Sensitivity Analysis of a Radiative-Convective Model by the Adjoint Method[1]

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ABSTRACT

The adjoint method of sensitivity analysis is demonstrated on a radiative-convective climate model. A single adjoint calculation, which requires about the same computation time as the original model, suffices to calculate sensitivities of surface air temperature to all 312 model parameters. The uses of these sensitivities are discussed and illustrated. The sensitivities accurately predict the effect on surface air temperature of small variations in the model parameters. Relative sensitivities are used to rank the importance of all the parameters. Several of the sensitivities to parameters customarily considered in previous works (e.g., solar constant, surface albedo, relative humidity, CO_2 concentration) are reproduced, but the largest sensitivities are to constants used to compute the saturation vapor pressure of water. The uncertainties in the model results are expressed formally in terms of all the sensitivities and parameter covariances. For results that cannot readily be compared with observation (for example, the results of a CO_2 doubling experiment), this method of uncertainty analysis is the only systematic way to estimate the reliability of model results.

The radiative-convective model contains complex nonlinear processes of the type found in general circulation models. Therefore, the fact that the adjoint method works successfully and efficiently for the radiative-convective model provides valuable information about subsequent application of the method to general circulation models.

1. Introduction

An essential part of climatic research consists of interpreting the results of climatic models. Valuable information for this purpose comes from analyzing sensitivities of the results to parameters involved in modeling physical processes. For example, the current concern about the climatic impact of CO_2 stems from the significant sensitivity that climatic models exhibit to the atmospheric concentration of CO_2 (e.g., Manabe and Wetherald, 1980). Also, sensitivities provide valuable insight about the modeling of physical phenomena. For example, the ice-albedo feedback mechanism corresponds to the observed negative sensitivity of surface air temperature to surface albedo (Manabe and Wetherald, 1967). Furthermore, sensitivities quantify the extent that uncertainties in parameters contribute to uncertainties in results of models. For example, subgrid processes need to be parameterized. These parameterizations are highly simplified approximations of complex processes, so the uncertainties in the parameters involved can be large. If the corresponding sensitivities are also large, then the results of the model will have large uncertainties. In fact, it is the need to analyze this type of uncertainty for CO_2 doubling experiments that provides the motivation for this work.

To date, the most common procedure to estimate sensitivities for climatic models has been to change the value of one of the parameters and recalculate the results. For complex models the large amount of computing time severely restricts the scope of this procedure. Physically meaningful changes in the parameters do not usually shift the climatic mean of the model outside the range of fluctuations of the model's dependent variables, so typically each rerun requires several hours computing time to improve signal-to-noise ratio. In practice this means that the modeler can investigate only a few parameters that he judges *a priori* to be important, although each rerun does provide sensitivities for all dependent variables everywhere to the one parameter that has been varied.

A way of investigating sensitivities to more parameters is to consider simplified models obtained by reducing the spatial resolution (i.e., number of grid points or wavenumbers) of complex three-dimensional models (Held and Suarez, 1978; Aleksandrov and Gates, 1981). Although this makes rerunning less expensive, the parameters must still be selected *a priori*, and consequently important sen-

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sitivities may be missed. Also, reducing the spatial resolution usually requires changes in the way subgrid processes are modeled, so it is not clear whether the respective sensitivities of the simplified and complex models are the same.

Leith (1975) has proposed a method of determining climatic sensitivity based on the fluctuation-dissipation theorem. This theorem relates covariances of the variables in certain statistical systems to the effect that small changes in forcing influences have on the mean values of the variables. But as Leith states, "the climate ensemble is not a normal equilibrium ensemble as required for the general fluctuation-dissipation theorem to hold exactly". Recently, Bell (1980) has reported an application of this theorem to a simple model based on the barotropic vorticity equation.

Marchuk (1974) has proposed a method of obtaining sensitivities to boundary conditions for a "baroclinic model of the atmosphere in the nonadiabatic approximation." He uses a conjugate system of equations, and expresses perturbations in regional average temperature in terms of the conjugate solution and variations in boundary conditions. His method, though, cannot be applied to models that include nonlinear source terms (e.g., any realistic climatic model), and the method is not designed to calculate sensitivities to parameters other than boundary conditions.

Sensitivity analysis techniques have also been developed in conjunction with applications in other fields. These techniques fall into three main categories: direct methods, Green's function methods, and adjoint methods. All these methods yield first derivatives of results with respect to parameters. Note that these methods do not give information on the physical effects of second and higher order derivatives that rerunning the model can reveal.

The direct method (Dickinson and Gelinas, 1976) involves differentiation of the equations describing the model with respect to a parameter. The resulting set of equations is solved for the derivative of all the model variables with respect to that parameter. These derivatives are used to obtain the sensitivity of the result to the parameter. Note that the actual form of the differentiated equations depends on the parameter under consideration. Consequently a different set of equations must be solved to obtain the sensitivity corresponding to each parameter.

The Green's function method uses Green's functions to avoid solving the differentiated equations anew for each parameter. Demiralp and Rabitz (1981) present theoretical aspects of this method and consider a space- and time-dependent chemical kinetics problem. For such a problem, though, a clear advantage of the Green's function method over the direct method remains to be demonstrated.

Adjoint methods were first developed for sensitivity analysis of specific problems in reactor physics. The mathematical foundations underlying the use of adjoint methods have been established by Cacuci (1981a), who presents a general method that can be applied to nonlinear problems. This method starts by expressing a result, customarily called the response, as a functional of the model variables. The sensitivity of this response is defined in terms of a functional derivative. A system of adjoint equations is developed from a differentiated form of the original equations. The sensitivity of the response *to all the model parameters* can be evaluated in terms of integrals involving the adjoint solution. Note that, in contrast to the direct method, the form of the adjoint equations remains unchanged regardless of the parameter under consideration. Consequently, a single adjoint calculation suffices to obtain sensitivities to all the parameters. The adjoint equations, though, include a source term that depends on the functional defining the response, so that for each response the adjoint equations must be solved anew. In practice, for about the same amount of computation, the adjoint method gives the sensitivity of a functional of the variables to all the parameters, while the direct method gives the sensitivity of all the variables to a single parameter.

This work demonstrates how the adjoint method can be applied to a radiative-convective model (RCM) of the climate. This model includes 312 parameters and contains the nonlinear phenomena characteristic of radiatively-coupled processes. Section 2 presents a brief mathematical description of the RCM. Section 3 presents the mathematical definition of sensitivity, the derivation of the adjoint equations satisfied by the adjoint functions, and the expressions of sensitivities in terms of these adjoint functions. Also, uncertainties in the model results are expressed formally in terms of sensitivities and parameter covariances. Section 4 presents an analysis of the numerical values obtained for the sensitivities, illustrates the use of sensitivities, and discusses computational requirements. Section 5 summarizes the results of this work, and discusses their relevance to future application of the adjoint method to the sensitivity analysis of general circulation models.

