for the $n=1$ Rossby wave, shear parameter $=1$.

<table>
<thead>
<tr>
<th>$y$ (deg)</th>
<th>Exact $N=2$, $S_y=8.45$</th>
<th>Error</th>
<th>Exact $N=2$, $S_y=6$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.5875</td>
<td>0.5360</td>
<td>-0.0515</td>
<td>0.5284</td>
</tr>
<tr>
<td>6</td>
<td>0.9455</td>
<td>0.9079</td>
<td>-0.0376</td>
<td>0.8895</td>
</tr>
<tr>
<td>9</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0</td>
<td>1.0000</td>
</tr>
<tr>
<td>12</td>
<td>0.8421</td>
<td>0.8255</td>
<td>-0.0166</td>
<td>0.8880</td>
</tr>
<tr>
<td>15</td>
<td>0.6133</td>
<td>0.5282</td>
<td>-0.0851</td>
<td>0.6548</td>
</tr>
<tr>
<td>18</td>
<td>0.4060</td>
<td>0.2652</td>
<td>-0.1408</td>
<td>0.4084</td>
</tr>
<tr>
<td>21</td>
<td>0.2485</td>
<td>0.1052</td>
<td>-0.1433</td>
<td>0.2162</td>
</tr>
<tr>
<td>24</td>
<td>0.1395</td>
<td>0.0332</td>
<td>-0.1063</td>
<td>0.0964</td>
</tr>
<tr>
<td>27</td>
<td>0.0716</td>
<td>0.0083</td>
<td>-0.0633</td>
<td>0.0352</td>
</tr>
<tr>
<td>30</td>
<td>0.0353</td>
<td>0.0017</td>
<td>-0.0336</td>
<td>0.0098</td>
</tr>
</tbody>
</table>

Table 2\(^7\) compares the coefficients of the Hermite solution of (4.13), the north-south velocity equation, for $S_y=8.45$ and $S_y=6$. Clearly, the smaller scale factor gives much better accuracy, but note that the coefficients are decreasing rapidly in either case. Table 3 and Fig. 8a compare the exact and approximate eigenfunctions. Note that the difference between the curves in Fig. 8a is smaller than that between the perturbed and exact eigenfunctions in Fig. 8b; the spectral method gives good approximations not only to the eigenfunction itself but also to the changes in the eigenfunction which are produced by shear. The spectral and perturbative approximations to the eigenvalue are compared in Table 4: one must carry the perturbation series to fourth order to obtain accuracy comparable to either spectral solution.

For many purposes, the $N=2$ Hermite approximations would be adequate. However, it should be noted that even if greater accuracy is needed, it is not necessary to explicitly solve a cubic or quartic equation for $\varepsilon$ or to call an elaborate sequence of algebraic eigenvalue library routines. The $N=2$ approximations are close enough to the true solution so that one can easily solve the algebraic eigenvalue problem for higher $N$ by using inverse iteration (Gourlay and Watson, 1973) instead, employing the $N=2$ solution as a first guess.

There is one caveat that must be applied to oceanographic basin calculations: Hermite series have the property of infinite order convergence only if the function decays exponentially at infinity. Moore (1968) analytically solved the problem of the reflection of a Kelvin wave from an eastern boundary, but because the reflected wave includes a component that propagates north and south along the boundary, the solution decays only algebraically in $y$ at a given longitude, so the Hermite series converges very slowly. Moore (1968) extracted useful information from the expansion only by introducing a clever summability method, and this trick is necessary in a number of other related oceanographic problems. With this caveat, however, Hermite spectral methods are highly efficient and accurate for equatorial wave problems. Although I have limited the discussion to one-dimensional eigenvalue problems here, Hermite series in latitude can also be used with finite differences in height to solve the two-dimensional equatorial wave equation when the method of multiple scales is not applicable.

8. The method of multiple scales, Part 2: Illustration and discussion

Now that all the various techniques for handling the effects of horizontal shear have been discussed (Sections 5–7) along with the formalism of the method of multiple scales itself (Section 2), it is appropriate to illustrate how the two are combined through an example. To do this, I choose the simplest: the Kelvin wave in linear shear, whose solution to zeroth order in $\sigma$ and first order in $\gamma_0$ is given by

$$\phi = A(\tau) \exp\left[\frac{k\beta y^2}{2\omega_0(\tau)}\right] \left[1 - k^2\gamma_0(\tau)y/\beta\right].$$

We define

$$\xi = \frac{y}{l_0(\tau)}$$

so that to lowest order $dy = l_0(\tau)d\xi$. Substituting these last two equations into (2.13), converting the averaging variable from $y$ to $\xi$, and dividing by $A(\tau)l_0(\tau)$ yields

$$\int_{-\infty}^{+\infty} d\xi e^{i\xi} \left\{ (1 - 2k^2\gamma_0(\xi)/\beta) \left( \frac{dA}{dt} + A\frac{df}{dt} \right) + 2l_0(\tau)A f \right\} = 0.$$  

Using the elementary integrals ($m$ is a non-negative integer)

$$\int_{-\infty}^{+\infty} d\xi e^{-i\xi^2} \xi^m = (\pi)^{1/2} (2m-1)!!/2^m,$$

Table 4. A comparison of spectral and perturbative approximations to the $n=1$ Rossby wave for $n=1, S=1$.

