# Climate Sensitivity from Fluctuation Dissipation: Some Simple Model Tests

THOMAS L. BELL

*Laboratory for Atmospheric Sciences, Goddard Space Flight Center, Greenbelt, MD 20771*

(Manuscript received 23 January 1980, in final form 28 April 1980)

ABSTRACT

Leith has suggested that climatic response to change in external forcing parameters of the climate system may be estimated via the fluctuation-dissipation theorem (FDT). The method, which uses the natural fluctuations of the atmosphere to probe its dynamics, is tested here using a twenty-variable truncation model of the barotropic vorticity equation. Dissipative terms are added to the equations, so that the model is pushed away from the region where it is expected to satisfy the FDT. It is found that, even though the FDT is no longer satisfied in every detail, the FDT continues to provide an excellent estimate of the climatic sensitivity of the model.



## 1. Introduction

Leith (1975, 1978) has proposed a method for determining climatic sensitivity to influences external to the atmospheric system from observations of the natural variability of the atmosphere. As a helpful though not strictly parallel example of the idea, consider a mass oscillating suspended at the end of a spring. Let its "climate" be its average height above the ground. Its "climatic sensitivity" is a measure of how far the midpoint of the oscillations would sink if, say, we were to increase the gravitational force acting on the mass. This depends on how elastic the spring is. But the elasticity of the spring can be determined by observing the period of oscillation of the mass, without interfering with its motion experimentally. We are therefore able to infer the "climatic sensitivity" of the mass and spring system by observing its undisturbed oscillations.

It will be convenient for our discussion to introduce here some of Leith's (1975) notation. Let us denote by $\{x_\alpha(t)\}$ the collection of dynamical variables describing the atmosphere. The label $\alpha$ indicates both the nature of the variable (whether $x_\alpha$ is a velocity, temperature, moisture, etc.) and its spatial position (grid-point). The atmospheric equations may be written

$$\frac{dx_\alpha}{dt} = Q_\alpha[x] + f_\alpha, \tag{1.1}$$

where $Q_\alpha$ contains linear and nonlinear terms in the variables and $f_\alpha$ represents external forcing of the system. Climatic means are estimated from time averages

$$\langle x_\alpha \rangle = \frac{1}{T}\int_0^T x_\alpha(t)dt, \tag{1.2}$$

where $T$ is some suitably long averaging time, typically of the order of decades.

We shall assume that when $T$ is large enough the climatic means are functions of the external forcing alone and not dependent on initial conditions.

The sensitivity matrix $\mathbf{M}_{\alpha\beta}$ describes shifts in the climatic means ($\langle x_\alpha\rangle \rightarrow \langle x_\alpha\rangle + \delta\langle x_\alpha\rangle$) due to small changes in the external forcing ($f_\beta \rightarrow f_\beta + \delta f_\beta$):

$$\delta\langle x_\alpha\rangle = \sum_\beta \mathbf{M}_{\alpha\beta}\delta f_\beta. \tag{1.3}$$

The sensitivity matrix, if known, has both predictive value in estimating climatic change and, since we should be able to calculate it, theoretical value as a test of our understanding of geophysical processes. It is not, however, easily accessible to observation, since nature rarely provides the observer with clean, well defined changes in external influences nor with responses that can be confidently distinguished from natural fluctuations of the atmospheric state. Leith (1975) has nevertheless suggested that these same natural fluctuations of the atmosphere may allow us to estimate its sensitivity. By Leith's hypothesis, summarized below, the sensitivity matrix is given in terms of the linear regression matrix $\mathbf{R}_{\alpha\beta}(\tau)$ for the atmosphere [defined in (1.13)] as

$$\mathbf{M}_{\alpha\beta} = \int_0^\infty \mathbf{R}_{\alpha\beta}(\tau)d\tau. \tag{1.4}$$

To explain his suggestion, we must introduce the mean response function $\mathbf{g}_{\alpha\beta}(\tau)$:

Let $x_\alpha(t)$ be a function that satisfies (1.1) and represents the unperturbed atmosphere. Suppose that at time $t = t_0$ an infinitesimal kick is given to variable $x_{\beta_0}$ produced by an infinitesimal perturbation in the forcing $f_{\beta_0}(t)$, which we may write as

$$\delta f_{\beta_0}(t) = \epsilon\delta(t - t_0), \qquad (1.5)$$

where $\epsilon$ is infinitesimal. Let $x_\alpha'(t)$ be the new solution to (1.1) with this change in the forcing. The difference between the perturbed and the unperturbed solutions is described by the impulse response function

$$x_\alpha'(t) - x_\alpha(t) = \delta x_\alpha(t) = \epsilon \hat{\mathbf{g}}_{\alpha\beta_0}(t,t_0). \qquad (1.6)$$

The impulse response function in general depends on $x_\alpha(t)$, and by definition must satisfy $\hat{\mathbf{g}}_{\alpha\beta}(t_0,t_0) = \delta_{\alpha\beta}$ and $\hat{\mathbf{g}}_{\alpha\beta}(t,t_0) = 0$ for $t < t_0$.