2. Mathematical representation of the radiative-convective model

The RCM used in this work determines the equilibrium vertical temperatures for a single atmospheric column and its underlying surface, subject to prescribed insolation, atmospheric composition and surface albedo. The RCM includes models for the transfer of solar and terrestrial radiation, the heat exchange between the earth's surface and atmosphere, the vertical redistribution of heat within the atmosphere, the atmospheric water vapor con-

tent, and clouds. The radiative transfer model is that of the Oregon State University two-level atmospheric general circulation model (Schlesinger and Gates, 1980). For the solar radiation, this model includes the effects of water vapor and ozone absorption, molecular scattering, and the scattering and absorption by clouds. For the terrestrial radiation, the emission and absorption by water vapor, carbon dioxide and clouds are included. The exchange of heat between the earth's surface and atmosphere is treated as a Newtonian heating with a prescribed heat transfer coefficient multiplied by the temperature difference between the earth's surface and surface air. The vertical redistribution of heat within the atmosphere is modelled following Manabe and Wetherald (1967) such that the lapse rate does not exceed a prescribed critical value of 6.5°C km^{-1}. In the present work, the atmospheric water vapor content is determined from the saturation vapor pressure of water as a function of temperature, and from a prescribed vertical profile of relative humidity taken as that of Manabe and Wetherald (1967). Consequently, the water vapor content depends on the temperature, but clouds do not exist for the results presented in Section 4.

The state of radiative-convective equilibrium is computed as an asymptotic state of an initial-value problem, by time integration of the model's set of three nonlinear, coupled differential equations:

$$\frac{du_i}{dt} = f_i(u_1, u_2, u_3, \alpha_1, \ldots, \alpha_P), \quad i = 1, 2, 3. \quad (1)$$

The variables u_1 and u_2 represent the temperatures of the two layers of the model atmosphere, and u_3 represents the temperature of the earth's surface. The functions f_i, $i = 1, 2, 3$ represent all processes that change the temperature. In practice, each f_i is a nonlinear function of u_1, u_2, and u_3. The real scalars $\alpha_1, \ldots, \alpha_P$ represent parameters of the model such as specific heats, transmission functions, albedos, and initial conditions. For this study, the total number of model parameters is 312 (i.e., $P = 312$). The physical and mathematical considerations leading to (1) are presented in Appendix A.

To apply the adjoint method of sensitivity analysis (Cacuci, 1981a), (1) is written as

$$N_i(\mathbf{u}, \alpha) = 0, \quad i = 1, 2, 3, \quad (2)$$

where the operator $N_i(\mathbf{u}, \alpha)$ is defined by

$$N_i(\mathbf{u}, \alpha) \equiv du_i/dt - f_i(\mathbf{u}, \alpha). \quad (3)$$

In (2) and (3) the components of the column vector

$$\mathbf{u}(t) = [u_1(t), u_2(t), u_3(t)] \quad (4)$$

are the dependent variables, and the components of the column vector

$$\alpha = (\alpha_1, \ldots, \alpha_P) \quad (5)$$

are the real scalars representing the parameters in the model.

The initial conditions at time $t = a$ associated with the set of differential equations represented by (2) are

$$B_i(\mathbf{u}, \alpha) = 0, \quad i = 1, 2, 3, \quad (6)$$

where

$$B_i(\mathbf{u}, \alpha) \equiv u_i(a) - \alpha_i. \quad (7)$$

Thus, as indicated by (6) and (7), the initial conditions are considered as the first three components of the vector of model parameters α.

For sensitivity analysis, the results of interest calculated with the model are customarily referred to as responses. As discussed in the next section, the method of sensitivity analysis to be presented relies on each response being expressed as a functional of $\mathbf{u}$ and α. The specific response functional $R(\mathbf{u}, \alpha)$ chosen in this work is of the form

$$R(\mathbf{u}, \alpha) = \int_a^b r(\mathbf{u}, \alpha, t)dt. \quad (8)$$

In this equation, b represents the final time value considered in the model, and $r(\mathbf{u}, \alpha, t)$ is a function of $\mathbf{u}$, α and t. The form of R given in (8) is sufficiently general to represent a wide variety of specific responses. For example, if the response is the surface air temperature [see (A5)] at some time c (where $a \leq c \leq b$), then r is given by

$$r(\mathbf{u}, \alpha, t) = \mathbf{d} \cdot \mathbf{u}\delta(t - c), \quad (9)$$

where $\mathbf{d} = (-0.5, 1.5, 0)$. Note that the traditional definition of the response for an RCM corresponds to $c \to \infty$. In practice, c is a sufficiently large but *finite* time value at which equilibrium is judged to have been reached.

3. Sensitivity analysis: Adjoint operator approach

The purpose of sensitivity analysis is to assess changes in each response (i.e., result) when variations are made in the parameters of the model. The adjoint method of sensitivity analysis presented in this section relies on each response being expressed as a functional of the dependent variables. It would be possible to define the response to be the fields of the dependent variables themselves, but in this case a more sophisticated version of the adjoint method is required (Cacuci, 1981b). Note that if the response R is solely a *function* of the scalar parameters α [i.e., $R = R(\alpha)$], then the sensitivity of R to variations $\mathbf{g}$ around the "base-case" parameter values α^0 can be expressed in terms of the partial derivatives $\partial R(\alpha)/\partial\alpha_i$ evaluated at α^0. These derivatives can then be used in a first-order Taylor expansion to estimate the change in the value of the response:

$$R(\alpha^0 + \mathbf{g}) - R(\alpha^0) = \sum_{i=1}^{P} (g_i\{\partial R(\alpha)/\partial\alpha_i\}_{\alpha^0} + O[g_i^2]). \quad (10)$$

[Here, and in the following, $\{\lambda\}_\mu$ means "λ evaluated at μ".] The response R given by (8) and (9), though, depends on both α and $\mathbf{u}(t)$, where $\mathbf{u}(t)$ and α are implicitly related through the system of nonlinear differential equations (2) and (6). Therefore the sensitivity can no longer be defined in terms of ordinary partial derivatives, and the more general concepts of functional analysis are used to define the sensitivity.

The most general definition of the sensitivity of a response to variation in the system parameters is the Gateaux-(G-) differential (see, e.g., Cacuci, 1981a). The G-differential $VR(\mathbf{u}^0, \alpha^0; \mathbf{h}, \mathbf{g})$ of $R(\mathbf{u}, \alpha)$ at $(\mathbf{u}^0, \alpha^0)$ with increment $(\mathbf{h}, \mathbf{g})$ is given by

$$VR(\mathbf{u}^0, \alpha^0; \mathbf{h}, \mathbf{g}) = \lim_{\epsilon \to 0} \{R(\mathbf{u}^0 + \epsilon\mathbf{h}, \alpha^0 + \epsilon\mathbf{g}) - R(\mathbf{u}^0, \alpha^0)\}/\epsilon, \quad (11)$$

or, equivalently, by

$$VR(\mathbf{u}^0, \alpha^0; \mathbf{h}, \mathbf{g}) = \{(d/d\epsilon)R(\mathbf{u}^0 + \epsilon\mathbf{h}, \alpha^0 + \epsilon\mathbf{g})\}_{\epsilon=0}, \quad (12)$$

where $\mathbf{h}(t) = [h_1(t), h_2(t), h_3(t)]$ and $\mathbf{g} = (g_1, \ldots, g_P)$ are column vectors that are at this stage arbitrary, and ϵ is a real scalar. When $R(\mathbf{u}, \alpha)$ is uniformly continuous in $\mathbf{u}$ and α at $(\mathbf{u}^0, \alpha^0)$, and when $VR(\mathbf{u}^0, \alpha^0; \mathbf{h}, \mathbf{g})$ is linear in $\mathbf{h}$ and $\mathbf{g}$, it can be shown (see, e.g., Cacuci 1981a) that the total variation of R is given by

$$R(\mathbf{u}^0 + \mathbf{h}, \alpha^0 + \mathbf{g}) - R(\mathbf{u}^0, \alpha^0) = VR(\mathbf{u}^0, \alpha^0; \mathbf{h}, \mathbf{g}) + O[\|\mathbf{h}\|^2] + O[\|\mathbf{g}\|^2], \quad (13)$$

where $\|\ \|$ denotes an appropriate norm, e.g.,

$$\|\mathbf{h}\|^2 = \sum_{i=1}^{3} \int_a^b [h_i(t)]^2 dt.$$

Note that (13) is a more general form of (10).