<table>
<thead>
<tr>
<th>Approximation</th>
<th>Eigenvalue</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact ($S=1$)</td>
<td>727</td>
<td>Difference = 4368</td>
<td></td>
</tr>
<tr>
<td>Exact ($S=0$)</td>
<td>5095</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First-order perturbation theory</td>
<td>-672</td>
<td>-1399</td>
<td>192%</td>
</tr>
<tr>
<td>Second-order perturbation theory</td>
<td>1480</td>
<td>753</td>
<td>103%</td>
</tr>
<tr>
<td>Two-term Hermite, $S_y=8.45$</td>
<td>531</td>
<td>-196</td>
<td>27%</td>
</tr>
<tr>
<td>Two-term Hermite, $S_y=6$</td>
<td>675</td>
<td>-52</td>
<td>7%</td>
</tr>
</tbody>
</table>
one finds
\[
\frac{dA}{d\tau} + Q(\tau)A = 0, \tag{8.5}
\]
where
\[
Q(\tau) = \frac{1}{2} \frac{d \ln f}{d\tau} + \frac{1}{2} \frac{d \ln \tilde{g}}{d\tau}, \tag{8.6}
\]
\[
A(\tau) = f(\tau)^{-1/2} \tilde{g}(\tau)^{-1}. \tag{8.7}
\]
The $f^{-1}$ coefficient in (8.7) is usual WKB amplitude factor for the vertical structure equation (see Nayfeh, 1973). The $\tilde{g}^{-1}$ term allows for the change of meridional scale with height as first found by Lindzen (1971).

Because the terms of the integrand in (8.4) which are multiplied by $\gamma_0$ are all antisymmetric with respect to the equator, Eq. (8.7) is in fact identical with Lindzen’s result. This could have been anticipated from the fact the first-order (in $\gamma_0$) solutions are all

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**Fig. 8a.** Comparison of the exact (solid) meridional wind for the $n=1$ Rossby wave with the two term Hermite spectral approximations for $S_I=8.45$ (dashed) and $S_I=6$ (dotted).

**Fig. 8b.** Comparison of the exact meridional wind for the $n=1$ Rossby wave for no shear (dashed) and observed ($S=1$) shear (solid).
of opposite symmetry with respect to the equator than their unperturbed counterparts; it is only at 
O(\gamma \sigma^2) that linear shear changes the vertical structure of the Kelvin wave. Despite its trivial conclusion, however, this example does show the great simplicity of the method of multiple scales, with or without latitudinal shear.

So far, the discussion has been confined to inviscid waves, but as observed by Lindzen (1971), the basic multiscaling formalism including modifications to handle horizontal shear is unchanged by dissipation in the form of linear friction and cooling. If the friction and cooling coefficients are equal, it is only necessary to add an imaginary part to the Dopplershifted frequency, as indicated in Appendix A. If the coefficients are unequal, it is straightforward to define two complex frequencies \( \omega_1 \) and \( \omega_2 \), which differ only in their imaginary parts, to replace \( \omega \) in the momentum and thermodynamic energy equations, respectively, and then to proceed through the derivation of the preceding sections keeping the distinction between \( \omega_1 \) and \( \omega_2 \). Holton (1975, p. 137) notes, however, that "although the inclusion of a linear friction is somewhat artificial ... calculations [specifically, comparison of Holton and Lindzen (1972) with Lindzen (1971)] indicate that for \( \alpha \ll |\omega| \) the perturbation fields computed ... will differ little from those computed using only linear cooling with a coefficient of 2\( \alpha \)." The exception to Holton's remark is the latitudinal profile of the net acceleration of the mean zonal flow by the dissipating wave, which, as shown by Andrews and McIntyre (1976a), is very sensitive to the ratio of the cooling and friction coefficients, especially for gravity and mixed Rossby-gravity waves (but not the Kelvin wave). Obviously, the formalism developed earlier will work when the imaginary part of the frequency as well as the real part is a function of latitude and pressure, so \( \gamma \) and \( \sigma \) dependent damping coefficients are no complication.

The equation of nondivergence of wave action flux used to find \( A(\tau) \) is no longer valid when damping is present, but Boyd (1976b) and Andrews and McIntyre (1976a) have derived independent generalizations which will serve the same purpose. The full expression [Eq. (3.9) of Boyd (1976b)] can be simplified by exploiting the fact that the explicit equatorial solutions show that some terms involve variables which are exactly out of phase in the absence of vertical shear and dissipation. Ignoring these terms, as is consistent with the zeroth-order multiple-scales approximation [see Andrews and McIntyre (1976a) for a careful discussion of this], and averaging in \( \gamma \),

\[
\begin{align*}
\frac{d}{ds} \left( e^{-s(\tilde{\phi}\omega/\omega)} \right) &= \left\{ \frac{1}{\omega} \left( \frac{\partial}{\partial y} \left( \tilde{\nu} \tilde{u} \right) - \tilde{v} \tilde{u} - \tilde{v} \tilde{u} - \alpha \phi^2 \right) / N^2 \right\},
\end{align*}
\]

where \( \nu \) is the coefficient of linear friction and \( \alpha \) the coefficient of linear cooling. Eq. (8.8) can be used to determine \( A(\tau) \) in the dissipative case in precisely the same way (2.10) was used in the inviscid case. [Note that the assumption of linear friction and cooling is made chiefly for convenience. In Boyd (1976b), the form of the momentum and thermal forcing are not specified, so multi-scaling via the wave action flux equation can be applied to very general forms of dissipation or external forcing.]