The impulse response function may be used to construct the linear response $\delta x_\alpha(t)$ to any perturbation $\delta f_\beta(t)$:

$$\delta x_\alpha(t) = \sum_\beta \int_{-\infty}^{t} \hat{\mathbf{g}}_{\alpha\beta}(t,t_0)\delta f_\beta(t_0)dt_0. \qquad (1.7)$$

In particular, if $\delta f_\beta(t)$ represents a constant shift in the external forcing,

$$\delta f_\beta(t) = \delta f_\beta, \qquad (1.8)$$

then the climatic response, combining (1.2) and (1.6), is

$$\delta\langle x_\alpha\rangle = T^{-1}\int_0^T dt \sum_\beta \int_{-\infty}^{t} dt_0 \hat{\mathbf{g}}_{\alpha\beta}(t,t_0)\delta f_\beta$$
$$= \sum_\beta \int_0^\infty d\tau \mathbf{g}_{\alpha\beta}(\tau)\delta f_\beta, \qquad (1.9)$$

where $\mathbf{g}_{\alpha\beta}(\tau)$ is the mean (linear) response function

$$\mathbf{g}_{\alpha\beta}(\tau) = T^{-1}\int_0^T \hat{\mathbf{g}}_{\alpha\beta}(t, t - \tau)dt. \qquad (1.10)$$

Thus we see from (1.3) and (1.9) that the sensitivity matrix may be obtained from the mean response function as

$$\mathbf{M}_{\alpha\beta} = \int_0^\infty \mathbf{g}_{\alpha\beta}(\tau)d\tau. \qquad (1.11)$$

The linear regression matrix for the system $\mathbf{R}_{\alpha\beta}(\tau)$ is constructed from the lagged covariance matrix

$$\mathbf{U}_{\alpha\beta}(\tau) = T^{-1}\int_0^T x_\alpha(t + \tau)x_\beta(t)dt. \qquad (1.12)$$

(Assume that $\langle x_\alpha\rangle = 0$ for each variable for simplicity.) The linear regression matrix predicts the average relaxation of fluctuations of the system back to the climatic mean, and is given by

$$\mathbf{R}_{\alpha\beta}(\tau) = \sum_\gamma \mathbf{U}_{\alpha\gamma}(\tau)[\mathbf{U}^{-1}(0)]_{\gamma\beta}, \qquad (1.13)$$

where $\mathbf{U}^{-1}(0)$ is the inverse of the zero-lag covariance matrix.

The fluctuation-dissipation theorem (FDT), which holds for many physical systems, states that

$$\mathbf{g}_{\alpha\beta}(\tau) = \mathbf{R}_{\alpha\beta}(\tau). \qquad (1.14)$$

In a system to which the theorem applies, the mean response function to external perturbations can be obtained by observing natural fluctuations of the unperturbed system. Eq. (1.4) follows from (1.11) and (1.14). The atmosphere is unfortunately not one of the physical systems for which a rigorous proof of (1.14) can be given, but Leith (1975) has argued that it might nevertheless be sufficiently well satisfied that it could serve as a useful tool for investigating climatic sensitivities. Since recourse to rigorous proof or to experiment is denied us, confidence in Eq. (1.4) will probably come only with the accumulation of tests using models. A given model may not correctly predict the climatic sensitivity of the real atmosphere, but if the characteristics of the model are in some sense close enough to those of the atmosphere, success of the FDT in predicting the mean response of the model may encourage us to trust in the applicability of the FDT to the atmosphere. Discrepant predictions may then be cause for reexamination of the physics of the model.

The FDT as proved by Leith (1975) for a system with two conserved quantities (e.g., energy and enstrophy) is stated in terms of averages over an ensemble of systems with a Gaussian probability distribution, whereas the statement of the theorem given here in (1.14) uses time-averaged quantities (1.10) and (1.12). It is shown in the statistical mechanics of conservative systems that averages over a Gaussian-distributed ensemble of systems and averages over time of a single system may be substituted for each other if two conditions are met: First, the number of variables in the system must be large enough that the highly peaked Gaussian probability distribution may be replaced by a distribution of systems all having the same energy (and enstrophy, in our case); second, the equations of motion of the system must be such that the ergodicity assumption may be invoked, so that the average over the ensemble of systems with identical energies (and enstrophies) may be replaced by an average over time of the behavior of a single system with that energy (and enstrophy). The formulation of the FDT in terms of time-averaged quantities is more convenient for us, since we intend to investigate the usefulness of the FDT in a forced, dissipative model, for which it would be difficult to specify *a priori* an ensemble with statistics stationary in time.

We present here the results of some tests of the FDT using a simple twenty-variable model. Although the model is based on a truncated version of the barotropic vorticity equation and omits many of the important dynamical processes occurring in the atmosphere, it has the advantage of allowing us to pass from a model for which the theorem can be proved to one for which it cannot be, simply by introducing dissipation and external forcing. When the model is pushed well away from the regime where the FDT is known to be satisfied, statistically significant differences appear in the detailed comparison of the response matrix and the regression matrix, but the sensitivity matrix $\mathbf{M}_{\alpha\beta}$, which involves an integral over the regression matrix [Eq. (1.4)], is still well predicted by the FDT.