The general concepts discussed above are used to determine the sensitivity of the response given by (8). The application to this equation of the definition of the G-differential given in (12) proceeds as follows:

$$VR(\mathbf{u}^0, \alpha^0; \mathbf{h}, \mathbf{g}) = \{(d/d\epsilon) \int_a^b r(\mathbf{u}^0 + \epsilon\mathbf{h}, \alpha^0 + \epsilon\mathbf{g}, t)dt\}_{\epsilon=0} = \int_a^b \mathbf{r}'_u \cdot \mathbf{h} dt + \int_a^b \mathbf{r}'_\alpha \cdot \mathbf{g} dt, \quad (14)$$

where $\mathbf{r}'_u$ is the column vector

$$\mathbf{r}'_u = \{(\partial r/\partial u_1, \partial r/\partial u_2, \partial r/\partial u_3)\}_{(\mathbf{u}^0, \alpha^0)}, \quad (15)$$

and $\mathbf{r}'_\alpha$ is the column vector

$$\mathbf{r}'_\alpha = \{(\partial r/\partial \alpha_1, \ldots, \partial r/\partial \alpha_P)\}_{(\mathbf{u}^0, \alpha^0)}. \quad (16)$$

Here $\mathbf{u}^0(t)$ represents the solution of (2) and (6) that is obtained using the base-case parameter values α^0. Note that the derivatives appearing in (15) and (16) will always exist in the applications to be presented in this work [cf. (9)]. Also note that (14) is linear in both $\mathbf{h}$ and $\mathbf{g}$. It follows that the response given by (8) and (9) satisfies the relationship given in (13).

When performing sensitivity analysis, the vector $\mathbf{g}$ of variations around the base case parameter values α^0 is chosen at the outset. The sensitivity $VR(\mathbf{u}^0, \alpha^0; \mathbf{h}, \mathbf{g})$, though, can be evaluated only after determining the corresponding vector $\mathbf{h}(t)$ of variations around the base-case solution $\mathbf{u}^0(t)$. The first-order relationship between $\mathbf{h}$ and $\mathbf{g}$ is obtained by taking G-differentials of (2) and (6). The explicit result of the application of the definition in (12) to (2) and (6) is

$$dh_i/dt - \sum_{j=1}^{3} h_j\{\partial f_i(\mathbf{u}, \alpha)/\partial u_j\}_{(\mathbf{u}^0, \alpha^0)} = \sum_{k=1}^{P} g_k\{\partial f_i(\mathbf{u}, \alpha)/\partial \alpha_k\}_{(\mathbf{u}^0, \alpha^0)}, \quad i = 1, 2, 3, \quad (17)$$

$$h_i(a) = g_i, \quad i = 1, 2, 3. \quad (18)$$

In principle, given an arbitrary vector of parameter variations $\mathbf{g}$, (17) and (18) can be solved to obtain $\mathbf{h}(t)$. This value of $\mathbf{h}(t)$ can then be used in (14) to evaluate the sensitivity VR. In practice, though, since there are P linearly independent choices of $\mathbf{g}$, a complete sensitivity analysis using this procedure would involve solving (17) and (18) anew P times. (This procedure would represent a generalization of the direct method mentioned in the Introduction.) An alternative procedure to evaluate the sensitivity VR that avoids the need to repeatedly solve (17) and (18) is developed by applying the theory presented by Cacuci (1981a). This theory is based on the use of adjoint operators.

To introduce these adjoint operators, (17) is rewritten in matrix form, i.e.

$$\mathbf{L}\mathbf{h} = \mathbf{Q}\mathbf{g}, \quad (19)$$

where the components L_{ij} of the 3×3 matrix $\mathbf{L}$ are

$$L_{ij} = \delta_{ij} d/dt - \partial f_i/\partial u_j, \quad i, j = 1, 2, 3, \quad (20)$$

and the components Q_{ik} of the $3 \times P$ matrix $\mathbf{Q}$ are

$$Q_{ik} = \partial f_i/\partial \alpha_k, \quad i = 1, 2, 3, \quad k = 1, \ldots, P. \quad (21)$$

[The quantity δ_{ij} appearing in (20) is the customary Kronecker delta.] The operator $\mathbf{L}^*$ that is adjoint to $\mathbf{L}$ is introduced through the relationship

$$\int_a^b \mathbf{h} \cdot (\mathbf{L}^* \mathbf{v}) dt = \int_a^b \mathbf{v} \cdot (\mathbf{L}\mathbf{h}) dt - [\mathbf{h} \cdot \mathbf{v}]_a^b, \quad (22)$$

where $\mathbf{v} = [v_1(t), v_2(t), v_3(t)]$ is a column vector that at this stage is arbitrary. In view of (22) and (20),

the elements L_{ij}^* of the 3×3 matrix $\mathbf{L}^*$ are given by

$$L_{ij}^* = -\delta_{ij}d/dt - \partial f_j/\partial u_i, \quad i, j = 1, 2, 3. \quad (23)$$

The vector $\mathbf{v}$ is now chosen by identifying the term on the left side of (22) with the first term on the right side of (14). Note that this identification is only possible if the response is defined by a functional. [Cacuci has discussed this point in detail (1981a,b)]. This identification gives the relationship

$$\mathbf{L}^*\mathbf{v} = \mathbf{r}'_u. \quad (24)$$

The value of $\mathbf{h}(a)$ is known from (18); the unknown value of $\mathbf{h}(b)$ can be eliminated from (22) by choosing

$$\mathbf{v}(b) = \mathbf{0}. \quad (25)$$

Eqs. (24) and (25) uniquely determine the adjoint function $\mathbf{v}$. Note that these equations are independent of $\mathbf{h}$ and $\mathbf{g}$.

In view of (24), (25), (18) and (19), Eq. (22) can be written

$$\int_a^b \mathbf{h}\cdot\mathbf{r}'_u dt = \int_a^b \mathbf{v}\cdot(\mathbf{Q}\mathbf{g})dt + \sum_{j=1}^{3} g_j v_j(a). \quad (26)$$

Comparison of (26) and (14) shows that the sensitivity VR is given by

$$VR = \int_a^b \mathbf{r}'_\alpha \cdot \mathbf{g}dt + \int_a^b \mathbf{v}\cdot(\mathbf{Q}\mathbf{g})dt + \sum_{j=1}^{3} g_j v_j(a). \quad (27)$$

Note that this expression is independent of $\mathbf{h}$. Once the single calculation to determine the adjoint function $\mathbf{v}$ has been performed, (27) can be used to efficiently evaluate the sensitivity VR of R for any vector of variations $\mathbf{g}$ around the base case parameter values α^0.

If a variation occurs solely in the nth parameter, then the vector of parameter variations $\mathbf{g}$ is denoted by $\mathbf{g}^n$, where

$$\mathbf{g}^n = (0, \ldots, g_n, \ldots, 0). \quad (28)$$

In this case, the corresponding sensitivity VR is denoted by VR^n, where

$$VR^n = VR(\alpha^0, \mathbf{u}^0; \mathbf{h}, \mathbf{g}^n). \quad (29)$$

The value of VR^n can be used in conjunction with (13) to predict the change in the response when the value of the nth parameter is changed by an amount g_n.