For most practical purposes, the lowest order multi-scaling solutions as described above are entirely adequate, but Andrews and McIntyre (1976b) calculate to first order in the vertical shear expansion parameter for two separate reasons. The first is to carry out a direct calculation of the net change in the mean zonal wind due to a damped wave. As first discussed by Charney and Drazin (1961) for quasi-geostrophic waves, the actual time rate of change of the mean zonal wind is not given simply by the sum of the divergences of the eddy momentum fluxes because the waves will also drive a mean meridional circulation. The "net" acceleration is the difference between the eddy momentum divergences and the Coriolis torque on the wave-driven mean north-south wind. As shown by Charney and Drazin for planetary waves and extended to equatorial waves by Holton (1974), the "net" acceleration is zero in the absence of damping, critical surfaces where the Doppler-shifted frequency is zero, and transience: the Charney-Drazin theorem.

For weak damping, this implies trouble because the Coriolis torque and eddy momentum divergences will still almost cancel so that the "net" acceleration is the small difference of large terms. Carrying the analysis further, Andrews and McIntyre (1976b) show that this implies that one must extend the method of multiple scales to first order in \( \alpha \) to have any hope of accurately calculating the "net" acceleration by taking the difference of the momentum divergences and the Coriolis torque on the mean north-south wind.

Fortunately, as independently discovered both by Boyd (1976b) and Andrews and McIntyre (1976a), there is another way to calculate the net acceleration. The whole point of Section 3 of Boyd (1976b) is to develop formulas for the net acceleration which involve only terms directly proportional to the friction or cooling coefficient. Simplifying (3.9) of Boyd (1976b) as before by ignoring products of variables that are

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*This is strictly true only for Kelvin and Rossby waves; for gravity and mixed Rossby-gravity waves of small \( \chi \), \( 2\alpha \) should be replaced by \( 4\alpha \).
out of phase when there is no shear or dissipation and specializing the momentum and thermal forcing to linear friction and cooling yields

$$U_t^{(net)} = \left(1/a_0\right) \left[ \frac{\partial}{\partial y} (\bar{v} \bar{u}) - \bar{v}^2 - \bar{v}_x^2 - \alpha \phi_e^2 / N^2 \right].$$  \hspace{1cm} (8.10)$$

For planetary waves in general, the left-hand side of (8.19) would have additional terms involving the mean meridional circulation, but Andrews and McIntyre (1976a), who give a similar but more general expression, show through a scaling argument that these terms can be safely ignored for equatorial waves. Note that although $u$ and $v$ are out of phase to lowest order, $\bar{u}$ and $\bar{v}$ are in phase, so (8.10) involves only averages of pairs of terms that are well correlated. To calculate this via (8.9), only the zeroth order multi-scaling solutions are necessary.

The second reason of Andrews and McIntyre (1976b) for going to first order in $\sigma$ is to perturbatively include the effects of weak horizontal shear. (They assume that the horizontal shear strength is of the same formal order as the vertical shear strength.) There is nothing mathematically incorrect with their procedure. However, the present work and its companion paper (Boyd, 1978a), show that first, it is unnecessary to calculate the effects of vertical shear to first order to compute those of horizontal shear to lowest order—the method outlined above includes horizontal shear at zeroth explicit order in $\sigma$—and second, that latitudinal shear effects are often so large (except for observed Kelvin waves) that nonperturbative techniques such as those developed in Sections 6 and 7 are needed to handle them. Thus, it is never necessary, either for computing the net acceleration or for incorporating horizontal shear, to carry the method of multiple scales beyond lowest order.

In the example I have used, the solution of the horizontal structure equation was given analytically, but there is no real complication if, as in Section 7, $P(\xi, \tau)$ and $\epsilon(\tau)$ are calculated level-by-level numerically for substitution into (2.5) and (2.6). It is unnecessary to find the eigenvalues and eigenfunctions from scratch at each level because those computed at the previous level will generally be good approximations to the eigenmodes of the current height. Thus instead of employing the relatively cumbersome and expensive QR algorithm, one can use the much simpler shifted inverse power iteration (often known simply as inverse iteration), which is discussed in Gourlay and Watson (1973). Whenever it is applicable, the method of multiple scales is orders of magnitude more efficient than purely numerical schemes, even when the horizontal and vertical structure equations of the two-scales approach must be solved numerically.

What does one do when the method of multiple scales is not applicable as appears to be the case for the troposphere and the tropical ocean? The answer is to attack the nonseparable wave equation directly by using second-order finite differences in height and Hermite functions in latitude, and then solving the resulting system of algebraic equations by modified Gaussian elimination as discussed in Lindzen and Kuo (1969) and Boyd (1978e). Previous numerical studies such as those of Lindzen (1970) and Holton (1970) imposed artificial sidewalls and used second-order finite differences in latitude, but this is extremely inefficient: Lindzen needed 60 grid points per level, whereas the results of the previous section suggest that one could obtain the same accuracy with ten or fewer Hermite polynomials; for some investigations, two or three might be quite adequate. For discussing observed atmospheric equatorial waves in Part II, a one-dimensional approach is sufficient, but it is important to realize that by using the Hermite methods of Section 7, one can develop an inexpensive and easily programmed two-dimensional numerical model to study many situations where the method of multiple scales is not enough.

9. Perturbation theory for higher modes

a. Background

As will be made clearer in Part II, the Kelvin wave perturbation theory of Section 5 is useful because 1) it is very simple and 2) it is quantitatively as well as qualitatively accurate for the Kelvin waves observed in the atmosphere, even for very strong shear. For the mixed Rossby-gravity and the $n=1$ Rossby modes observed, however, a perturbative approach to horizontal shear is both less accurate and much more complicated, which explains the necessity for the "gamma-plane" approximation and Hermite spectral methods of Sections 6 and 7. Nonetheless, it is straightforward to derive perturbation series in the horizontal shear parameters for these higher modes and because of this and the qualitative usefulness of the expansions, no treatment of latitudinal shear and equatorial waves would be complete without such series.

