Efficient Estimation of Feedback Effects with Application to Climate Models

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ABSTRACT

This work presents an efficient method for calculating the sensitivity of a mathematical model's result to feedback. Feedback is defined in terms of an operator acting on the model's dependent variables. The sensitivity to feedback is defined as a functional derivative, and a method is presented to evaluate this derivative using adjoint functions. Typically, this method allows the individual effect of many different feedbacks to be estimated with a total additional computing time comparable to only one recalculation. The effects on a CO_2-doubling experiment of actually incorporating surface albedo and water vapor feedbacks in a radiative-convective model are compared with sensitivities calculated using adjoint functions. These sensitivities predict the actual effects of feedback with at least the correct sign and order of magnitude. It is anticipated that this method of estimating the effect of feedback will be useful for more complex models where extensive recalculations for each of a variety of different feedbacks is impractical.

1. Introduction

A simplification that frequently occurs in climate models is the use of experimentally observed values for what should be prognostically determined variables (e.g., variables describing ocean currents or cloud cover). Such a simplification is useful because the interactive modeling of a particularly complicated or ill-understood process can be postponed, while it is still possible to make physically meaningful comparisons between experimental observations and quantities that remain prognostically determined. When a climate model is used predictively, the experimentally prescribed quantities are in reality subject to change due to forcing influences. When the effect of such a change is taken into account, this is usually referred to as including the effect of feedback because quantities that are normally input in the model (e.g., parameters or data) are allowed to depend on the model output (i.e., the dependent variables). For example, the importance of allowing a temperature dependence of the atmospheric water vapor mixing ratios was discussed as early as 1967 by Manabe and Wetherald.

When there is more than one plausible way of incorporating feedback in a model, it is useful to experiment with various forms of the feedback. But for more complex models, such as atmospheric general circulation models, recalculation for each of several different forms of the feedback can be prohibitively expensive. Therefore the purpose of this work is to develop and demonstrate an efficient method of estimating the effect of feedback for use with models where extensive recalculation is impractical.

Using a radiative–convective model, Hall *et al.* (1982) have demonstrated the efficiency of the adjoint method when estimating the effect of prescribed variations in the model's parameters. However, the adjoint method cannot be applied directly to estimate the effect of feedback because the variations in the parameters are not prescribed, but depend on the output of the model. Consequently, in Section 2 the use of the term *sensitivity to feedback* is defined and justified. It is then shown how this sensitivity can be estimated using the adjoint method. To provide guidance for judging when the sensitivity will provide a realistic estimate of the actual effects of feedback, Section 3 compares sensitivities with the actual effect of incorporating water vapor and surface albedo feedback in a radiative-convective model. The summary and conclusions are presented in Section 4.

2. Definition and estimation of sensitivity to feedback

Many physical processes, including climate processes, are modeled mathematically by equations of the form

$$\left.\begin{aligned} \mathbf{N}[\mathbf{u}(\mathbf{x}), \boldsymbol{\alpha}(\mathbf{x})] &= \mathbf{0} \\ \mathbf{B}[\mathbf{u}(\mathbf{x}), \boldsymbol{\alpha}(\mathbf{x})] &= \mathbf{0} \end{aligned}\right\}, \tag{1}$$

where the components of $\mathbf{x}$ are the independent variables (e.g., time and/or position) taking values in a region of physical interest Ω; the components of $\mathbf{u}$ are the dependent or state variables; the components of $\boldsymbol{\alpha}$ are the prescribed parameters or data in the model; the components of $\mathbf{N}$ are operators representing the physical processes in the model; and the components of $\mathbf{B}$ are operators representing the initial and/or boundary conditions of the model. Although (1) is general enough to represent even the most complex of climate models, a simple zero-dimensional climate model illustrates the following general development. This illustrative model is described by the equations

$$\left.\begin{aligned} du/dt + \alpha_1 u^4 + \alpha_2 = 0 \\ u(a) - u_a = 0 \end{aligned}\right\}. \tag{2}$$

For this model, the only independent variable is time t which varies from a to b, the only dependent variable is the temperature $u(t)$, and the two parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ are constants that depend on the physical properties of the system such as heat capacity, incident radiation, albedo and emissivity. The initial value of u is u_a.

