Variational Formulation of Budyko-Sellers Climate Models

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(Manuscript received 16 August 1978, in final form 10 October 1978)

ABSTRACT

A class of simple climate models including those of the Budyko-Sellers type are formulated from a variational principle. A functional is constructed for the zonally averaged mean annual temperature field such that extrema of the functional occur when the climate satisfies the usual energy-balance equation. Local minima of the functional correspond to stable solutions while saddle points correspond to unstable solutions. The technique can be used to construct approximate solutions from trial functions and to carry out finite-amplitude stability analyses. A spectral example is given in explicit detail.

1. Introduction

Though many properties of the simple energy-balance climate models (Budyko, 1968, 1969; Sellers, 1969) are now well understood, there are some interesting problems which remain unsolved. This study attempts to add to the knowledge of these models by analytical techniques and can therefore be compared to the earlier studies by Held and Suarez (1974), North (1975a,b), Ghil (1976), Su and Hsieh (1976), Drazin and Griffel (1977) and Cahalan and North (1979). These works have provided analytical solutions and linear stability theorems for various versions of the Budyko–Sellers models. It is, of course, hoped that some of the properties discovered to hold for these models can be generalized to the more comprehensive members of the model hierarchy [for a discussion of this point the review by Schneider and Dickinson (1974) is recommended].

Paltridge (1975) has recently suggested that a variational principle might hold for global climatological systems. His reasoning was based upon the theory of non-equilibrium thermodynamics proposed by Prigogine (1968). While we do not take this approach it provided some of the motivation that caused us to undertake this study. As it turns out our approach is purely mathematical and seems to bear almost no connection with rates of change of the usual thermodynamical functions.

The purpose of this paper is to present a functional of the temperature field which takes on an extreme value if the temperature field satisfies the energy-balance equation for a Budyko–Sellers model. The extremum has the property that it is a relative minimum for stable steady-state solutions, and a saddle point for unstable solutions. Since it can be shown that the time rate of change of the functional is always negative or vanishing if the temperature field obeys a time-dependent energy-balance equation, one can establish a finite-amplitude stability theory as oppposed to the linear stability analyses previously given. Such a variational approach to the finite-amplitude stability theory of climate models has also been suggested by Ghil (1976), who even provided a suitable variational principle for the Sellers model. Our treatment extends Ghil's and provides a more detailed study of the extrema and a rigorous basis for some of his conjectures.

In addition it can be shown that the time rate of change of the temperature field components (Fourier–Legendre amplitudes) are related to certain correspond-

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0022-4928/79/020255-05$05.25

ing components of the gradient of the functional so that the functional serves as a kind of generalized potential. This property could be useful in various applications which are discussed in the conclusion.

2. Variational principle

Consider a climate model defined by the heat-balance equation

$$CT_t-[D(1-x^2)T_x]_x+I(T)=QS(x)a(T), \quad (1)$$

with the boundary conditions $(1-x^2)^{\frac{1}{2}}T_x=0$ at $x=\pm 1$; here t is the time, x the sine of the latitude, $T(x,t)$, the sea level mean annual zonally averaged temperature, D the horizontal thermal conductivity which may be a function of x, $I(T)$ the outgoing infrared radiation, $4Q$ the solar constant, $S(x)$ the normalized distribution of solar input and $a(T)$ the coalbedo. The model is nonlinear because of the possibly complicated prescribed T dependence of $a(T)$ and $I(T)$. The model reduces to the model studied previously by North (1975a,b) if the function $I(T)$ is linear as suggested by Budyko (1969) and if $a(T)$ is a step function discontinuous at the ice-cap edge, also first suggested by Budyko (1969).

We seek a functional of T such that extrema of this functional occur when T satisfies the steady-state version of Eq. (1). We have found such a functional, and it is given by

$$F[T]=\int dx[\tfrac{1}{2}D(1-x^2)T_x^2+R(T)-QSA(T)], \quad (2)$$

where

$$R(T)=\int^{T} I(T)dT, \quad (3)$$

$$A(T)=\int^{T} a(T)dT. \quad (4)$$

To demonstrate that $F[T_x,T]$ satisfies our requirement, we consider a small arbitrary variation $T(x)\rightarrow T(x)+\delta T(x)$, leading to

$$\delta F=\int dx\{-[D(1-x^2)T_x]_x+I(T) -QSA(T)\}\delta T(x)=0, \quad (5)$$

which can only hold for arbitrary $\delta T(x)$ if the integrand vanishes which condition leads to Eq. (1) with $T_t=0$. In deriving Eq. (5), use was made of the relation $\delta T_x=(\delta T)_x$ and an integration by parts was performed on the first term; end-point terms vanish because of the boundary conditions. These manipulations are standard and can be found in any textbook covering Lagrangian mechanics or variational calculus.