As before, the wind will be modeled by a parabola, the perturbation series will be calculated directly from the system of primitive equations rather than from a single derived equation for the meridional velocity alone, and the eigenfunctions at each order will be polynomials in $y$. However, to avoid being buried by the algebra, it is convenient to adopt the following nondimensional scaling in which both the unperturbed eigenvalue and $\beta$ are set equal to one, thus reducing the number of independent parameters by 2.

Let $\lambda$ be the nondimensional eigenvalue (Lamb’s parameter) for a given choice of $(k, \omega_0)$ when there is
no shear. Then set

$$Y = \varepsilon_0^3 \theta,$$

$$E = \varepsilon^* / \varepsilon^*_{*},$$

$$\Phi = \varepsilon_0^6 \Phi/(20a)^2,$$

$$U = \varepsilon_0^6 u/(20a),$$

$$V = -i\varepsilon_0^6 v/(20a),$$

$$\Delta = \delta_0 / \beta,$$

$$T = \gamma \varepsilon_0^6 t/(20a),$$

$$K = (\varepsilon_0^6)^{-1} k a,$$

$$\Omega = \varepsilon_0^6 \omega/(20a).$$

(9.1a) (9.1b) (9.1c) (9.1d) (9.1e) (9.1f) (9.1g) (9.1h) (9.1i)

Except for the left-hand side of (9.1f), \( \Omega \) is used to denote the angular frequency of the earth's rotation in (9.1), but in the rest of the section it will be used to represent the Doppler-shifted frequency of the wave as nondimensionalized according to (9.1i). This scaling is common in oceanography (e.g., Moore and Philander, 1977), but is rather unorthodox in meteorology, which is why it has not been used in the rest of the paper. It differs from the alternative nondimensionalization of Holton (1975) and Longuet-Higgins (1968), which uses the earth's radius as the length scale and the reciprocal of twice the earth's angular frequency as the time scale, only through the various factors of \( \varepsilon_0 \). In order that (9.2)–(9.4) can be written in terms of real quantities only, a factor \( i \) has been included in the nondimensionalization of \( v \) [Eq. (9.1e)]. To evaluate the solutions given below, one simply computes \( \varepsilon_0 \) from \( k \) and \( \omega_0 \) using the standard resting atmosphere formulas of Holton (1975) for the mode in question, and then from (9.1), the conversion to and from nondimensional variables is trivial. The primitive equations, after elimination of \( w \) using the separability relation, become

\[
(\Omega + K Y + K \Delta Y^2/2) U - (Y - \Gamma - \Delta Y) V + K \Phi = 0,
\]

(9.2)

\[
- (\Omega + K Y + K \Delta Y^2/2) V + Y U + \Phi Y = 0,
\]

(9.3)

\[
K U + V Y + (\Omega + K Y + K \Delta Y^2/2) E \Phi = 0.
\]

(9.4)

Solving this system for nonzero shear via perturbation theory is straightforward, so only an outline of the procedure will be given. The first step is to introduce a formal expansion parameter \( \xi \) to replace \( \Gamma \) and \( \Delta \) wherever they appear by \( \xi \Gamma \) and \( \xi \Delta \), respectively. Next, assume that \( U, V, \Phi \) and \( E \) can be expanded as Taylor series in \( \xi \) and substitute these expansions into (9.2)–(9.4). Matching powers of \( \xi \) then gives the order-by-order perturbation equations. After this has been done, \( \xi \) can be set equal to 1 since it is only a formal parameter. The three perturbation equations at \( n \)th order can then be reduced to a single, inhomogeneous equation for \( V^n \) of the form

\[
\left( \frac{d^2}{dY^2} + 2n + 1 - Y^2 \right) V^n = f^n(Y, E^n, K, \Omega, \Gamma, \Delta),
\]

(9.5)

where \( n \) is the mode number of the zeroth-order solution. Since the operator of the differential equation has a non-trivial homogeneous solution, i.e., \( V^h = H_n(Y) \exp(-Y^2/2) \), where \( H_n(Y) \) is the \( n \)th Hermite polynomial, Eq. (9.5) is solvable if and only if \( f^n \) is orthogonal to \( V^h \). This solubility condition is what determines the eigenvalue \( E \).

The great advantage of approximating the mean wind by a polynomial is that \( f^n, V^n, U^n \) and \( \Phi^n \) must be polynomials in \( Y \) [multiplied by \( \exp(-Y^2/2) \)] for all values of \( N \). [Using (9.8) below and the fact that the zeroth-order solutions are polynomials multiplied by a Gaussian, the reader can easily verify this by induction on \( N \).] Consequently, \( f^n \) can be rewritten as a truncated Hermite series by simple algebraic rearrangement, i.e.,

\[
f^n = \sum_{m=0}^{2N-n} a_m^N(E^n, K, \Omega, \Gamma, \Delta) H_m(Y) e^{-Y^2/2}.
\]

(9.6)

The eigenvalue relation is then simply

\[
a_m^N(E^n, K, \Omega, \Gamma, \Delta) = 0,
\]

(9.7)

which is always a linear equation in \( E^n \) and

\[
V^n = e^{-Y^2/2} \sum_{m=0}^{2N+n} a_m^N H_m(Y)
\]

(9.8)

It is trivial to derive expressions for \( U^n \) and \( \Phi^n \) in terms of \( V^n \) by eliminating one of the unknowns in the \( n \)th order perturbation equations, so knowing \( V^n \) and \( E^n \), it is easy to determine the other wave fields.