A scalar result R of a model described by (1) can in general be expressed as a functional of $\mathbf{u}$ and $\boldsymbol{\alpha}$. For example, the average longwave radiation in the illustrative model described by (2) is proportional to the functional

$$R(\mathbf{u}, \boldsymbol{\alpha}) = \int_a^b dt(\alpha_1 u^4). \tag{3}$$

In the following, $\boldsymbol{\alpha}^0$ denotes the parameters' nominal values, and $\mathbf{u}^0$ denotes the nominal solution of (1). Thus the nominal solution satisfies

$$\left.\begin{aligned} \mathbf{N}(\mathbf{u}^0, \boldsymbol{\alpha}^0) = \mathbf{0} \\ \mathbf{B}(\mathbf{u}^0, \boldsymbol{\alpha}^0) = \mathbf{0} \end{aligned}\right\}, \tag{4}$$

and the nominal result is $R(\mathbf{u}^0, \boldsymbol{\alpha}^0)$.

Feedback can be introduced into the model by allowing some of the parameters $\boldsymbol{\alpha}$ to depend on the components of $\mathbf{u}$. Without loss of generality, this feedback can be specified by adding an operator $\mathbf{A}(\mathbf{u})$ to the parameters' nominal values $\boldsymbol{\alpha}^0$. Thus, in the presence of feedback, the parameters' values become $\boldsymbol{\alpha}^0 + \mathbf{A}(\mathbf{u})$, and the solution $\mathbf{u}^f$ with feedback satisfies

$$\left.\begin{aligned} \mathbf{N}[\mathbf{u}^f, \boldsymbol{\alpha}^0 + \mathbf{A}(\mathbf{u}^f)] = \mathbf{0} \\ \mathbf{B}[\mathbf{u}^f, \boldsymbol{\alpha}^0 + \mathbf{A}(\mathbf{u}^f)] = \mathbf{0} \end{aligned}\right\}. \tag{5}$$

The result with feedback is $R[\mathbf{u}^f, \boldsymbol{\alpha}^0 + \mathbf{A}(\mathbf{u}^f)]$. In the illustrative model, feedback can be introduced by allowing the emissivity to depend on temperature. For example, when the value of α_1 is allowed to be $\alpha_1{}^0 + \lambda(u - u_a)$, where λ is a constant specifying the strength of the feedback, then the feedback operator $\mathbf{A}(u)$ is the vector

$$\mathbf{A}(u) = [\lambda(u - u_a), 0]. \tag{6}$$

The solution u^f with feedback now satisfies

$$\left.\begin{aligned} du^f/dt + [\alpha_1{}^0 + \lambda(u^f - u_a)](u^f)^4 + \alpha_2{}^0 = 0 \\ u^f(a) - u_a = 0 \end{aligned}\right\},$$

and the result (3) with feedback is

$$\int_a^b dt[\alpha_1{}^0 + \lambda(u^f - u_a)](u^f)^4.$$

The difference

$$R[\mathbf{u}^f, \boldsymbol{\alpha}^0 + \mathbf{A}(\mathbf{u}^f)] - R(\mathbf{u}^0, \boldsymbol{\alpha}^0) \tag{7}$$

gives the actual effect of the feedback $\mathbf{A}$ on the result $R(\mathbf{u}^0, \boldsymbol{\alpha}^0)$. In practice, this difference can be evaluated exactly only by introducing feedback into the model explicitly, calculating $\mathbf{u}^f$, and re-evaluating the result. For more complex climate models such as atmospheric general circulation models, rerunning the model more than once can be prohibitively expensive. This provides the motivation to develop a more efficient method of evaluating (7). In his development of the adjoint method, Cacuci (1981) describes an efficient method of evaluating the change in the functional $R(\mathbf{u}^0, \boldsymbol{\alpha}^0)$ arising from a prescribed but arbitrary set of variations in $\boldsymbol{\alpha}^0$. However, this method cannot be applied directly to the evaluation of (7) because, with feedback, the variation $\mathbf{A}(\mathbf{u}^f)$ in $\boldsymbol{\alpha}^0$ is not prescribed but depends on $\mathbf{u}^f$. The purpose of the following development is to provide an approximate expression for (7) that can be evaluated efficiently using the adjoint method.