An immediate application of this formulation is a method of obtaining approximate solutions to the steady-state equation. One inserts a *trial* function $T_\alpha(x)$ with adjustable parameters into F then sets the derivatives of F with respect to these parameters equal to zero and solves the resulting algebraic equations for the parameters. For example, one might try $T_\alpha(x)=b_0+b_2P_2(x)$ and solve for b_0 and b_2.

We now examine the behavior of F in the neighborhood of an extremum. Consider a small departure from a particular extremum

$$T(x)=T_e(x)+\Psi(x), \quad (6)$$

where $T_e(x)$ satisfies the steady-state version of Eq. (1), and $\Psi(x)$ is taken to be small. Then, for example, $R(T)$ can be expanded in a Taylor series

$$R(T)=R(T_e)+I(T_e)\Psi(x)+\tfrac{1}{2}I'(T_e)\Psi^2(x)+\ldots \quad (7)$$

Inserting Eq. (6) into Eq. (2) and making use of expansions like (7) for $R(T)$ and $A(T)$, we obtain

$$F[T]=F[T_e]+\int_{-1}^{1} dx\{-[D(1-x^2)T_{ex}]_x+I(T_e) -QSa(T_e)\}\Psi(x)+\int_{-1}^{1} dx\tfrac{1}{2}\Psi(x)\left[-\frac{d}{dx}D(1-x^2)\frac{d}{dx} +I'(T_e)-QSa'(T_e)\right]\Psi(x), \quad (8)$$

where we have only retained terms up to the second order in $\Psi(x)$. The first integral in Eq. (8) vanishes from the definition of $T_e(x)$. To study the second integral we consider the eigenvalue problem

$$-[D(1-x^2)\Psi_x^k(x)]_x+(I'-QSa')\Psi^k(x)=\lambda_k\Psi^k(x). \quad (9)$$

Eq. (9), along with the boundary condition $(1-x^2)^{\frac{1}{2}}\Psi_x^k=0$ at $x=\pm 1$, form a Sturm–Liouville system. The $\Psi_k(x)$ then can be made to form an orthonormal basis set and the λ_k are bounded from below, etc. (Courant and Hilbert, 1953).

Assuming the $\Psi^k(x)$ form a complete basis set, we may expand

$$\Psi(x)=\sum_k C_k\Psi^k(x) \quad (10)$$

and insert this into (8) with the result

$$F[T]=F[T_e]+\tfrac{1}{2}\sum_k C_k^2\lambda_k. \quad (11)$$

It is convenient to think of the C_k as coordinates in an infinite-dimensional space. If all of the coefficients λ_k are positive the "surface" defined by Eq. (11) is concave upward [i.e., the sum in Eq. (11) is positive definite]. However, if one of the coefficients λ_k is negative, the surface is a saddle point. The eigenvalues of Eq. (9) then determine the geometry of the surface near an extremum of F. We next turn to the physical significance of the λ_k.

We now consider the stability of the steady-state solution $T_e(x)$. By stability we mean the behavior of solutions to Eq. (1) for infinitesimal departures from a particular steady state solution $T_e(x)$. To this end we set

$$T(x,t)=T_e(x)+\Psi(x)e^{-\lambda t}. \quad (12)$$

Inserting Eq. (12) into Eq. (1) and dropping terms quadratic and higher in $\Psi(x)$, we obtain

$$-[D(1-x^2)\Psi_x(x)]_x+[I'(T_e)-QSa'(T_e)]\Psi(x) = \lambda\Psi(x), \quad (13)$$

which is identical to Eq. (9) except for the index k. Eq. (13) is an eigenvalue problem the signs of whose eigenvalues tell us the stability of the system. If all the eigenvalues are positive, the solution is stable. If one or more of the eigenvalues are negative the solution is unstable. Clearly, from this discussion stable steady-state solutions lead to local minima in the functional $F(T)$ and unstable solutions correspond to saddle points.

It is interesting to examine the time dependence of $F(T)$ along system trajectories [$T(x,t)$ satisfies Eq. (1)]. Taking the time derivative of Eq. (2) leads to

$$\frac{d}{dt}J(t)=\int_{-1}^{1} dxT_t\{-[D(1-x^2)T_x]_x+I(T)-QSa(T)\}$$

$$=-\int_{-1}^{1} T_t^2dx\leqslant 0, \quad (14)$$

where $J(t)=F[T(x,t)]$ with $T(x,t)$ satisfying (1). If the system is out of equilibrium the function $J(t)$ decreases until the system comes to rest at an extremum. Stability occurs if the extremum is a local minimum. If the extremum is a saddle point, only an infinitesimal perturbation in an unstable direction [along an eigenvector $\Psi^k(x)$ with negative eigenvalue] leads to further decreases in $J(t)$ until another extremum is encountered. The functional $F[T]$ is known in the mathematical literature as a Lyapunov functional (Hahn, 1963).