The principal difficulty in executing the algorithm above is that, even with the optimal scaling introduced in (9.1), one is still confronted with four parameters \( (k, \omega_0, \delta_0, \gamma_0) \) in dimensional form) plus the spatial coordinate \( Y \). Thus, the algebra is rather messy even at first order. In consequence, the perturbative solutions were computed on the University of Michigan Amdahl 470 using the algebraic manipulation language REDUCE2. Since relatively few institutions have adequate facilities for algebraic manipulation, a full discussion of this language and its application to equatorial waves is reserved for Boyd (1979).

First, REDUCE2 employs arithmetic of unlimited precision, in contrast to the fixed precision of FORTRAN, and expresses fractions as the ratio of two integers. Furthermore, the perturbative solutions here are polynomial functions of \( Y, \Gamma \) and \( \Delta \) and rational functions of \( K \) and \( \Omega \). Algebraic manipulation languages in general and REDUCE2 in particular can add, multiply, subtract and divide such functions.
very efficiently without approximation or truncation, except at the user’s option. Consequently, the results presented below for each perturbative order are exact. Second, since REDUCE2 offers automatic polynomial differentiation and power matching, it is easy (15 statements) to substitute the approximate solution up to given order directly into the original equation (9.2)–(9.4), factor the residual into powers of \( \Gamma \) and \( \Delta \), and check the computed perturbation series. With this final step, the machine computation is much less time-consuming than the equivalent hand calculation. Further, it is probably more reliable (especially at second order) since experience [see Boyd (1979) for an amusing example and references to several comprehensive reviews] has shown that errors in published results from lengthy hand calculations are depressingly common. The only caveat is that the greatest common divisor algorithm, which is important in simplifying results, is sometimes too costly to apply and other simplifying manipulations, which would be noticed and made in a hand calculation, may be missed by both program and programmer in the corresponding machine calculation. Thus, no guarantee is given that the results below are in the simplest possible form or even the simplest possible form that would be derived by paper and pencil, if the mind guiding the pencil were careful and alert. Barton and Fitch (1972) give a thorough review of physical applications of algebraic manipulation languages.

Although \( K \) and \( \Omega \) are treated as independent parameters for purposes of presenting results, they are in fact related through the eigenvalue formula, which can be written as

\[
\varphi = 1 - K \Omega \quad \text{(mixed Rossby-gravity,)} \quad (9.9a)
\]

\[
\varphi = (2n+1 + K^2) - K \quad (n \geq 1). \quad (9.9b)
\]

Eq. (9.9b) obviously has three solutions for a given value of \( K \), corresponding physically to an easterly Rossby wave plus easterly and westerly gravity waves. The perturbation series given below for \( n = 1 \) are valid for all three classes of waves and will give a different answer for each, depending upon which branch of the solutions of (9.9b) is used to relate \( \Omega \) to \( K \).

The eigenvalue formula (9.9) has been used directly to simplify expressions in \( K \) and \( \Omega \) so that they are at most linear in \( \omega \) for the mixed Rossby-gravity wave. This makes it easy to derive small \( K \) approximations to the full perturbative solutions which are given below.

b. First-order for the mixed Rossby-gravity wave

The zeroth-order eigenvalue formula is (9.9) above; the rest of the zeroth-order solution is

\[
V^0 = e^{-\frac{\Delta}{2}} \quad \text{(9.10)}
\]

\[
U^0 = e^{-\frac{\Delta}{2}} \left[ \frac{Y}{Y + \Lambda} \right],
\]

\[
\Phi^0 = e^{-\frac{\Delta}{2}} \left[ \frac{Y}{Y + \Lambda} \right].
\]

The first-order solutions can be written in the form

\[
E^1 = \frac{-\Delta (K^2 + 4 \Lambda \Omega + 2)}{2 \Delta}, \quad (9.12)
\]

\[
V^1 = e^{-\frac{\Delta}{2}} \left[ 4a_1 \Gamma Y + b_1 \Delta (1 - 2Y^2) \right], \quad (9.13)
\]

\[
a_1 = -K^2 - 5K^2 + 4K + \Lambda,
\]

\[
b_1 = 3K^2 - 6K - 6 \Omega,
\]

\[
\Phi^1 = e^{-\frac{\Delta}{2}} \left[ 4 \Lambda (C_1 + C_2 Y^2) + \Delta (d_1 Y + 2d_2 Y^3) \right], \quad (9.14)
\]

\[
C_1 = 4K^2 \Omega - 5K^2 + 2K + 1,
\]

\[
C_2 = -K^2 - 5K^2 + 5K^2 - 1,
\]

\[
d_1 = 6K^2 - 9K^2 + 2K + 2,
\]

\[
d_2 = 2K^2 + 5K^2 - 2,
\]

\[
U^1 = e^{-\frac{\Delta}{2}} \left[ 4 \Lambda (e_1 + e_2 Y) \right], \quad (9.15)
\]

\[
e_1 = -K^2 \Omega + 4K^2 - 6K^2 + K^2 + 5K^2 - 2,
\]

\[
e_2 = 4K^2 \Omega - 4K^2 + 10K^2 - 5K^2 - \Lambda K + 1,
\]

\[
f_1 = -6K^2 \Omega - 6K^2 - 17K^2 + 11K^2 + 16K^2 - 10,
\]

\[
f_2 = 2K^2 \Omega - 2K^2 + 7K^2 - 5K^2 - 2K^2 + 2,
\]

\[
Q = 8(K^2 - 2K^2 - \Omega).
\]

Several comments are in order. First, using \( x = K \Omega \) and small \( x \) form of (9.9a), \( \Omega = 1 - K^2 / 2 \), it is easy to show that (9.12)–(9.15) reduce to the first-order-in-\( x \) solutions which are given in dimensional form in the last subsection of Section 6.