## 2. The model

The model is derived from the barotropic vorticity equation

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = 0, \qquad (2.1)$$

which describes the motion of a two-dimensional, incompressible, inviscid fluid with velocity components $u(x,y,t)$, $v(x,y,t)$ and vorticity $\zeta(x,y,t) = \partial v/\partial x - \partial u/\partial y$. A set of truncated equations is derived following the classic work of Lorenz (1960). The flow is assumed spatially periodic, and expanded in Fourier modes

$$\zeta(\mathbf{x},t) = \sum_{\mathbf{k}} \tilde{\zeta}(\mathbf{k},t)\exp(i\mathbf{k}\cdot\mathbf{x}), \qquad (2.2)$$

where the wavenumbers $\mathbf{k}$ have the discrete values

$$\mathbf{k} = \frac{2\pi}{L}(n_x,n_y), \qquad (2.3)$$

$n_x$, $n_y$ are integers, and $L$ is the periodicity length. (Results reported here use $L = 2\pi$.) Since the vorticity $\zeta$ is real, the complex Fourier amplitudes $\tilde{\zeta}(\mathbf{k},t)$ are not independent:

$$\tilde{\zeta}(-\mathbf{k},t) = \tilde{\zeta}^*(\mathbf{k},t). \qquad (2.4)$$

The equations for $\tilde{\zeta}$ which follow from (2.1) are

$$\frac{d}{dt}\tilde{\zeta}(\mathbf{k}) = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (p_yq_x - p_xq_y)p^{-2}\tilde{\zeta}(\mathbf{p})\tilde{\zeta}(\mathbf{q}), \qquad (2.5)$$

where energy

$$E = \tfrac{1}{2}\sum_{\mathbf{k}} k^{-2}|\tilde{\zeta}(\mathbf{k})|^2 \qquad (2.6)$$

and enstrophy

$$F = \tfrac{1}{2}\sum_{\mathbf{k}} |\tilde{\zeta}(\mathbf{k})|^2 \qquad (2.7)$$

are conserved.

If the only non-zero variable is $\tilde{\zeta}_R(0,1)$, the real part of $\tilde{\zeta}(0,1)$, the corresponding vorticity field is

$$\zeta(\mathbf{x}) = 2\tilde{\zeta}_R(0,1)\cos y$$

and the velocity field is

$$u(\mathbf{x}) = -2\tilde{\zeta}_R(0,1)\sin y, \quad v = 0.$$

If instead of $L = 2\pi$ we were to use units typical of geophysical scales, say $L = 10^4$ km and time units $T = 4$ days, then unit strength in $\tilde{\zeta}_R(0,1)$ (which has units $1/T$) would correspond to zonal velocities $u$ of the order of 10 m s$^{-1}$.

A truncated set of equations is obtained from the infinite set (2.5) by setting all but a finite set of variables in the equations equal to zero and insisting that they remain zero. Energy and enstrophy are still conserved. In Lorenz's (1960) original work, three variables were retained. Flows periodic in time were generated by the three equations because of the two conservation laws. Kells and Orszag (1978) have investigated truncations with larger numbers of variables, keeping all Fourier modes $\tilde{\zeta}(\mathbf{k})$ for which $k^2 \leqslant K^2$. They find that when $K^2$ is increased to 5 the flows generated appear ergodic and have statistics similar to the statistics that can be derived for very large systems.

It is this model with ten complex variables $\tilde{\zeta}(\mathbf{k})$, $k^2 \leqslant 5$, or twenty real variables, that we shall study here. We investigate first how well the FDT is satisfied by the inviscid model. If the statistics were obtained by averaging over a Gaussian-distributed ensemble of runs of the model instead of from the time average of a single run, we would expect the FDT to be exactly obeyed by the model. Although the FDT was not investigated by them, Kells and Orszag (1978) show that the number of variables is large enough and the assumption of ergodicity good enough that time-averaged quantities agree with ensemble-averaged predictions to a few percent (at least for low-order statistics of the model). We may therefore expect the time-averaged statistics of the model to conform to the FDT to within a few percent.

The model is integrated using a fourth order Runge-Kutta scheme, with initial conditions chosen randomly but constrained to have total energy $E = 7$ and total enstrophy $F = 20$. This choice is made in order that all of the modes be significantly excited and that the characteristic times of evolution of the various modes not be spread over a range of scales differing by more than a factor of 4 or 5. The time step is chosen small enough for energy and enstrophy to be conserved to one part in a million over the length of the run.

To test the FDT, as expressed in (1.14), we require the response matrix and the regression matrix. We shall refer to the 20 real variables of the model as $\zeta_\alpha$, $\alpha = 1, 2, \ldots, 20$, whenever identification by wavenumber and "real" or "imaginary" is unnecessarily cumbersome. The covariance matrix, needed to compute the regression matrix (1.13), is obtained by averaging over a single run of length $T = 4000$. Because the statistics of the system is expected to be spatially homogeneous, the covariance matrix $\mathbf{U}_{\alpha\beta}$ should be diagonal. This is observed to be the case to within expected statistical uncertainty. We therefore compare the diagonal elements of the regression matrix, which are just the correlation functions

$$C_{\alpha\alpha}(\tau) = \mathbf{U}_{\alpha\alpha}(\tau)/\mathbf{U}_{\alpha\alpha}(0), \tag{2.8}$$

to the corresponding response functions.