The *relative* sensitivity of R to α_n^0, denoted by s_n, is defined as the dimensionless quantity

$$s_n = \frac{VR^n}{R} \Big/ \frac{g_n}{\alpha_n^0}. \quad (30)$$

In view of (13) and (30), the relative sensitivity s_n is a measure of the ratio between the fractional change in R and the corresponding fractional change in α_n^0. In practice, relative sensitivities are used to rank the importance of parameters.

Sensitivities can also be used to calculate the uncertainty in R due to uncertainties in the parameters. If α^0 represents the nominal values of the parameters, and $\mathbf{g}$ represents the error in the parameters, then the corresponding error in R is given, to first order in $\mathbf{g}$, by $VR(\mathbf{u}^0, \alpha^0; \mathbf{h}, \mathbf{g})$. Consequently, the standard deviation $\sigma(R)$ of R is given by

$$\sigma(R) = (\langle VR^2 \rangle)^{1/2}, \quad (31)$$

where the angle braces denote expected value. Since VR is linear in $\mathbf{g}$ [see (27)], VR can be expressed as the sum of VR^n over n, i.e.

$$VR = \sum_{n=1}^{P} VR^n. \quad (32)$$

Using this relationship, (31) can be recast in terms of relative sensitivities:

$$\sigma(R)/R = [\mathbf{s}^T\mathbf{C}\mathbf{s}]^{1/2}. \quad (33)$$

In this equation the elements of $\mathbf{s}$ are the relative sensitivities defined in (30), and $\mathbf{C}$ is the relative covariance matrix with elements given by

$$C_{nm} = \langle g_n g_m \rangle / (\alpha_n^0 \alpha_m^0), \quad n, m = 1, \ldots, P. \quad (34)$$

Note that the square root of the nth diagonal element of $\mathbf{C}$ is the relative standard deviation of the nth parameter. This procedure for evaluating $\sigma(R)/R$ is customarily called uncertainty analysis.

For illustrative purposes, the explicit form of the adjoint system and the expression for VR are given below for the particular case when $r(\mathbf{u}, \alpha, t)$ is specified by (9). Since $\mathbf{r}'_u = \mathbf{d}\delta(t - c)$, the adjoint system [i.e. (24) and (25)] becomes

$$\left.\begin{aligned} -dv_i/dt - \sum_{j=1}^{3} v_j\{\partial f_j/\partial u_i\}_{(\mathbf{u}^0,\alpha^0)} &= d_i\delta(t-c) \\ v_i(b) &= 0, \quad i = 1, 2, 3 \end{aligned}\right\}. \quad (35)$$

Since $\mathbf{r}'_\alpha = \mathbf{0}$ and $Q_{ik} = \partial f_i/\partial \alpha_k$, the expression for VR [i.e. (27)] becomes

$$VR = \int_a^b \sum_{i=1}^{3}\sum_{j=1}^{P} v_i g_j \{\partial f_i/\partial \alpha_j\}_{(\mathbf{u}^0,\alpha^0)} dt + \sum_{j=1}^{3} g_j v_j(a). \quad (36)$$

Since radiative-convective processes do not depend explicitly on initial conditions, f_1, f_2 and f_3 are independent of α_1, α_2 and α_3. In this case, (36) yields the following expressions for the sensitivities VR^n:

$$VR^n = g_n v_n(a), \quad n = 1, 2, 3, \quad (37)$$

TABLE 1. Sensitivity of surface air temperature to initial conditions.

Time after $t = a$ (days)		3		300		800	
Surface air temperature (Response)		292.63 K		283.32 K		283.28 K	
		Change in response (K)		Change in response (K)		Change in response (K)	
Parameter description	Fractional variation in parameter[a]	Predicted change[b]	Actual change[c]	Predicted change[b]	Actual change[c]	Predicted change[b]	Actual change[c]
Initial value $u_1(a)$	0.1%	9.42 − 2	8.37 − 2	3.29 − 4	3.39 − 4	2.45 − 8	2.45 − 8
Initial value $u_2(a)$	0.1%	2.00 − 1	2.24 − 1	7.10 − 4	9.24 − 4	5.28 − 8	6.70 − 8
Initial value $u_3(a)$	0.1%	4.53 − 3	4.61 − 3	1.63 − 5	1.58 − 5	1.21 − 9	1.15 − 9

[a] Fractional variation in parameter = g_n/α_n^0.
[b] Predicted change = VR^n. The second part of the number represents a power of 10; e.g., $9.42 - 2 = 9.42 \times 10^{-2}$.
[c] Actual change = $R(\mathbf{u}^0 + \mathbf{h}, \alpha^0 + \mathbf{g}) - R(\mathbf{u}^0, \alpha^0)$; also see footnote b.

$$VR^n = \int_a^b \sum_{i=1}^{3} v_i g_n \{\partial f_i/\partial \alpha_n\}_{(\mathbf{u}^0,\alpha^0)} dt, \qquad n = 4, \ldots, P. \tag{38}$$

In view of (13) and (37), the value of the adjoint function v_n at $t = a$ is the ratio of the variation in the response to the variation in the initial value of the dependent variable u_n at $t = a$. Numerical aspects of the solution of the adjoint system given by (35), and of the evaluation of the sensitivities given by (38) are discussed in Appendix B.

For a response that is a linear combination of two other responses, i.e.,

$$R = \gamma R_1 + \beta R_2, \tag{39}$$

application of (12) gives

$$VR = \gamma VR_1 + \beta VR_2. \tag{40}$$

This equation can be used to obtain the sensitivities for a CO_2 doubling experiment by taking R_1 as the response with the normal concentration of CO_2, R_2 as the response with double this concentration, and by setting γ to -1 and β to 1.

4. Results

The theory developed in Section 3 has been used to evaluate the sensitivities of several responses to all 312 parameters in the model. Presented in Table 1 are the sensitivities of the surface air temperature to the initial conditions; the surface air temperature and the corresponding sensitivities are evaluated at 3, 300 and 800 days after $t = a$. Table 1 serves two purposes. As outlined in Section 3, Eqs. (13) and (37) enable the accuracy of the numerical calculations of the adjoint functions to be assessed by considering the sensitivities to the initial conditions. The results presented in the columns labeled "Actual change" represent the changes in the surface air temperature after actually varying each initial condition in turn by 0.1% and rerunning the model. [The small variations of 0.1% were chosen to reduce the effect of higher order terms in (13)]. The results in the columns labeled "Predicted change" are the values of the sensitivities VR^n calculated by using (37). These results are related directly to the value of the adjoint functions at $t = a$. The good agreement between the actual and predicted changes shown in Table 1 gives confidence in the adequacy of the numerical method used to solve the adjoint system.

The other purpose served by Table 1 is to show that as time increases, the absolute values of the sensitivities decrease. This confirms the fact that if the calculation is extended over sufficiently long times, the results become independent of initial conditions.

Presented in Table 2 are the sensitivities of the equilibrium surface air temperature to the solar constant, the surface albedo, the relative humidity at the surface, and the atmospheric CO_2 concentration. As shown in this table, the relative sensitivity of the surface air temperature to the solar constant is 0.381. Note that if the Stefan-Boltzmann law were valid for the surface air temperature, then the relative sensitivity would be exactly 0.25. The value of 0.381 obtained for this model indicates the existence of a feedback mechanism that enhances temperature changes. Such a mechanism is the effect of constant relative humidity: an increase in temperature will increase the moisture content of the atmosphere, and hence increase the "greenhouse" effect. To verify this, the sensitivity was calculated with fixed absolute humidity, and the value was found to be 0.234. [The change in sensitivity due to fixed relative humidity was first pointed out by Manabe and Wetherald (1967).] Note that the agreement between predicted changes [calculated using (38)] and the actual changes is excellent for both 0.1% and 10% variations in the solar constant. This indicates that the surface temperature varies linearly with the solar constant, at least for variations in this range.