We wish to point out one further generalization of the functional (2). Stone (1973) has suggested that the diffusion coefficient used in simple models be proportional to T_x. Since this class of models is then of some interest we consider the diffusion coefficient an arbitrarily given function of T_x, $D(T_x)$. Without going through the details of the manipulations involved we simply state that the first term in the brackets of the integrand of Eq. (2) is replaced by

$$(1-x^2)E(T_x), \quad (15)$$

where

$$E(T_x)=\int^{T_x} D(T_x)T_xdT_x. \quad (16)$$

The new functional obtained from this interchange has all of the properties of the one just discussed.

3. Spectral example

In this section we present an example of the use of the functional [Eq. (2)] for the special case of an ice-cap model with discontinuous albedo at the ice-cap edge and a linear infrared law (Budyko, 1969). In this case analytical solutions can be given for the equilibrium solution (North, 1975a,b), so that we will be able to see the structure of the functional explicitly. In this case it will also be possible to relate the functional to a kind of generalized potential.

The specialization of Eq. (1) is then that

$$I(T)=A+BT \quad (17)$$

and

$$a(T)=a_0\theta(T_s-T)+a_1\theta(T-T_s), \quad (18)$$

where $\theta(Z)$ is the unit stepfunction, zero for $Z<0$, unity for $Z\geqslant 0$; A and B are empirical constants, T_s is the critical temperature below which ice forms [usually taken to be $-10°C$ (Budyko, 1969)], a_0 and a_1 are the absorption fractions over ice-covered and ice-free areas, respectively, and we shall further specify that D be a constant independent of latitude.

It is convenient to expand the temperature field into even indexed Legendre polynomials, i.e.,

$$T(x)=\sum_{\text{even}} T_nP_n(x), \quad (19)$$

where we are restricting ourselves for simplicity to north–south symmetric solutions. Inserting Eq. (19) into Eq. (2) and making use of the orthogonality of the P_n, we have

$$F(T_0, T_2, \ldots)=AT_0+\sum_n \frac{L_nT_n^2}{2(2n+1)} - M(T_0, T_2, \ldots), \quad (20)$$

where

$$\left.\begin{aligned} L_n&=[n(n+1)D+B] \\ M(T_0, T_2, \ldots)&=Q\int S(x)(T-T_s)[a_0\theta(T_s-T) \\ &\quad +a_1\theta(T-T_s)]dx \end{aligned}\right\}, \quad (21)$$

and in Eq. (21), it is implied that Eq. (19) is to be substituted for $T(x)$.

The extremum of Eq. (20) occurs for the values of $T_0, T_2, \ldots$ for which the $\partial F/\partial T_n$ vanish simultaneously. This condition reads

$$A\delta_{n0}+L_nT_n=(2n+1)Q\int_0^1 dxS(x)P_n(x)[a_0\theta(T_s-T) +a_1\theta(T-T_s)]\equiv QH_n(x_s), \quad (22)$$

where x_s is the value of x at the ice-cap edge, i.e., $T(x_s)=T_s$. Note that for a given x_s the $H_n(x_s)$ can be

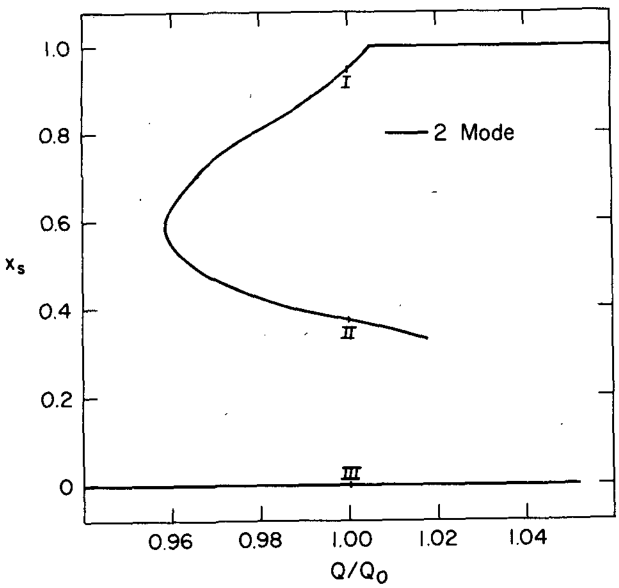

FIG. 1. Sine of latitude of ice-cap edge x_s versus solar constant Q normalized to its present value Q_0 for the two-mode approximation to a simple diffusive transport model taken from North (1975b). The present climate corresponds to I, the intermediate solution to II and the ice-covered earth to III.