The response function is calculated in a manner suggested by (1.6). A basic solution $\zeta_\alpha(t)$ is followed during the entire course of the run. At $t = t_1$ mode $\zeta_\beta$ is perturbed by an amount $\epsilon$ and the response function $\hat{\mathbf{g}}_{\alpha\beta}(t_1 + \tau, t_1)$ is calculated from (1.6) as integration of the two solutions $\zeta_\alpha(t)$ and $\zeta_\alpha'(t)$ proceeds. At $t = t_2$ the basic solution $\zeta_\alpha(t)$ is again perturbed and a new sample response function $\hat{\mathbf{g}}_{\alpha\beta}(t_2 + \tau, t_2)$ is calculated. This is repeated $N$ times, with $t_{n+1} - t_n = 3$. The mean response function is the average over the $N$ trials:

$$\mathbf{g}_{\alpha\beta}(\tau) = N^{-1} \sum_{j=1}^{N} \hat{\mathbf{g}}_{\alpha\beta}(t_j + \tau, t_j). \tag{2.9}$$

The perturbation $\epsilon$ is chosen small enough to obtain a linear response.

Correlation functions and response functions are graphed in Figs. 1 and 2. Because of the symmetry of the system, all variables $\tilde{\zeta}(\mathbf{k})$ (real and imaginary components) with wavenumbers of the same length should have identical statistical properties. This is confirmed numerically, and consequently only one graph for each value of $k^2$ is shown in Figs. 1 and 2.

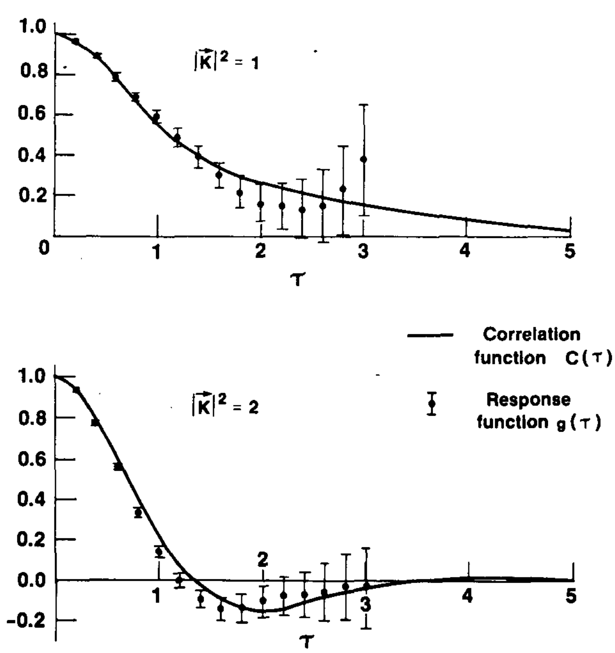


FIG. 1. Lagged correlation functions and mean response functions for the 20-component barotropic model with energy $E = 7$ and enstrophy $F = 20$. Correlation functions are obtained from a sample of length $T = 4000$, and are averages of the correlation functions of the real and of the imaginary parts of all $\tilde{\zeta}(k)$ with given $k^2$ ($k^2 = 1, 2$). Response functions show mean response of a variable to 286 successive kicks, separated 3 units of time apart. Error bars are estimated standard deviations of the mean, and are explained in the text.

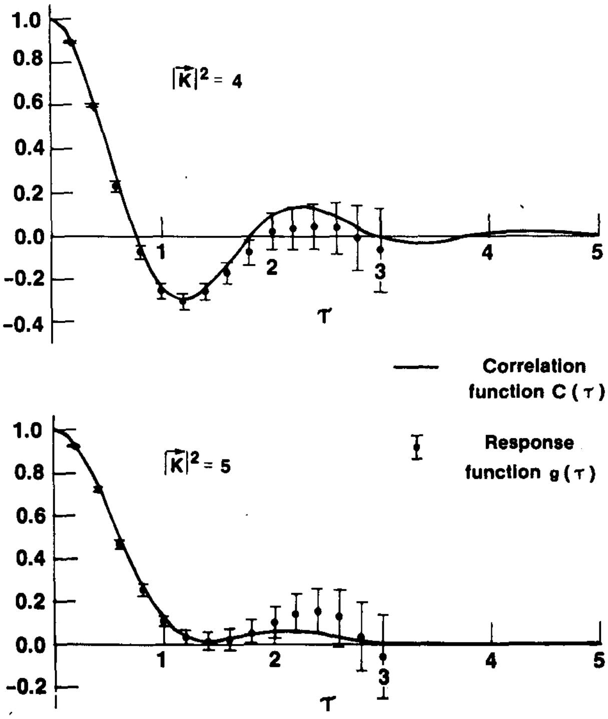


FIG. 2. As in Fig. 1 except for $k^2 = 4, 5$.

The correlation functions shown are averages over all variables with the same $k^2$ and are estimated to be accurate to $\pm 0.01$ or better. The error bars on the response functions indicate the uncertainty in $\mathbf{g}_{\alpha\beta}(\tau)$ due to finite sample size, estimated as

$$\{[N(N-1)]^{-1} \sum_{j=1}^{N} [\hat{\mathbf{g}}_{\alpha\beta}(t_j + \tau, t_j) - \mathbf{g}_{\alpha\beta}(\tau)]^2\}^{1/2}. \tag{2.10}$$

Since neighboring solutions diverge exponentially with time, the error in $\mathbf{g}_{\alpha\beta}(\tau)$ also grows exponentially with $\tau$.

The agreement between the correlation functions and the response functions is excellent. This is encouraging, but, as discussed earlier, not entirely surprising. With this experience in hand we are ready to turn to a situation more like the real atmosphere, where the FDT cannot be proved.