Table 2 also shows that, as expected, the sensitivity

TABLE 2. Sensitivity of equilibrium surface air temperature to physically significant parameters.

Parameter description	Relative sensitivity[a]	Fractional variation in parameter	Predicted change[b] in response (K)	Actual change[c] in response (K)
		Equilibrium surface temperature = 283.28 K (Response)		
Solar constant	0.381	0.1%	0.108	0.108
		10%	10.8	10.8
Surface albedo	−0.0299	0.1%	−0.00847	−0.00850
		10%	−0.847	−0.853
Relative humidity at surface	0.0285	0.1%	0.00807	0.00811
		10%	0.807	0.795
Atmospheric CO_2 concentration	0.0121	0.1%	0.00343	0.00343
		100%	3.43	2.42

[a] Relative sensitivity = $(VR^n/R)/(g_n/\alpha_n^0)$.
[b] Predicted change = VR^n.
[c] Actual change = $R(\mathbf{u}^0 + \mathbf{h}, \alpha^0 + \mathbf{g}) - R(\mathbf{u}^0, \alpha^0)$.

to the surface albedo is negative; in more complicated models this causes the ice-albedo feedback mechanism. The sensitivity to the relative humidity at the surface is positive confirming the greenhouse effect caused by water vapor. The sensitivity to the CO_2 concentration was used to predict the result of a CO_2 doubling experiment (i.e., a change of 100%). The discrepancy between the predicted change of 3.43 K and the actual change of 2.42 K arises from neglecting the higher order terms in (13).

The equilibrium surface temperature is most sensitive to a parameter that is used to calculate the saturation vapor pressure of water. The results given in Table 3 illustrate the nonlinear dependence of the equilibrium surface air temperature on this parameter. For small variations in this parameter (i.e., 0.1%), the predicted change [again calculated using (38)] agrees well with the actual change in the response. By contrast, for larger variations (i.e., 1.0%), the nonlinear behavior becomes clear; for positive parameter variations the change is overpredicted, while for negative parameter variations the change is underpredicted. Once again the differences between predicted and actual change arise from neglecting higher order terms in (13).

In general, the magnitude of the relative sensitivities can be used to rank the parameters. In the table appearing in Appendix C, the 152 parameters with non-zero sensitivities are ranked in order of decreasing magnitude of relative sensitivity. The four largest sensitivities all correspond to parameters used to calculate the saturation vapor pressure of water. The next largest sensitivities correspond to parameters used to characterize the solar radiation and to determine transmission functions. Almost an order of magnitude smaller are the sensitivities to parameters that determine lapse rate adjustment, surface albedo, stratospheric temperature, and surface heat exchange. Only 89 of the sensitivities were greater than 0.001.

To use sensitivities for uncertainty analysis requires a knowledge of parameter covariances [see (33) and (34)]. Typically, parameter covariances are not readily available, and they must be obtained by referring to the origin of the parameters' values. Usually the largest contribution to the uncertainty in the response arises from parameters that have the largest product of standard deviation and sensitivity. Although ultimately all the parameters should be considered in an uncertainty analysis, a practical approach is to consider first those parameters that either have a high relative sensitivity, or that are suspected to have a high standard deviation.

In this vein for the RCM, the parameters identified in Section 4 and Appendix C as having the highest relative sensitivities [*viz.*, coefficients in (A14)] were investigated to obtain their uncertainties. These parameters were obtained from the work of Lowe and Ficke (1974). Lowe and Ficke evaluated the parameters from basic data obtained from Goff and Gratch (1946). Although Lowe and Ficke discuss uncertainties in the basic data, the quoted uncertainty "does not apply below 0°C where no experi-

TABLE 3. Sensitivity of equilibrium surface air temperature to coefficient $A_{1,3}$ in Eq. (A14) used to calculate the saturation vapor pressure of water.

Relative sensitivity	Fractional variation in parameter	Predicted change in response (K)	Actual change in response (K)
1.19	−1%	−3.37	−6.73
	−0.1%	−0.337	−0.390
	0.1%	0.337	0.328
	1%	3.37	2.10

mental data were available" (both atmospheric layers in the RCM have mid-level temperatures below 0°C). This lack of experimental data illustrates that a major cooperative effort between modelers and data evaluators is needed to obtain parameter uncertainties, even if only the parameters with large sensitivities are considered.

Once covariance information has been obtained for the parameters, the corresponding sensitivities can be used [see (33)] to estimate the resulting uncertainty in the response. For example, the relative sensitivity of equilibrium surface air temperature to the surface albedo is −0.0299. Assuming that the surface albedo (0.1 for this model) has a standard deviation of 10% leads to a standard deviation of 0.299% in equilibrium surface air temperature (i.e., 283.3 ± 0.8 K).

Uncertainty analysis can be applied equally well to the response of a CO_2 doubling experiment. To illustrate this, the response is considered to be the increase in surface air temperature after doubling CO_2. Using (40), the relative sensitivity of this response to surface albedo was found to be −0.0275. Assuming again that the surface albedo has a standard deviation of 10% leads to a standard deviation of 0.275% in the response (i.e., 2.42 ± 0.007 K). This indicates that an uncertainty of 10% in suface albedo leads to an insignificant uncertainty for the surface air temperature in this radiative-convective CO_2 doubling experiment.

Uncertainty analysis using a complete set of parameter covariances would enable assessment of the response uncertainty arising from the uncertainties in all the parameters. A knowledge of this response uncertainty is useful in comparative studies between results of the model and observation. If the discrepancy between the model and nature is much larger than the model uncertainty arising from the parameters, then important physical processes must have been modeled inadequately. If, though, all important physical processes and feedback mechanisms have, however crudely, been included, then uncertainty analysis will yield a realistic estimate of the uncertainty in the results of the model. Furthermore, when a comparison between the results of the model and observation is impractical (for example with a CO_2 doubling experiment), uncertainty analysis is the only systematic way to estimate the reliability of the model results.

The total computing time required for this model to reach equilibrium was 8 s on an IBM 3033. The additional computing time to calculate 312 sensitivities using the adjoint approach was 12 s (i.e., 5 s to calculate the adjoint functions and 7 s to calculate the derivatives $\partial f_i/\partial \alpha_j$, $i = 1, 2, 3$; $j = 1, \ldots, 312$). The additional computing time needed to obtain the same information by rerunning, with each of the parameters changed in turn, would have been about 40 min.

5. Summary and conclusions

This work has demonstrated the adjoint method of sensitivity analysis on a radiative-convective model (RCM). The results for this model have shown the method to be both efficient and accurate. A single adjoint calculation, involving a similar amount of computation as the original model, sufficed to obtain sensitivities of surface air temperature to all 312 model parameters. The uses of sensitivities have been discussed and illustrated. In this context, the sensitivities accurately predicted the effect of small variations in the model parameters. Relative sensitivities were used to rank the importance of all the parameters. Sensitivities to parameters that have customarily been investigated in previous works were reproduced, although sensitivities to constants used to compute the saturation vapor pressure of water were found to be the largest. Finally it has been shown that both sensitivities and covariances for all the parameters are needed to assess the uncertainties in model results. The difficulty of estimating parameter covariances has been illustrated, but the importance of obtaining this information has been emphasized. For a CO_2 doubling experiment, uncertainty analysis using sensitivities and covariances for all model parameters is the only systematic way to estimate the reliability of the model results. This estimate, though, will only be realistic if all important physical processes and feedback mechanisms have been represented, even if approximately, in the model.