The functional $VR_{\mathbf{A}}(\mathbf{h})$ is defined by

$$VR_{\mathbf{A}}(\mathbf{h}) = \lim_{\epsilon \to 0} \{R[\mathbf{u}^0 + \epsilon\mathbf{h}, \boldsymbol{\alpha}^0 + \epsilon\mathbf{A}(\mathbf{u}^0 + \epsilon\mathbf{h})] - R(\mathbf{u}^0, \boldsymbol{\alpha}^0)\}/\epsilon, \tag{8}$$

where $\mathbf{h}$ is an arbitrary set of increments to the dependent variables $\mathbf{u}^0$, and ϵ is a real number. The functional $\Delta R_{\mathbf{A}}(\mathbf{h})$ is defined by

$$\Delta R_{\mathbf{A}}(\mathbf{h}) = \{R[\mathbf{u}^0 + \mathbf{h}, \boldsymbol{\alpha}^0 + \mathbf{A}(\mathbf{u}^0 + \mathbf{h})] - R(\mathbf{u}^0, \boldsymbol{\alpha}^0)\} - VR_{\mathbf{A}}(\mathbf{h}). \tag{9}$$

Note that if the arbitrary value of $\mathbf{h}$ is chosen to be $\mathbf{u}^f - \mathbf{u}^0$, then (9) can be written

$$R[\mathbf{u}^f, \boldsymbol{\alpha}^0 + \mathbf{A}(\mathbf{u}^f)] - R(\mathbf{u}^0, \boldsymbol{\alpha}^0) = VR_{\mathbf{A}}(\mathbf{h}) + \Delta R_{\mathbf{A}}(\mathbf{h}). \tag{10}$$

Thus for $\mathbf{h} = \mathbf{u}^f - \mathbf{u}^0$, $\Delta R_{\mathbf{A}}(\mathbf{h})$ is the discrepancy between the actual effect of feedback (7) and the functional (8).

The properties of the functional (9) can be determined as follows. Since both $\mathbf{A}$ and $\mathbf{h}$ are arbitrary in (9) they can be replaced respectively by $\epsilon\mathbf{A}$ and $\epsilon\mathbf{h}$ to give

$$\Delta R_{\epsilon\mathbf{A}}(\epsilon\mathbf{h}) = \{R[\mathbf{u}^0 + \epsilon\mathbf{h}, \boldsymbol{\alpha}^0 + \epsilon\mathbf{A}(\mathbf{u}^0 + \epsilon\mathbf{h})] - R(\mathbf{u}^0, \boldsymbol{\alpha}^0)\} - VR_{\epsilon\mathbf{A}}(\epsilon\mathbf{h}). \tag{11}$$

The definition in (8) shows that $VR_{\epsilon\mathbf{A}}(\epsilon\mathbf{h}) = \epsilon VR_{\mathbf{A}}(\mathbf{h})$, and so dividing (11) by ϵ and letting $\epsilon \to 0$ gives

$$\lim_{\epsilon \to 0} \Delta R_{\epsilon\mathbf{A}}(\epsilon\mathbf{h})/\epsilon = 0. \tag{12}$$

This equation shows that $\Delta R_{\mathbf{A}}(\mathbf{h})$ contains no first-order terms in $\mathbf{A}$ or $\mathbf{h}$. Thus, for $\mathbf{h} = \mathbf{u}^f - \mathbf{u}^0$, (10) and (12) show that the functional $VR_{\mathbf{A}}(\mathbf{h})$ given by

(8) is an estimate of the actual effect of feedback (7) correct to first order in **A** and **h**. Consequently, $VR_{\mathbf{A}}(\mathbf{h})$ can be called the *sensitivity of R to feedback* **A**.

In practice, it is more convenient to use the following alternative definition of $VR_{\mathbf{A}}$ which is equivalent to (8):

$$VR_{\mathbf{A}} = \{(d/d\epsilon)R[\mathbf{u}^0 + \epsilon\mathbf{h}, \alpha^0 + \epsilon\mathbf{A}(\mathbf{u}^0 + \epsilon\mathbf{h})]\}_{\epsilon=0}. \quad (13)$$