## 3. Forced, dissipative model

The viscous and forcing terms in the atmospheric equations prevent a rigorous proof of the validity of the FDT for the atmosphere. Leith (1975) has nevertheless argued that it might still provide an adequate estimate of the sensitivity of the atmosphere to external changes. There is unfortunately very little in the way of direct numerical evidence on the subject. In order to obtain some impression of how useful the

TABLE 1. Means and variances of selected variables for the forced, dissipative model. The characteristic time $\tau_\alpha$ is defined in (3.7).

| Variable | $\langle\zeta_\alpha\rangle$ | $\langle(\zeta_\alpha - \langle\zeta_\alpha\rangle)^2\rangle$ | $\tau_\alpha$ |
|---|---|---|---|
| $\tilde{\zeta}_R(0,1)$ | 0.603 ± 0.017 | 1.08 ± 0.018 | 0.65 ± 0.05 |
| $\tilde{\zeta}_R(-1, 1)$ | −0.058 ± 0.044 | 4.05 ± 0.09 | 1.17 ± 0.12 |
| $\tilde{\zeta}_I(0,2)$ | 0.032 ± 0.020 | 2.10 ± 0.03 | 0.47 ± 0.03 |
| $\tilde{\zeta}_R(-1, 2)$ | 0.009 ± 0.013 | 1.356 ± 0.015 | 0.32 ± 0.02 |

FDT might be when the system studied in Section 2 is pushed well away from the regime where the FDT is known to apply, the system is modified by adding viscous and forcing terms to Eqs. (2.5):

$$\frac{d}{dt}\tilde{\zeta}(\mathbf{k}) = \text{nonlinear terms} - \nu(\mathbf{k})\tilde{\zeta}(\mathbf{k}) + F(\mathbf{k}). \quad (3.1)$$

Energy and enstrophy are no longer conserved.

The forces $F(\mathbf{k})$ are all set equal to zero except for $\mathbf{k} = (0,1)$, for which we choose

$$F(0,1) = 1.0. \quad (3.2)$$

This introduces a stress on the fluid in the $x$ direction proportional to sin$y$. Usual choices for the viscosities $\nu(\mathbf{k})$, as a constant independent of $k^2$, or increasing with $k^2$, prove unsatisfactory for the model. The solution $\zeta_\alpha(t)$ is always observed numerically to relax to a steady state. A stability analysis of the model confirms this. In order to augment the instabilities of the model so that a turbulent flow is maintained, the viscosities for the modes with $k^2 = 2$ and $k^2 = 4$ are made negative. This can be imagined as an attempt to mimic the contributions from baroclinic instabilities in the real atmosphere, although these generally operate at higher wavenumbers. With the choice

$$\left.\begin{aligned}\nu(k^2 = 1) = \nu(k^2 = 5) = 1.0\\ \nu(k^2 = 2) = \nu(k^2 = 4) = -0.6\end{aligned}\right\}, \quad (3.3)$$

the flow is turbulent, lagged correlations decay on a time scale of order 1, and the statistics of the flow do not seem to be qualitatively sensitive to the precise values of the viscosities. Energies of the various modes are maintained well away from equipartition values.

Statistics for the system are gathered from a run of length $T = 5000$. Means and variances for some of the variables are given in Table 1. Under the action of the externally imposed stress, a mean flow in the $x$-direction develops proportional to sin$y$. In the geophysical units proposed in Section 2, maximum mean zonal winds are of the order of 6 m s$^{-1}$. Estimated errors are computed as in Leith (1973). Denoting by $\sigma[\langle\zeta_\alpha\rangle]$ the expected error in the mean $\langle\zeta_\alpha\rangle$, the estimated standard deviation of the mean is

$$\sigma^2[\langle\zeta_\alpha\rangle] = (2\tau_\alpha/T)\langle\zeta_\alpha{}^2\rangle_c, \quad (3.4)$$

where the subscript $c$ indicates variance about the mean,

$$\langle\zeta_\alpha{}^2\rangle_c = \langle(\zeta_\alpha - \langle\zeta_\alpha\rangle)^2\rangle, \quad (3.5)$$

and the estimated standard deviation of the variance is

$$\sigma^2[\langle\zeta_\alpha{}^2\rangle_c] = (2\tau_\alpha/T)\langle\zeta_\alpha{}^2\rangle_c{}^2. \quad (3.6)$$

The characteristic time $\tau_\alpha$ for a given variable $\zeta_\alpha$ is

$$\tau_\alpha = \int_0^\infty C_{\alpha\alpha}(\tau)d\tau, \quad (3.7)$$

where $C_{\alpha\alpha}$ is the lagged correlation function defined in (2.8). The characteristic times $\tau_\alpha$ must be estimated from a finite sample of length $T$. It is not difficult to show that if the true ($T \to \infty$) correlation function decays exponentially as $\exp(-\tau/\tau_\alpha)$, and the variables $\zeta_\alpha$ have Gaussian statistics, then replacing the upper integration limit in (3.7) with a finite cutoff $\tau_L$ will yield an estimate of $\tau_\alpha$, i.e.,

$$\tau_\alpha \approx \int_0^{\tau_L} C_{\alpha\alpha}(\tau)d\tau, \quad (3.8)$$

with uncertainty

$$\sigma^2[\tau_\alpha] = (4\tau_L/T)\tau_\alpha{}^2. \quad (3.9)$$

This error increases with $\tau_L$. The cutoff $\tau_L$ must therefore be chosen as small as possible, but large enough to encompass the range of $\tau$ for which $C_{\alpha\alpha}(\tau)$ is significantly different from zero. The errors listed in Table 1 for the characteristic times are estimated in this manner.