Radiative-convective models include many of the complex nonlinear processes that are found in general circulation models (GCM's). This work has shown that the adjoint method can deal with such complex time-dependent processes successfully and efficiently, a finding that provides valuable information for subsequent application of the adjoint method to the sensitivity analysis of a GCM. This subsequent application, though, raises several new issues; GCM's involve many variables, the results can be functions of position, and the solution only reaches a statistical "quasi-equilibrium." It is expected that high sensitivities, such as those found for parameters used to calculate the saturation vapour pressure of water, will also require consideration of second and higher order effects. Further research is underway to investigate these issues.

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APPENDIX A

Description of the Radiative-Convective Model (RCM)

In the RCM the atmosphere is divided into two layers of equal mass between the earth's surface with pressure p_4 (=1000 mb), and the level 0 surface, with pressure p_0 (=200 mb) (see Fig. 1). The pressure thickness of each layer is given by $\Delta p = (p_4 - p_0)/2$. The mass-averaged temperatures of the layers are determined prognostically at their mid-levels by

$$\left.\begin{aligned} C_1 \frac{\partial T_1}{\partial t} &= R_0 - R_2 + Q_2 \\ C_3 \frac{\partial T_3}{\partial t} &= R_2 - R_4 + Q_4 - Q_2 \end{aligned}\right\}, \qquad \text{(A1)}$$

where $C_k = c_{pk}\Delta p/g$, with c_{pk} the equivalent specific heat for moist air at constant pressure and g the acceleration of gravity; t is the time; R_l the net downward flux of solar plus terrestrial radiation at level l (=0, 2, 4); Q_4 the upward heat flux from the earth's surface to the atmosphere; and Q_2 the upward heat flux from the lower to the upper layer required to prevent the lapse rate $\Gamma = -(\partial T/\partial z)$ from exceeding a prescribed critical value Γ_c (=6.5°C km^{-1}).

The state of radiative-convective equilibrium is computed as an asymptotic state by integrating (A1) in time from some prescribed initial state at $t = a$. For each time step Δt, the temperatures are first changed in response to the radiative heating, i.e.,

$$T_k^{(n+1)*} = T_k^{(n)} + \frac{\Delta t}{C_k}\left[R_{k-1}^{(n)} - R_{k+1}^{(n)}\right], \qquad k = 1, 3, \qquad \text{(A2)}$$

where the superscripts are the time-step index, and $t^{(n)} = a + n\Delta t$. The solar and terrestrial radiative fluxes which comprise R_l (l = 0, 2, 4) are determined from the radiative transfer model of the Oregon State University atmospheric general circulation model (Schlesinger and Gates, 1980) for the atmospheric state at time $t^{(n)}$.

Next, the temperature of the lower layer is changed in response to the heating from the earth's surface:

$$T_3^{(n+1)**} = T_3^{(n+1)*} + \frac{\Delta t}{C_3} Q_4^{(n)}, \qquad \text{(A3)}$$

where $Q_4^{(n)}$ is obtained as a Newtonian heating

$$Q_4^{(n)} = C_s[T_s^{(n)} - T_4^{(n)}] \qquad \text{(A4)}$$

with prescribed heat transfer coefficient C_s (=10 ly day^{-1}K^{-1}). The surface air temperature $T_4^{(n)}$ is obtained by linear extrapolation of the temperature profile with respect to pressure, i.e.,

$$T_4^{(n)} = \frac{3}{2} T_3^{(n)} - \frac{1}{2} T_1^{(n)}. \qquad \text{(A5)}$$

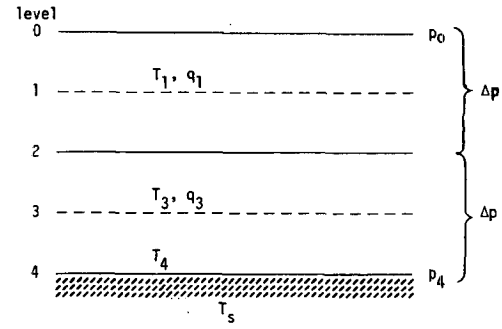

FIG. 1. Schematic of the radiative-convective model.

The temperature of the earth's surface $T_s^{(n)}$ satisfies (at equilibrium) the surface energy budget

$$\sigma[T_s^{(n)}]^4 + Q_4^{(n)} = DR_4^{(n)}, \qquad \text{(A6)}$$

where $\sigma[T_s^{(n)}]^4$ is the upward flux of terrestrial radiation, with σ the Stefan-Boltzmann constant, which, along with $Q_4^{(n)}$, is required to balance the absorbed solar plus downward terrestrial radiation at the earth's surface $DR_4^{(n)}$. Because of the dependence of $Q_4^{(n)}$ on $T_s^{(n)}$, Eq. (A6) is an implicit equation which must be solved by iteration. Rather than do that at each time step during the integration toward equilibrium, (A6) is made explicit as

$$\sigma[T_s^{(n)}]^4 = DR_4^{(n-1)} - Q_4^{(n-1)}; \qquad \text{(A7)}$$

this computational expedient is justified because (A7) is identical to (A6) at equilibrium.

Finally, the temperature of each layer is changed due to the interlayer heat flux Q_2 which is required to prevent a supercritical lapse rate $\Gamma > \Gamma_c$, i.e.,

$$\left.\begin{aligned} T_1^{(n+1)} &= T_1^{(n+1)*} + \frac{\Delta t}{C_1} Q_2 \\ T_3^{(n+1)} &= T_3^{(n+1)**} - \frac{\Delta t}{C_3} Q_2 \end{aligned}\right\}. \qquad \text{(A8)}$$

Using the hydrostatic equation it can be shown that the temperature $(T_1)_c$ at the isobaric level 1 for the critical lapse rate Γ_c is related to a given T_3 by

$$(T_1)_c = T_3\left(\frac{P_1}{P_3}\right)^{\Gamma_c R/g}, \qquad \text{(A9)}$$

where R is the gas constant. Thus Q_2 is zero, except when $T_1^{(n+1)*}$ is less than the value of $(T_1)_c$ given by (A9) with T_3 set to $T_3^{(n+1)**}$. In the latter case, Q_2 is determined by (A8) and the condition for the critical lapse rate is

$$\frac{T_1^{(n+1)}}{T_3^{(n+1)}} = \left(\frac{P_1}{P_3}\right)^{\Gamma_c R/g}. \qquad \text{(A10)}$$