For nearly all physical models, performing the differentiation in (13) gives

$$VR_{\mathbf{A}} = R_1'\mathbf{h} + R_2'\mathbf{A}(\mathbf{u}^0), \quad (14)$$

where R_1' and R_2' denote, respectively, the partial Gateaux derivatives at $(\mathbf{u}^0, \alpha^0)$ of $R(\mathbf{u}, \alpha)$ with respect to its first and second arguments. For example, with the result R defined by (3) and the feedback **A** defined by (6), $VR_{\mathbf{A}}$ is obtained as follows:

$$VR_{\mathbf{A}} = \left\{(d/d\epsilon)\int_a^b dt[\alpha_1{}^0 + \epsilon\lambda(u^0 + \epsilon h - u_a)] \times (u^0 + \epsilon h)^4\right\}_{\epsilon=0}$$

$$= \int_a^b dt[\alpha_1{}^0 4(u^0)^3 h] + \int_a^b dt[(u^0)^4\lambda(u^0 - u_a)].$$

Note that for this model R_1' is the operator

$$R_1'(\ \ ) = \int_a^b dt[\alpha_1{}^0 4(u^0)^3(\ \ )],$$

and R_2' is the operator

$$R_2'(\ \ ) = \left[\int_a^b dt[(u^0)^4], 0\right]\cdot(\ \ ).$$

To evaluate the sensitivity to feedback given by (14), Eqs. (4) and (5) are needed to determine $\mathbf{h} = \mathbf{u}^f - \mathbf{u}^0$. Subtracting (4) from (5) gives

$$\left.\begin{aligned}\mathbf{N}[\mathbf{u}^f, \alpha^0 + \mathbf{A}(\mathbf{u}^f)] - \mathbf{N}(\mathbf{u}^0, \alpha^0) = \mathbf{0}\\ \mathbf{B}[\mathbf{u}^f, \alpha^0 + \mathbf{A}(\mathbf{u}^f)] - \mathbf{B}(\mathbf{u}^0, \alpha^0) = \mathbf{0}\end{aligned}\right\}. \quad (15)$$

Relationships equivalent to (10) hold for the operators **N** and **B**; applying these relationships to (15) gives

$$\left.\begin{aligned}\mathbf{VN_A(h)} + \Delta\mathbf{N_A(h)} = \mathbf{0}\\ \mathbf{VB_A(h)} + \Delta\mathbf{B_A(h)} = \mathbf{0}\end{aligned}\right\}, \quad (16)$$

where

$$\lim_{\epsilon\to 0}\Delta\mathbf{N}_{\epsilon\mathbf{A}}(\epsilon\mathbf{h})/\epsilon = \lim_{\epsilon\to 0}\Delta\mathbf{B}_{\epsilon\mathbf{A}}(\epsilon\mathbf{h})/\epsilon = \mathbf{0}. \quad (17)$$

Also, relationships equivalent to (14) hold for $\mathbf{VN_A}$ and $\mathbf{VB_A}$; applying these relationships to (16) gives

$$\left.\begin{aligned}\mathbf{N}_1'\mathbf{h} + \mathbf{N}_2'\mathbf{A}(\mathbf{u}^0) + \Delta\mathbf{N_A(h)} = \mathbf{0}\\ \mathbf{B}_1'\mathbf{h} + \mathbf{B}_2'\mathbf{A}(\mathbf{u}^0) + \Delta\mathbf{B_A(h)} = \mathbf{0}\end{aligned}\right\}. \quad (18)$$

For the illustrative model, with N and B defined by (2) and **A** defined by (6), $VN_{\mathbf{A}}(h)$ and $VB_{\mathbf{A}}(h)$ can be obtained as follows:

$$VN_{\mathbf{A}}(h) = ((d/d\epsilon)\{(d/dt)(u^0 + \epsilon h) + [\alpha_1{}^0 + \epsilon\lambda(u^0 + \epsilon h - u_a)](u^0 + \epsilon h)^4 + \alpha_2{}^0\})_{\epsilon=0}$$

$$= [d/dt + \alpha_1{}^0 4(u^0)^3]h + (u^0)^4\lambda(u^0 - u_a),$$

$$VB_{\mathbf{A}}(h) = \{(d/d\epsilon)[u(a) + \epsilon h(a) - u_a]\}_{\epsilon=0}$$