Graphs of the correlation functions $C_{\alpha\alpha}(\tau)$ for the four variables in Table 1 are shown as smooth curves in Figs. 3 and 4. The correlation functions have an estimated error of ±0.015 or less. Regression matrices $\mathbf{R}_{\alpha\alpha}(\tau)$ were also computed and found to be identical to $C_{\alpha\alpha}(\tau)$ to within statistical uncertainty. Off-diagonal correlations are small, in spite of the spatial inhomogeneity imposed on the system by the external force. This may be explained by the relatively small energy input from the forcing compared to that generated by the instability of the modes with negative viscosities (see Table 2).

To test the FDT, the response function is obtained for each of the four variables listed in Table 1. The mean responses $\mathbf{g}_{\alpha\alpha}(\tau)$ are plotted in Figs. 3 and 4 with error bars indicating the estimated standard deviations of the means. As before, the errors grow exponentially.

It is evident that in this case there are statistically significant deviations of the response curve from the lagged correlation curve. However, the correlation functions still approximate the response functions to within 25% over the range of $\tau$ where statistical uncertainty in the response function is not so large as to make the comparison uninformative.

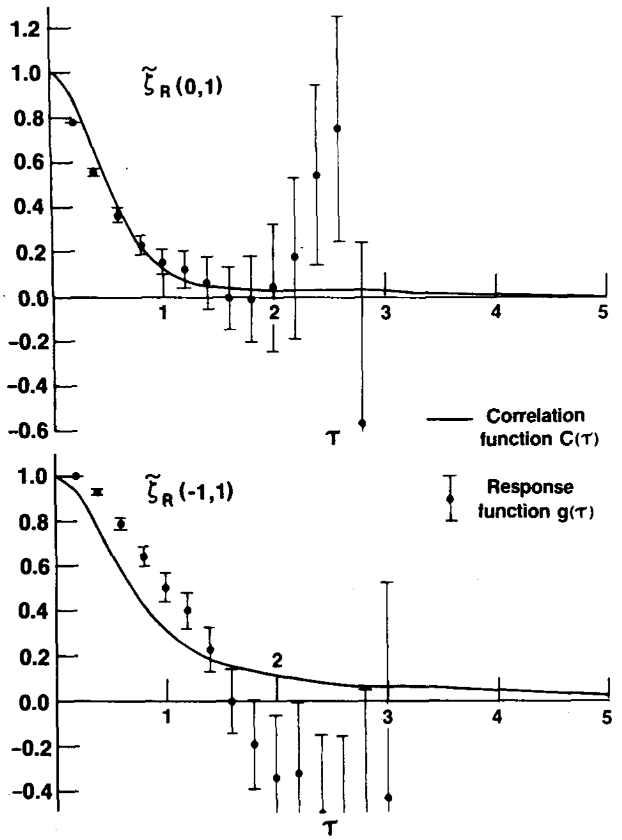


FIG. 3. Lagged correlation functions and mean response functions for the 20-variable model with forcing and dissipative terms added, for variables $\tilde{\zeta}_R(0,1)$ and $\tilde{\zeta}_R(-1, 1)$. Correlation functions are obtained from a sample of length $T = 5000$. Their estimated error is less than 0.015. Response functions are averages over responses to 227 successive impulsive kicks.

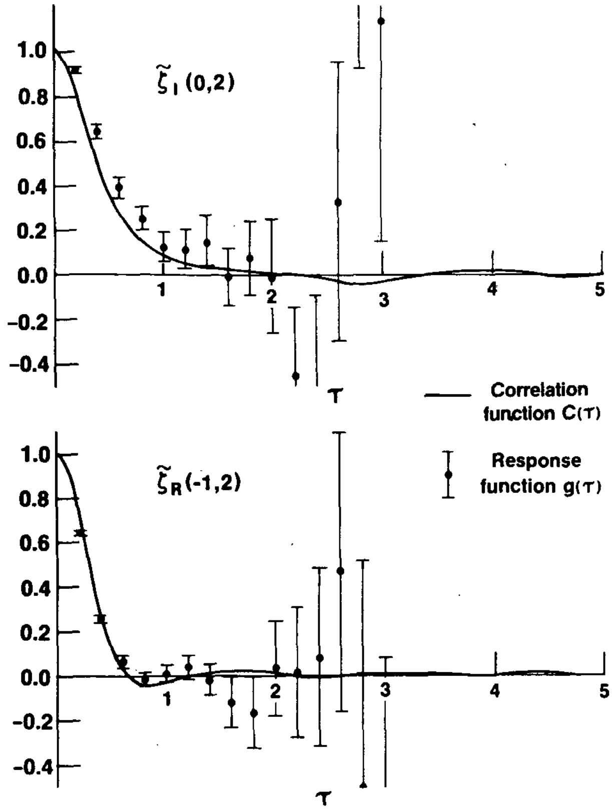


FIG. 4. As in Fig. 3, except for variables $\tilde{\zeta}_I(0,2)$ and $\tilde{\zeta}_R(-1, 2)$.