In the RCM the insolation at the top of the atmosphere (p = 0 mb) is

$$S_0 = \bar{S}_0\left(\frac{\bar{r}_e}{r_e}\right)^2 F_d \overline{\cos\zeta}, \tag{A11}$$

where $\bar{S}_0$ (=2793.6 ly day^{-1}) is the solar constant at one astronomical unit, $\bar{r}_e$; r_e is the earth-sun distance ($r_e = \bar{r}_e$); $\overline{\cos\zeta}$ (=0.5) is an averaged cosine of the solar zenith angle ζ; and F_d (=0.5) is an averaged fraction of a day that the sun is above the horizon. The insolation is absorbed by a prescribed amount of ozone Ω (=367 Dobson units) above the p_0 surface, by water vapor below the p_0 surface, and by the earth's surface with a prescribed albedo α_s (=0.1). The terrestrial radiation is emitted and absorbed by water vapor and a prescribed CO_2 concentration μ_{CO_2} (=320 ppmv). The water vapor mixing ratios q_k are determined from

$$q_k = q_k^* \, \mathrm{RH}_k, \quad k = 1, 3, \tag{A12}$$

where RH_k is the relative humidity and

$$q_k^* = 0.622 \frac{e^*(T_k)}{p_k - e^*(T_k)} \tag{A13}$$

is the saturation mixing ratio. The saturation vapor pressure e^* is obtained from the relation given by the polynomial fit of Lowe and Ficke (1974), i.e.,

$$e^*(T) = \sum_{i=1}^{7} A_{j,i} T_c^{i-1}, \tag{A14}$$

where

$$T_c = T - 273.155 \text{ K} \quad \text{and}$$
$$j = \begin{cases} 1 & \text{if } T_c \geq -50°\text{C} \\ 2 & \text{if } T_c < -50°\text{C}. \end{cases} \tag{A15}$$

The relative humidities are prescribed from the profile of Manabe and Wetherald (1967),

$$\mathrm{RH}_k = 0.77\left(\frac{\frac{p_k}{p_4} - 0.02}{1 - 0.02}\right), \quad k = 1, 3. \tag{A16}$$

By prescribing the relative humidities rather than the mixing ratios, the water vapor content becomes a variable which depends on temperature. Although the radiative transfer model treats both convective (cumuloform) and large-scale (stratiform) clouds in both the solar and terrestrial radiative calculations, the clouds are prescribed not to exist in the present study. (In a subsequent investigation the stratiform clouds are made time dependent through an existence criterion that RH_k exceed some prescribed critical RH, and by prescribing q_k rather than RH_k.)

The difference equations (A.2), (A.3) and (A.8) for T_1 and T_3, and the explicit equation (A7) for T_s, can all be written in the form

$$T_k^{(n+1)} = T_k^{(n)} + F_k[T_1^{(n)}, T_3^{(n)}, T_s^{(n)}, \alpha]\Delta t,$$
$$k = 1, 3, s, \tag{A17}$$

where the F_k are functions of T_1, T_3, T_s and the vector $\alpha = (\alpha_1, \ldots, \alpha_p)$ contains the system parameters. Eq. (A17) represents the numerical method of solving the set of coupled first-order equations

$$\frac{d\mathbf{u}}{dt} = f(\mathbf{u}, \alpha), \tag{A18}$$

where $\mathbf{u} = (T_1, T_3, T_s)$, and $f = (F_1, F_3, F_s)$.

APPENDIX B

Numerical Aspects of the Adjoint Method

The adjoint system is given in Section 3 [*viz.* (35)]. These equations are solved numerically by using the modified Euler's method. The true solution $\mathbf{v}(t)$ is approximated at $(M + 1)$ evenly spaced values of time, $t = a, t^1, \ldots, t^{M-1}, b$. The time interval is given by

$$\Delta t = (b - a)/M, \tag{B1}$$

and the discrete times are given by

$$t^m = a + m\Delta t, \quad m = 0, \ldots, M. \tag{B2}$$

Integrating (35) from t^{m-1} to t^m gives

$$-[v_i(t^m) - v_i(t^{m-1})] - \int_{t^{m-1}}^{t^m} \sum_{j=1}^{3} v_j(t)\{\partial f_j/\partial u_i\}_{(\mathbf{u}^0,\alpha^0,t)} dt = \begin{cases} d_i, & t^m > c > t^{m-1} \\ 0, & \text{otherwise.} \end{cases} \tag{B3}$$

If the approximate value of $v_i(t^m)$ is denoted by v_i^m, then the modified Euler's method gives the following relationship between v_i^m and v_i^{m-1}:

$$-(v_i^m - v_i^{m-1}) - \Delta t \sum_{j=1}^{3} \bar{v}_j\{\partial f_j/\partial u_i\}_{(\mathbf{u}^0,\alpha^0,t^{m-1/2})} = \begin{cases} d_i, & t^m > c > t^{m-1} \\ 0, & \text{otherwise,} \end{cases} \tag{B4}$$

where $\bar{v}_j$ is given by

$$-(v_j^m - \bar{v}_j) - (\Delta t/2) \sum_{k=1}^{3} v_k\{\partial f_k/\partial u_j\}_{(\mathbf{u}^0,\alpha^0,t^m)} = \begin{cases} d_j, & t^m > c > t^{m-1/2} \\ 0, & \text{otherwise.} \end{cases} \tag{B5}$$

These equations are solved sequentially for v_i, $i = 1$, 2, 3 and $m = M, \ldots, 0$, starting from the final time condition

$$v_i^M = 0, \quad i = 1, 2, 3. \tag{B6}$$

Note that the adjoint equations are solved by incrementing backward in time. Also note that (B4) and (B5) require information about the base-case solution through the values of $\{\partial f_j/\partial u_i\}_{(\mathbf{u}^0,\alpha^0,t)}$. In practice, these values are obtained during the base-case

calculation using relationships of the type

$$\partial f_j/\partial u_i = [f_j(\ldots, u_i^0 + \epsilon, \ldots, \alpha^0) - f_j(\mathbf{u}^0, \alpha^0)]/\epsilon \quad \text{(B7)}$$

for small ϵ; in this work the value of ϵ was chosen to be $f_i \Delta t$.

The sensitivities themselves are calculated using (37) and (38). Eq. (38) is approximated using the trapezoidal rule to give

$$VR^n = \Delta t \sum_{m=0}^{M} w_m \sum_{i=1}^{3} v_i^m g_n \{\partial f_i/\partial \alpha_n\}_{(\mathbf{u}^0, \alpha^0, t^m)}, \quad \text{(B8)}$$

where

$$w_m = \begin{cases} \frac{1}{2}, & m = 0, M, \\ 1, & \text{otherwise.} \end{cases}$$

The values of $\partial f_i/\partial \alpha_n$ are obtained in the same way as are $\partial f_j/\partial u_i$, but with $\epsilon = 0.001\alpha_n^0$.

APPENDIX C

Detailed Sensitivity Results for Equilibrium Surface Air Temperature

Table 4 ranks the parameters in the radiative-convective model in order of decreasing absolute value

TABLE 4. Parameters ranked in order of decreasing magnitude of relative sensitivity.