$$= h(a).$$

Note that for this model, N_1' is the operator

$$N_1'(\ \ ) = [d/dt + \alpha_1{}^0 4(u^0)^3](\ \ ),$$

N_2' is the operator

$$N_2'(\ \ ) = [(u^0)^4, 1]\cdot(\ \ ),$$

and the boundary conditions in (16) and (18) become

$$h(a) = -\Delta B_A(h). \quad (19)$$

The problem of efficiently evaluating the sensitivity (14) where $\mathbf{A}(\mathbf{u}^0)$ is known and **h** is determined by (18) is precisely the problem addressed by Cacuci (1981) in his development of the adjoint method. The purpose of this method is to evaluate the sensitivity (14) without explicitly evaluating **h**, thereby avoiding the need to solve (18) anew for every different feedback $\mathbf{A}(\mathbf{u}^0)$. The adjoint method starts by defining an operator $\mathbf{L}^*$ adjoint to $\mathbf{N}_1'$ as follows:

$$\langle\mathbf{q}|\mathbf{N}_1'\mathbf{r}\rangle = \langle\mathbf{r}|\mathbf{L}^*\mathbf{q}\rangle + P(\mathbf{q}, \mathbf{r}), \quad (20)$$

where **q** and **r** are arbitrary functions of **x**, $\langle\mathbf{q}|\mathbf{r}\rangle$ denotes the scalar product of **q** and **r** in the region of physical interest Ω, and $P(\mathbf{q}, \mathbf{r})$ is a term evaluated on the boundary of this region. For the illustrative model, the scalar product is

$$\langle q|r\rangle = \int_a^b dt[q(t)r(t)].$$

For this model, (20) can be written

$$\int_a^b dt\{q[d/dt + \alpha_1{}^0 4(u^0)^3]r\} = \int_a^b dt\{r[-d/dt + \alpha_1{}^0 4(u^0)^3]q\} + [qr]_a{}^b.$$

Thus L^* is the operator

$$L^* = [-d/dt + \alpha_1{}^0 4(u^0)^3],$$

and $P(q, r)$ is the term

$$P(q, r) = [qr]_a{}^b.$$

The adjoint solution $\mathbf{v}(\mathbf{x})$ is the solution to the system of equations

$$\left.\begin{aligned}\mathbf{L}^*\mathbf{v} = \mathbf{s}\\ \mathbf{B}^*\mathbf{v} = \mathbf{0}\end{aligned}\right\}, \quad (21)$$

where $\mathbf{s}$ is a source term defined by

$$\langle \mathbf{s}|\mathbf{q}\rangle = R_1'\mathbf{q}, \tag{22}$$

and $\mathbf{B}^*$ is an operator representing the adjoint boundary conditions that will be defined later. For the illustrative model, (22) becomes

$$\int_a^b dt[sq] = \int_a^b dt\alpha_1{}^0 4(u^0)^3 q.$$

Thus s is the term $\alpha_1{}^0 4(u^0)^3$, and the first of Eqs. (21) becomes

$$[-d/dt + \alpha_1{}^0 4(u^0)^3]v = \alpha_1{}^0 4(u^0)^3. \tag{23}$$

The adjoint method concludes by expressing the sensitivity $VR_\mathbf{A}$ in terms of an adjoint solution as follows:

$$VR_\mathbf{A} = R_2'\mathbf{A}(\mathbf{u}^0) - \langle \mathbf{v}|\mathbf{N}_2'\mathbf{A}(\mathbf{u}^0)\rangle - P(\mathbf{h}, \mathbf{v}) + \Delta_\mathbf{A}(\mathbf{h}), \tag{24}$$

where

$$\lim_{\epsilon \to 0} \Delta_{\epsilon\mathbf{A}}(\epsilon\mathbf{h})/\epsilon = 0.$$

The adjoint boundary conditions are chosen to eliminate the unknown values of $\mathbf{h}$ from $P(\mathbf{h}, \mathbf{v})$ in (24). For example, with the illustrative model Eq. (24) becomes

$$VR_\mathbf{A} = \int_a^b dt[(u^0)^4\lambda(u^0 - u_a)] - \int_a^b dt[v(u^0)^4\lambda(u^0 - u_a)] - [hv]_a{}^b + \Delta_\mathbf{A}(h). \tag{25}$$