The most important use of the FDT is in estimating the sensitivity matrix $\mathbf{M}_{\alpha\beta}$ to predict shifts in climatic means via Eq. (1.3). It is therefore interesting to compare the predicted shift in the climatic mean of the model, obtained using the FDT, to the actual shift under a change in the external stress acting on the model. The diagonal elements of the sensitivity matrix are predicted to be

$$\begin{aligned} \mathbf{M}_{\alpha\alpha} &= \int_0^\infty \mathbf{g}_{\alpha\alpha}(\tau)d\tau \\ &= \int_0^\infty \mathbf{R}_{\alpha\alpha}(\tau)d\tau \\ &\approx \int_0^\infty C_{\alpha\alpha}(\tau)d\tau \\ &= \tau_\alpha, \end{aligned}$$

where the first equation is from (1.11), the second equality assumes the FDT, and the third approximate equality is possible because of the very small off-diagonal correlations observed for the systems. The final equality follows from the definition (3.7). We may therefore read off the predicted sensitivities of the modes listed in Table 1 under the column labelled $\tau_\alpha$.

To check an FDT-estimated sensitivity, the force $F(0,1)$ is increased from 1.0 to 1.5. The mean $\langle\tilde{\zeta}(0,1)\rangle$ increases as a result from 0.603 to 0.886 (corresponding to an increase in the zonal wind, in the geophysical units suggested in Section 2, from 6 m s$^{-1}$ to 8.9 m s$^{-1}$). The *observed* shift in the climatic mean is therefore

$$\delta\langle\zeta\rangle_{\rm obs} = 0.28 \pm 0.05.$$

TABLE 2. Energy budget for the forced, dissipative model. The energy rate $dE/dt$ is the total energy lost or gained due to the viscosity term by all modes with a given value of $k^2$. The deficit −1.21 is made up by energy input from the external stress $F(0,1)$.

| $k^2$ | $dE/dt$ (viscosity term) |
|---|---|
| 1 | −9.28 |
| 2 | 9.91 |
| 4 | 2.54 |
| 5 | −4.38 |
| Total | −1.21 |

TABLE 3. Observed sensitivity of the forced, dissipative model versus the sensitivity predicted by the FDT. The force $F_\alpha$ acting on variable $\zeta_\alpha$ was changed by the amount $\delta F_\alpha$. The new mean $\langle \zeta_\alpha \rangle$, obtained from averaging period $T$, substracted from the old mean (given in Table 1), is shown as $\delta\langle \zeta_\alpha \rangle$. The FDT prediction $M_{\alpha\alpha}\delta F_\alpha$ is given in the last column, where $M_{\alpha\alpha}$ is obtained from the integral of the lagged correlation function, given in Table 1 as $\tau_\alpha$.

| Variable $\zeta_\alpha$ | $\delta F_\alpha$ | $T$ | Observed $\delta\langle \zeta_\alpha \rangle$ | Predicted $\delta\langle \zeta_\alpha \rangle$ |
|---|---|---|---|---|
| $\tilde{\zeta}_R(0,1)$ | 0.5 | 700 | 0.28 ± 0.05 | 0.33 ± 0.02 |
| $\tilde{\zeta}_R(-1, 1)$ | 0.5 | 700 | 0.55 ± 0.14 | 0.58 ± 0.06 |
| $\tilde{\zeta}_I(0,2)$ | 0.5 | 700 | 0.21 ± 0.06 | 0.23 ± 0.02 |
| $\tilde{\zeta}_R(-1, 2)$ | 0.5 | 700 | 0.16 ± 0.04 | 0.16 ± 0.01 |
| $\tilde{\zeta}_R(-1, 2)$ | 0.25 | 3000 | 0.040 ± 0.021 | 0.079 ± 0.006 |

The *predicted* shift is given by

$$\begin{aligned}\delta\langle \zeta_\alpha \rangle_{\text{pred.}} &= \mathbf{M}_{\alpha\alpha}\delta F_\alpha \approx \tau_\alpha \delta F_\alpha \\ &= 0.65 \times 0.5 \\ &= 0.33 \pm 0.02.\end{aligned}$$

The agreement is excellent. Table 3 summarizes the results of a series of experiments to determine the sensitivity of the model climate to the external force acting on each of the variables listed in Table 1.[1] The observed sensitivities are well predicted by the FDT over a range of nearly a factor of four.

## 4. Discussion

The model tests described here serve two purposes. They indicate that useful information about the application of the FDT to sensitivity studies can be obtained using models with relatively small numbers of variables. The models must, of course, be detailed enough to describe dynamical fluctuations of the atmosphere. The tests also offer considerable encouragement for further studies of the use of the FDT, since, in the cases investigated here, the theorem provides excellent estimates of the model sensitivities even when dissipation has pushed the system far from the regime where the theorem can be rigorously proved. Work is already under way on tests of the theorem with more realistic models.

New problems will appear in testing the FDT on larger models. The number of variables will increase enormously, and some means of reducing the size of the matrices involved will be needed. Leith (1975) has suggested using empirical orthogonal functions (EOF's) as a basis. The usefulness of this approach depends on whether the lagged covariance matrix $\mathbf{U}_{\alpha\beta}(\tau)$ has too many large, off-diagonal elements in this representation, and on whether the external forcing $f_\alpha$ for which the sensitivity of the system is being investigated can be well represented by a small number of EOFs. The FDT analysis of large scale atmospheric sensitivities should certainly benefit from an approach using EOFs. Herring (1977, 1980) has in any case suggested, on the basis of work with turbulence closure models, that the FDT may overestimate response times for the small-scale dynamics of the atmosphere, and so may be less useful as a guide to small-scale sensitivities.