computed directly. Now division through (22) by L_n, multiplication by $P_n(x_s)$ and summation leads to

$$A+BT_s=Q\sum_n H_n(x_s)P_n(x_s)/L_n, \tag{23}$$

which may be solved for $Q=Q(x_s)$, and constitutes a solution to the steady-state problem (North, 1975b). Fig. 1 shows a typical solution truncated at $n=2$. The present climate corresponds to the point I on the curve. Other solutions are represented by II and III.

It follows that the spectral representation (19) truncated at some $n=N$ does minimize the functional when the T_n satisfy (22). Furthermore, in this case it can be shown directly that the time-dependent equation (1) can be written in spectral form as

$$C\dot{T}_n=-\frac{\partial F}{\partial T_n}. \tag{24}$$

Therefore, the function $F(T_0,T_2,\ldots)$ is a generalized potential; the time derivative of component n is just proportional to the corresponding component of the gradient of $F(T_0,T_2,\ldots)$.

Fig. 2 shows a contour plot of $F(T_0,T_2)$ with $Q/Q_0=1$ corresponding to the two mode truncation depicted in Fig. 1. The minima corresponding to the present climate and the ice-covered earth climate are denoted by I and III, respectively, while the saddle point corresponds to the unstable intermediate solution II.

The time dependence of a climate perturbed far from equilibrium may now be followed with the help of Fig. 2 and Eq. (24). This analysis then constitutes a finite-amplitude stability theory for the models.

4. Conclusion

We suspect that the variational principle presented in this paper can be generalized to more comprehensive models, for example, a matrix formulation should work for multiple-level models. Such a principle can be useful in extracting simple few parameter approximations to the solutions of the energy-balance equation. In addition, a qualitative study of the finite-amplitude stability problem is simply a matter of charting the functional as in Fig. 2.

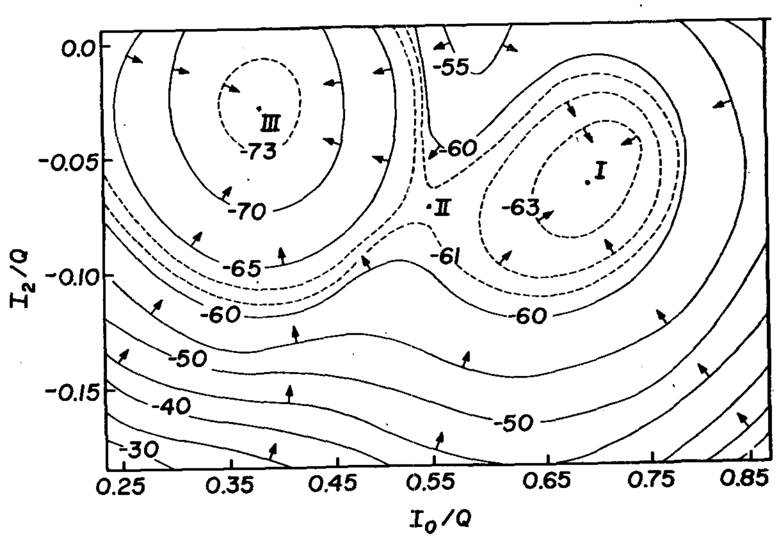

FIG. 2. Contour plot of the two-mode approximation to the functional $F(T)$ [Eq. (2) or Eq. (20)] in the I_0, I_2 plane, where $I_0=A+BT_0$ and $I_2=BT_2$. Values of parameters are the same as in Fig. 1 and are taken from North (1975b).

Perhaps the most important application of the theory is in the statistical theory of climate, where the functional can be used as a generalized potential. It is likely, for example, that a theory of fluctuations (Hasselmann, 1977) can be developed by analogy with the motion of a harmonically bound Brownian particle.

Acknowledgments. The authors would like to express their thanks to the 1976 Geophysical Fluid Dynamics Summer Program at the Woods Hole Oceanographic Institution for its hospitality during the course of completion of this work. The 1976 GFD program was supported in part by grants from the Office of Naval Research and the National Science Foundation. The authors further wish to acknowledge helpful discussions with Prof. A. Ingersoll and Prof. C. Rooth. One of us (G. R. N.) was supported in part by a research grant from the Office for Climate Dynamics of the National Science Foundation and in part by funds from Goddard Laboratory for Atmospheric Sciences, National Aeronautics and Space Administration.

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