Second, when \( K = \Omega = \sqrt{2}/2 \), which corresponds to \( s = k^2 / \alpha^2 \) in dimensional form, the denominators and each term in the numerators of \( U^1 \) and \( \Phi^1 \) are zero so that these expressions must be evaluated by L'Hopital's rule. These values of \( K \) and \( \Omega \) are the only pair of values consistent with the zeroth-order eigenvalue relation for which the denominators of \( V^1 \) and \( \Phi^1 \) vanish, so the first order solutions are well-behaved for all values of the parameters.

To demonstrate this and give some feeling for convergence of the series for the observed stratospheric mixed Rossby wave, the general first-order solution for \( K = \Omega = \sqrt{2}/2 \) is given below:

\[
E^1 = -4.5 \Delta
\]

\[
V^1 = \left[ \frac{13}{16} \Delta (2Y^2 - 1) + \frac{\Gamma \Delta}{Y} \right] e^{-\frac{\Delta}{2}}, \quad (9.16)
\]

\[
\Phi^1 = \left[ \frac{16}{16} \Delta (2Y^2 - 13Y) + \frac{\Gamma (Y^2 - 3)}{2} \right] e^{-\frac{\Delta}{2} \sqrt{2}},
\]

\[
U^1 = \left[ -\frac{16}{16} (2Y^2 - 45Y) + \frac{\Gamma (Y^2 - 1)}{2} \right] e^{-\frac{\Delta}{2} \sqrt{2}}.
\]

Comparing these corrections with those of the "away from the critical level" case of Part II, one finds
errors of roughly 10% in the first-order eigenvalue for the values $\Delta$ and $\Gamma$ used in Section 4 of that paper (−0.375 and 0.707, respectively). It is well known that errors in the eigenfunctions are usually larger than to those in the eigenvalue but comparable in magnitude, but no detailed error comparisons will be made here or in the next subsection because the exact solutions for the chosen parameters are illustrated in Part II. The second-order solution for the $n=0$ mode was also computed, but these results will not be presented here because (i) they are rather long, significantly longer than the first-order solutions for the $n=1$ mode given below; and (ii) they have second- or even third-order poles when $K=\Omega=\sqrt{2}/2$ (and corresponding zeros), which makes their evaluation in this parametric neighborhood rather tricky.

c. First-order for $n=1$ modes

$K$ and $\Omega$ are related through (9.9b), and the zeroth-order solutions are

$$V^0 = 2Y e^{-\gamma_{1/2}}$$
$$U^0 = \frac{2[(K-\Omega)Y^2-K]e^{-\gamma_{1/2}}}{(K^2-\Omega^2)}$$
$$\varphi^0 = \frac{2[(K-\Omega)Y^2+\Omega]e^{-\gamma_{1/2}}}{(K^2-\Omega^2)}$$

(9.17)

The first-order solutions are

$$E^1 = -\frac{(3K^3-6K\Omega^2+13K^2-24K^3\Omega+3K\Omega^4+9K^2\Omega^2+6K^3\Omega+6\Omega^3)}{2\Omega(3K^2-2K\Omega+3\Omega)}$$

(9.18)