	Surface air temperature (Response) = 283.28 K	
Ranking	Parameter description	Relative sensitivity*
1	Coefficient $A_{1,3}$ = 1.43 − 2 mb (°C)$^{-2}$ in (A14) for saturation vapor pressure of water	1.19
2	Coefficient $A_{1,2}$ = 4.44 − 1 mb (°C)$^{-1}$ in (A14) for saturation vapor pressure of water	−1.04
3	Coefficient $A_{1,4}$ = 2.65 − 4 mb (°C)$^{-3}$ in (A14) for saturation vapor pressure of water	−9.49 − 1
4	Conversion constant 273.155 from K to °C in (A15)	−6.80 − 1
5	F_d = (Total daylight hours)/24 = 0.5 in (A11)	4.16 − 1
6	Average cosine of solar zenith angle $\overline{\cos \zeta}$ = 0.5 in (A11)	4.16 − 1
7	Coefficient $A_{1,5}$ = 3.03E − 6 mb (°C)$^{-4}$ in (A14) for saturation vapor pressure of water	4.03 − 1
8	Stefan-Boltzman constant σ = 1.17 − 7 ly day^{-1} K^{-1} in (A7)	−4.00 − 1
9	Constant 0.930 in CO_2 transmission function τ_{CO_2} for terrestrial radiation [see Eq. (II.38) of Katayama (1972)]	−3.93 − 1
10	Constant 1.09 in CO_2 transmission function τ_{CO_2} for terrestrial radiation [see Eqs. (II.40, 41) of Katayama (1972)]	−3.92 − 1
11	Solar constant S_0 = 2793.6 ly day^{-1} in (A11)	3.81 − 1
12	Earth-sun distance factor $(\bar{r}_e/r_e)^2$ = 1 in (A11)	−3.80 − 1
13	Coefficient $A_{1,1}$ = 6.11 mb in (A14) for saturation vapor pressure of water	3.68 − 1
14	Constant 0.297 in water vapor transmission function $\tilde{\tau}_{H_2O}$ for terrestrial radiation [see Eq. (II.37) of Katayama (1972)]	−2.34 − 1
15	Constant 0.373 in water vapor transmission function $\tau^4_{H_2O}$ for terrestrial radiation [see Eq. (II.34) of Katayama (1972)]	−1.13 − 1
16	Coefficient $A_{1,6}$ = 2.03 − 8 mb (°C)$^{-5}$ in (A14) for saturation vapor pressure of water	−1.09 − 1
17	Constant 0.028 in water vapor transmission function $\tilde{\tau}_{H_2O}$ for terrestrial radiation [see Eq. (II.37) of Katayama (1972)]	8.13 − 2
18	Tropospheric pressure difference $p_4 - p_0$ = 800 mb	7.61 − 2
19	Constant 0.259 in water vapor transmission function $\tau^4_{H_2O}$ for terrestrial radiation [see Eq. (II.34) of Katayama (1972)]	−7.09 − 2
20	Fraction of solar insolation subject to water vapor absorption [constant $1 - f$ = 0.366 in Eqs. (33), (34) of Schlesinger and Gates (1979)]	6.00 − 2
21	Constant 0.194 in water vapor transmission function $\tilde{\tau}_{H_2O}$ for terrestrial radiation [see Eq. (II.37) of Katayama (1972)]	−4.15 − 2
22	Pressure p_0 = 200 mb	4.01 − 2
23	Stratospheric temperature T_{st} = 218K for terrestrial radiation calculation [see Eq. (27a) of Schlesinger and Gates (1979)]	3.81 − 2

TABLE 4. (*Continued*)

Ranking	Parameter description	Relative sensitivity*
	Surface air temperature (Response) = 283.28 K	
24	Constant 0.247 in albedo α_0 of cloudless sky due to Rayleigh scattering [see Eq. (39) of Schlesinger and Gates (1979)]	−3.73 − 2
25	Acceleration of gravity g = 9.8 m s^{-2}	−3.68 − 2
26	Constant 0.066 in CO_2 transmission function τ_{CO_2} for terrestrial radiation [see Eq. (II.38) of Katayama (1972)]	3.40 − 2
27	Constant 0.16 in water vapor plus CO_2 transmission function $\bar{\tau}_T$ for terrestrial radiation [see Eqs. (II.18, 23, 27) of Katayama (1972)]	3.34 − 2
28	Constant 9 × 10^{-5} in CO_2 transmission function τ_{CO_2} for terrestrial radiation [see Eqs. (II.40, 41) of Katayama (1972)]	3.23 − 2
29	Surface albedo α_s = 0.1	−2.99 − 2
30	Constant 0.622 in the saturation mixing ratio equation (A13)	2.86 − 2
31	Relative humidity at the surface, constant 0.77 in (A16)	2.86 − 2
32	Gas constant R = 287 m^2 s^{-2} K^{-1}	2.47 − 2
33	Critical lapse rate Γ_c = 6.5°C km^{-1}	2.47 − 2
34	Constant 0.9 in calculation of effective amount of water vapor $u^*_{H_2O}$ [see Eq. (29a) of Schlesinger and Gates (1979)]	−2.33 − 2
35	Constant 0.085 in albedo α_0 of cloudless sky due to Rayleigh scattering [see Eq. (39) of Schlesinger and Gates (1979)]	−2.13 − 2
36	Constant f = 0.634 in the ozone transmission function T_{O_3} for solar radiation [see Eq. (A.108) of Schlesinger (1976)]	2.07 − 2
37	Constant $(1 - f)$ = 0.366 in the water vapor transmission function T_{H_2O} for solar radiation [see Eq. (A.100) of Schlesinger (1976)]	−1.52 − 2
38	Constant p_{st} = 100 mb in the calculation of the water vapor mixing ratio profile [see Eq. (30) of Schlesinger and Gates (1979)]	−1.49 − 2
39	Surface heat transfer coefficient C_s = 10 ly day^{-1} K^{-1} in Eq. (A4)	1.39 − 2
40	Constant −1.66 in water vapor plus CO_2 transmission function $\bar{\tau}_I^+$ for terrestrial radiation [see Eqs. (II.19, 23, 24) of Katayama (1972)]	−1.37 − 2
41	Coefficient $A_{1,7}$ = 6.14 − 11 mb (°C)$^{-6}$ in (A14) for saturation vapor pressure of water	1.28 − 2
42	CO_2 concentration μ_{CO_2} = 320 ppmv	1.21 − 2
43	CO_2 molecular weight × 10^{-7} (44 × 10^{-7}) in calculation of total CO_2 amount h [see Eq. (IV.18) of Katayama (1972)]	1.21 − 2
44	CO_2 density at NTP × 10^3 (1.977 g cm^{-3}) in calculation of total CO_2 amount h [see Eq. (IV.18) of Katayama (1972)]	−1.21 − 2
45	Molecular weight of air (28.966) in calculation of total CO_2 amount h [see Eq. (IV.18) of Katayama (1972)]	−1.21 − 2
46	Constant in calculation of total CO_2 amount h [see Eq. (IV.18) of Katayama (1972)]	1.21 − 2
47	Coefficient b = 4.08 − 2 (cm-NTP)$^{-1}$ in O_3 absorption function A'_{O_3} for solar radiation [see Table A.5 of Schlesinger (1976)]	−1.18 − 2
48	Total ozone Ω =0.367 atm-cm	−1.17 − 2
49	Coefficient b = 1.44 − 1 (g cm^{-2})$^{-1}$ in water vapor absorption function A'_{H_2O} for solar radiation [see Table A.4 of Schlesinger (1976)]	1.14 − 2
50	Constant 15.78 in water vapor plus CO_2 transmission function $\bar{\tau}_T$ for terrestrial radiation [see Eqs. (II.19, 23, 27) of Katayamas (1972)]	1.13 − 2
51	Constant −42.59 in water vapor plus CO_2 transmission function $\bar{\tau}_T$ for terrestrial radiation [see Eqs. (II.19, 23, 27) of Katayama (1972)]	−1.01 − 2
52–89	Remaining parameters; mostly constant in transmission functions — magnitude between	1.01 − 2
	and	1.00 − 3
90–152	magnitude between	1.00 − 3
153–312	and	0
		0

* The second part represents a power of 10; e.g., −9.49 − 1 = −9.49 × 10^{-1}.

of relative sensitivity. The response is the equilibrium surface air temperature; its base-case value is 283.28 K. A relative sensitivity of unity indicates that a 1% variation in the parameter causes a 1% change in the response (i.e., 2.83 K).

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