Limitation to smaller sample sizes and problems such as handling seasonal trends in the model output will complicate tests of the FDT using more realistic models of the atmosphere. External forcing of the atmosphere is not always constant; for instance, sea-surface temperature, when considered as external to the atmosphere, influences the atmosphere through couplings that fluctuate depending on surface wind speeds, and if the FDT is to be used in such situations, it must be verified that the fluctuating external "forces" can be replaced by mean quantities. None of these problems seems insurmountable, however, and experience gained in making these tests will be valuable when the FDT is applied to the real atmosphere.

The FDT offers the prospect of direct access to climatic sensitivities without having to construct a detailed model of the atmosphere with all of the physics correctly accounted for. Such models will of course still be needed even if the FDT approach is successful, since large climatic swings will probably not be adequately handled by linear sensitivity forecasts; nor will the FDT provide any of the physical understanding that a model can bring of the processes that underlie a climatic shift. But the FDT can also serve as a useful diagnostic tool for models of the atmosphere. The FDT can extract model sensitivities to a multiplicity of parameters with much more efficiency than the usual method of estimating sensitivities. The latter, straightforward method requires that each parameter be changed by a small amount, and that two model runs, one before and one after the change, be compared in order to extract a statistically significant response. Since the parameter change cannot usually be made so large as to shift the climatic mean of the model outside the normal range of fluctuations of the model variables (because the physical parameterizations used in the model may not be trustworthy outside the regime where they were developed), long runs are required to assure adequate sample sizes. When the FDT is applicable, these long multiple runs can be replaced by a single run from which the necessary regression statistics are collected.

Pratt (1979) and Blackmon and Lau (1980) have

[1] The experiment with variable $\tilde{\zeta}_R(-1, 2)$ was repeated with a smaller force ($\delta F = 0.25$) because a substantial (three standard deviation) shift in the mean of $\tilde{\zeta}_R(-1, 1)$ occurred with the larger force. This off-diagonal response, not predicted by the FDT, was reduced to 1.3 standard deviations at the smaller force; it may therefore be an example of a nonlinear response of the system which cannot be predicted by the linear sensitivity estimate obtained from the FDT.

recently compared the spectral variance of several variables in large general circulation models to the corresponding atmospheric data, finding that variances at low frequencies on planetary scales tend to be significantly underestimated by the models. If the FDT proves to be as useful a guide to climatic sensitivity as it promises to be, it suggests another sort of comparison between models and the atmosphere. According to Eq. (1.4), the sensitivity matrix is determined by an integral over lag of the regression matrix. If we define the time-spectral covariance matrix as

$$\tilde{\mathbf{U}}_{\alpha\beta}(\omega) = \int_{-\infty}^{\infty} e^{i\omega\tau}\mathbf{U}_{\alpha\beta}(\tau)d\tau, \quad (4.1)$$

then the sensitivity matrix is given by

$$\mathbf{M}_{\alpha\beta} = \sum_{\gamma}\left[\tfrac{1}{2}\,\mathrm{Re}\tilde{\mathbf{U}}_{\alpha\gamma}(\omega=0) + \pi^{-1}\int_{0}^{\infty} d\omega\,\mathrm{Im}\tilde{\mathbf{U}}_{\alpha\gamma}(\omega)/\omega\right][\mathbf{U}^{-1}(\tau=0)]_{\gamma\beta}. \quad (4.2)$$

Thus it is not the absolute variances of the model and of the atmosphere that must agree for the model to predict *climatic change* well; interpreting (4.2) roughly, it is rather the ratio of the low-frequency variance to the total variance that must be accurately reproduced by the model.

*Acknowledgments.* I am very grateful to R. F. Cahalan, D. Chesters, U. Frisch, J. R. Herring, C. E. Leith, D. M. Straus and J. J. Tribbia for many helpful comments. I have especially benefitted from much useful discussion with G. R. North.

## REFERENCES

Blackmon, M. L., and N.-C. Lau, 1980: Regional characteristics of the Northern Hemisphere wintertime circulation: a comparison of the simulation of a GFDL general circulation model with observations. *J. Atmos. Sci.*, **37**, 497–514.

Herring, J. R., 1977: On the statistical theory of two-dimensional topographic turbulence. *J. Atmos. Sci.*, **34**, 1731–1750.

——, 1980: A note on Owens' mesoscale eddy simulation. *J. Phys. Oceanogr.*, **10**, 804–806.

Kells, L. C., and S. A. Orszag, 1978: Randomness of low-order models of two-dimensional inviscid dynamics. *Phys. Fluids*, **21**, 162–168.

Leith, C. E., 1973: The standard error of time-average estimates of climatic means. *J. Appl. Meteor.*, **12**, 1066–1069.

——, 1975: Climate response and fluctuation dissipation. *J. Atmos. Sci.*, **32**, 2022–2026.

——, 1978: Predictability of climate. *Nature*, **276**, 352–355.

Lorenz, E. N., 1960: Maximum simplification of the dynamic equations. *Tellus*, **12**, 243–254.

Pratt, R. R., 1979: A space-time spectral comparison of the NCAR and GFDL general circulation models to the atmosphere. *J. Atmos. Sci.*, **36**, 1681–1691.