$$V^1 = e^{-\gamma_{1/2}}[4\Gamma(A_0+A_1Y^2+\Delta B_0(3Y-2Y^2))/Q_s,$n
$$A_0 = 3K^2-2K\Omega-3K^3\Omega+9K^3+4K^3\Omega^2-21K^4\Omega-3K^4\Omega^2+28K^5\Omega^2-2K^5\Omega-18K^5\Omega^2+3K^6\Omega+7K^6\Omega^2+3\Omega^2),$n
$$A_1 = -A_1-6K^2\Omega^2+4K^2\Omega^2-6K^3\Omega^2,$n
$$B_0 = -K^3+3K^4\Omega-2K^5+2K^5\Omega^2-3K^4\Omega^2-6K^4\Omega^3+25K^5\Omega^4-K^5\Omega^3-31K^5\Omega^3+3K^6\Omega+13K^6\Omega^2+6\Omega^3,$n
$$Q_s = 4\Omega(3K^2-2K\Omega+2K^2\Omega-3\Omega^2),$n
$$\varphi^1 = e^{-\gamma_{1/2}}[4\Gamma(C_1Y+C_1Y^3+\Delta (D_0+D_2Y^2+2D_1Y^4)]/Q_s,$n
$$C_1 = 3K^3+11K^3\Omega+3K^4\Omega^2+9K^3+13K^4\Omega^2-48K^4\Omega^2-15K^4\Omega^3+91K^4\Omega^3+7K^5\Omega^3-102K^5\Omega^3+9K^6\Omega^3+61K^6\Omega^3-9K^7\Omega^3-18K^8\Omega^3-9\Omega^4,$n
$$C_2 = -3K^3+5K^3\Omega+K^4\Omega^2-9K^3+7K^4\Omega^2+24K^5\Omega^2+7\Omega^3-39K^4\Omega^3-3K^4\Omega^3+36K^4\Omega^3-5K^5\Omega^4-19K^5\Omega^4+3K^4\Omega^4+4K^4\Omega^4+3\Omega^3,$n
$$D_0 = -\Omega(3K^3+3K^4\Omega^2+6K^3-6K^4\Omega^2+9K^4\Omega^2-6K^5\Omega^2-23K^5\Omega^2-3K^5\Omega^2-3K^6\Omega^2-3K^6\Omega^3+3\Omega^3,$n
$$D_2 = -3K^3+6K^4\Omega^2-6K^4-9K^4\Omega^2-3K^5\Omega^2-12K^5\Omega^2+58K^5\Omega^2+27K^6\Omega^2+146K^6\Omega^2+6K^7\Omega^2+162K^7\Omega^2+15K^7\Omega^2-51K^8\Omega^3-3\Omega^4,$n
$$D_4 = K\Omega^4-4K^3\Omega^6+2K^3\Omega^6+5K^4\Omega^6+8K^4\Omega^6-12K^5\Omega^6+12K^5\Omega^6+36K^6\Omega^6-4K^6\Omega^6-28K^6\Omega^6+3K^7\Omega^6+3K^7\Omega^6-6\Omega^8,$n
$$Q_s = 4\Omega(3K^2-2K\Omega+2K^2\Omega-3\Omega^2)(K^2-\Omega^2),$n
$$U^1 = -e^{-\gamma_{1/2}}[4\Gamma(E_1Y+E_2Y^3+\Delta (F_0+F_2Y^2+2F_1Y^4)]/Q_s,$n
$$E_1 = 9K^3\Omega^2-9\Omega^2-16K^4\Omega^3+27K^4\Omega^3-27K^4\Omega^3-64K^5\Omega^3-84K^5\Omega^3-14K^5\Omega^3+24K^6\Omega^3-72K^6\Omega^3+92K^6\Omega^3+48K^6\Omega^3+3\Omega^8-6K^7\Omega^8+9\Omega^8,$n
$$E_2 = -3K^3\Omega^4+5K^4\Omega^4-9K^4\Omega^4+12K^5\Omega^4+24K^5\Omega^4+6K^6\Omega^4+30K^5\Omega^4+6K^6\Omega^4+12K^6\Omega^4+20K^6\Omega^4+4K^6\Omega^4-32K^7\Omega^6+5K^7\Omega^6+22K^7\Omega^6-3K^8\Omega^6-4K^8\Omega^6-3\Omega^8,$n
$$F_0 = K(3K^3+3K^4\Omega^2+6K^4\Omega^2-9K^4\Omega^2-9K^4\Omega^2-29K^5\Omega^2+9K^5\Omega^2+12K^5\Omega^2+9K^6\Omega^2+20K^6\Omega^2-3K^7\Omega^2+9K^7\Omega^2+3K^7\Omega^2+6\Omega^8,$n
$$F_1 = 3K^4\Omega^2+18K^4\Omega^2-18K^4\Omega^2-18K^4\Omega^2-31K^4\Omega^2-54K^4\Omega^2+104K^5\Omega^2+36K^4\Omega^2-253K^5\Omega^2+54K^6\Omega^2+88K^7\Omega^2-30K^8\Omega^2-229K^8\Omega^2-18K^8\Omega^2+21K^8\Omega^2+7K^8\Omega^2+42\Omega^8,$n
$$F_2 = K\Omega^4-4K^4\Omega^6+2K^4\Omega^6+5K^5\Omega^6-14K^5\Omega^6-6K^6\Omega^6+41K^6\Omega^6+12K^6\Omega^6-16K^7\Omega^6+8K^7\Omega^6.-35K^8\Omega^6+4K^8\Omega^6+34K^8\Omega^6-3K^9\Omega^6-7\Omega^6-6\Omega^8,$n
$$Q_s = 4\Omega(3K^2-2K\Omega-6K^2\Omega^2-6K^2\Omega^2+6K^2\Omega^2+2K^2\Omega^2-3\Omega^2).$$

(9.19)

(9.20)

(9.21)
10. Conclusions

Lindzen (1971, 1972) has already discussed the accuracy of the multi-scaling method for equatorial waves in vertical shear through comparison both with numerical solutions (Lindzen, 1970) and with observations. I have shown here and in Part II that the analytic, horizontal shear approximations of earlier sections give good to moderate accuracy for the observed Kelvin and mixed Rossby gravity waves. Thus, there is strong evidence that the generalized multi-scaling method of Sections 2 and 8 is accurate for all stratospheric equatorial waves.

For tropospheric and oceanic waves where the scales of the vertical shear and vertical wavelength are roughly the same, the multi-scaling approximation is probably not accurate. However, solutions to the one-dimensional, pure meridional shear model of earlier sections still give qualitative insight into the role of horizontal shear for these waves. Further, I have shown how effective and efficient the Hermite spectral method can be for such waves when numerical calculations are necessary.

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APPENDIX A

List of Symbols

\( a \) radius of the earth
\( A(r) \) amplitude factor for multiscaling solutions
\( c \) wave phase speed
\( E \) \( \epsilon/\omega_0 \) [Eq. (9.1)]
\( f(r) \) \( \partial z/\partial s \) [see (8.2) and (8.4)]
\( f'' \) inhomogeneous term for perturbation theory [see (9.5)]
\( g \) gravitational constant \((10 \text{ m s}^{-2})\)
\( h \) equivalent depth (explained in Section 3)
\( H_n(\xi) \) the \( n \)th Hermite polynomial, \( n = 0, 1, 2 \ldots \)

\( k \) dimensional zonal wavenumber \((= s/a)\)
\( K \) nondimensional zonal wavenumber \([\text{Eq. (9.1)}]\)
\( l_0(r) \) \( y/\xi \)
\( L \) wave meridional length scale
\( M \) linear operator defined by (3.11)
\( n \) meridional mode number \((= 0, 1, 2, \ldots)\) in beta-plane terminology
\( N \) perturbation order (Section 9)
\( N^2 \) static stability in log-pressure coordinates
\( \{ = R[(2/7)T(z) + dT/\partial z] \}, \) where \( R \) is the gas constant for air and \( T \) the zonally averaged temperature
\( Q \) linear operator defined by (7.2)
\( s \) integral zonal wavenumber \((= 0, 1, 2, \ldots)\)
\( S \) tropospheric shear parameter (Section 6)
\( w, \psi \) wave zonal and meridional winds respectively
\( U, V \) mean zonal wind
\( w \) vertical "velocity" in log-pressure coordinates
\( [ = Dz/Dt ] \)
\( \gamma(y) \) nondimensional north–south coordinate \((= 0)\)
\( \alpha \) log-pressure vertical coordinate \([= -\ln p/p_0], \) where \( p \) is pressure and \( p_0 \) a reference pressure
\( \beta \) "fast" vertical variable [see also (8.2)]
\( \gamma(y) \) coefficient of linear cooling
\( \gamma \) \( y \) derivative of the Coriolis parameter, evaluated at the equator \([= 2\Omega/a] \)
\( \delta(y) \) \( y \) derivative of the mean wind
\( \delta_0 \) nondimensional \( \gamma_0 \) \((9.1)\)
\( \Lambda \) equatorial value of \( \gamma_0 \)
\( \epsilon \) \( y \) derivative of \( \gamma \)
\( \epsilon_0 \) equatorial value of \( \delta \)
\( \Delta \) \( \beta_0^2 - 2\beta \gamma - \omega^2 \) (Section 3); nondimensional \( \delta_0 \)
\( \xi \) perturbation parameter for meridional shear (Section 5)

\( \lambda \) coupling constant for anharmonic oscillator (Section 5)
\( \lambda \) nondimensional parameter \((= k \omega_0 / \beta)\)
\( \chi \) dimensional eigenvalue of meridional structure equation
\( \epsilon^* \) nondimensional eigenvalue \([= \epsilon(4\Omega p a^2)]\)
\( \sigma \) dummy parameter \([ = 1 \text{ (gravity waves)} \text{ or 3 (Rossby waves)} \) (Section 3)
\( \sigma \) vertical shear parameter [see also (8.1), Section 8]
\( \theta \) latitude
dummy parameter \([ = 0 \text{ (gravity waves)} \text{ or 1 (Rossby waves)} \) (Section 3)
\( \tau \) "slow" vertical variable [see also Eq. (8.1)] (Section 8)
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ω(y) Doppler-shifted frequency \( [=i\kappa(\bar{U} - c)] \), if \( \gamma \)
and \( a \) are equal where \( \bar{U}(y) \) is the mean
zonal wind.

\( \omega_0 \) equatorial value of \( \omega \).

\( \Omega \) nondimensional \( \omega_0 \) (Section 9), angular fre-
quency of the earth (elsewhere).

Superscripts on \( u, v, \phi \) and \( e \) denote the perturbation
order for the Kelvin wave [see also Eqs.
(5.9)–(5.11)].

APPENDIX B

Reduction to the Harmonic Oscillator Equation

As described in Carrier and Pearson (1968), any
differential equation of the form

\[
v_{yy} + a_1(y)v_y + a_0(y)v = 0
\]

(B1)
can be transformed into

\[
w_{yy} + r(y)w = 0,
\]

(B2)

where

\[
w = v(y) \exp \left[ \int_B a_1(t)/2dt \right],
\]

(B3)

\[
r = a_0(y) - a_1(y)^2/4 - \frac{1}{2} \frac{d}{dy} \left[ a_1(y) \right].
\]

(B4)

When (B1) has the special form of (3-1), i.e., when
\( a_1(y) \) is a linear polynomial and \( a_0(y) \) is a quadratic
polynomial, then \( r(y) \) and the argument of the exponen-
tial in (B3) will both be quadratic polynomials.

Writing the transformed equation (B2) for this special case as

\[
v_{yy} + (A + By - Cy^2)v = 0,
\]

(B5)
one can change it into Schroedinger’s equation for the
harmonic oscillator by making the change of variable

\[
\xi = -B/(2C^2) + Cy
\]

(B6)

substituting this in (B5), and multiplying through by
\( C^{-1} \) to obtain

\[
v_{yy} + \left[ E - \xi^2 \right]v = 0,
\]

(B7)

\[
E = AC^{-1} + BC^{-4}/4.
\]

(B8)

The characteristic solutions of (B7), the harmonic oscillator
equation, are discussed in most elementary quantum
mechanics texts, for example, Merzbacher (1970), and also (with more
identities but fewer explanations) in Abramowitz and Stegun (1965). The
eigencondition is

\[
E = 2n + 1
\]

(B9)

and the eigenfunctions are

\[
v = e^{-\xi^2/2} H_n(\xi),
\]

(B10)

where \( H_n(\xi) \) is the \( n \)th Hermite polynomial
\((n = 0, 1, 2, \ldots)\).

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