The value of $h(a)$ in this equation is known from the initial conditions (19). Thus the only unknown value of h, i.e., $h(b)$, can be eliminated from (25) by choosing the adjoint boundary condition

$$v(b) = 0. \tag{26}$$

The advantage of the adjoint method is that the adjoint solution is independent of the feedback being considered, and all values of $\mathbf{h}$ in the expression (24) for $VR_\mathbf{A}$ are known without having to solve (18). Thus, once the adjoint solution $\mathbf{v}$ has been calculated, it is possible to estimate the effect of many different feedbacks without solving any additional differential equations. This advantage of the adjoint method can be seen for the illustrative model. The adjoint equations (23) and (26) do not contain any terms arising from the feedback operator. Moreover, the equivalent of (25) can be derived from (24) for a general feedback operator $[A_1(u), A_2(u)]$:

$$VR_\mathbf{A} = \left[\int_a^b dt(u^0)^4, 0\right] \cdot [A_1(u^0), A_2(u^0)] - \int_a^b dt\{v[(u^0)^4, 1] \cdot [A_1(u^0), A_2(u^0)]\} + \Delta B_\mathbf{A}(h)v(a) + \Delta_\mathbf{A}(h).$$

Thus, once (23) and (26) have been solved, the above equation can be used to estimate the effect of any feedback where the Δ terms are neglected.

3. Results for a radiative-convective model

As shown in Section 2, the actual effect (7) of feedback on a result differs from the sensitivity (8) by second- and higher-order terms in the strength of the feedback. For certain types of feedback, it is physically meaningful to vary the strength of the feedback continuously, compare the actual effect (7) with the sensitivity (8), and verify that as the strength of the feedback tends to zero, the actual effect (7) tends to the sensitivity (8). Such a comparison determines whether the numerical implementation of (21) and (24) is correct. For any feedback, even for one that may lose physical meaning if varied continuously, a comparison between the actual effect of feedback and the corresponding sensitivity shows the effect of neglecting second- and higher-order terms in the strength of the feedback, and provides guidance for judging when the sensitivity to feedback provides a realistic estimate of the actual effects of feedback.

In this section, the radiative convective model described by Hall *et al.* (1982) is used both to illustrate the evaluation, by the adjoint method, of sensitivity to feedback, and to compare this sensitivity to the effect of actually introducing feedback in the model. The nominal solution u^0 is that described by Hall and Cacuci (1983). The equations underlying this radiative convective model are a particular form of (1) with

1) $x \to t$,
2) $\Omega \to (a, b)$,
3) $\mathbf{u} \to [u_1(t), u_2(t), u_3(t)]$,
4) $\boldsymbol{\alpha} \to [\alpha_1, \ldots, \alpha_P]$,
5) $N_i \to du_i/dt - f_i(u, \alpha), \quad i = 1, 2, 3,$
6) $B_i \to u_i(a) - \alpha_i, \quad i = 1, 2, 3.$

The result $R(\mathbf{u}^0, \boldsymbol{\alpha}^0)$ considered in this section is the increase in the average surface air temperature which occurs after the atmospheric CO_2 concentration in the model is doubled. The averaging period τ for this temperature is the last day of the model integration. As discussed by Hall and Cacuci (1984), this result can be written as $R = R_2 - R_1$, where R_1 and R_2 are, respectively, the average surface air temperatures with the normal and twice the normal atmospheric CO_2 concentrations. The average surface air temperature (i.e., R_1 or R_2) is expressed as

$$\frac{1}{T}\int_a^b dt[\mathbf{d} \cdot \mathbf{u}H(t - b + \tau)],$$

where $\mathbf{d} = (-0.5, 1.5, 0)$, and H is the Heaviside step function.

The sensitivity of R to a feedback $\mathbf{A}$ is obtained by applying the method presented in Section 2. This gives

$$VR_A = \int_a^b dt \sum_{i=1}^{3} [v_i \sum_{p=1}^{P} A_p(\mathbf{u}^0)\{\partial f_i/\partial \alpha_p\}_{(\mathbf{u}^0,\alpha^0)}] + \Delta,$$

where $v_i(t)$ satisfy, for $i = 1, 2, 3$,

$$\left.\begin{aligned} -\frac{dv_i}{dt} - \sum_{j=1}^{3} v_j\{\partial f_j/\partial u_i\}_{(\mathbf{u}^0,\alpha^0)} &= H(t-b+\tau)d_i/\tau \\ v_i(b) &= 0 \end{aligned}\right\}.$$

Note that in this model the result R does not depend explicitly on α, and the initial conditions α_1, α_2 and α_3 are not involved in the feedback A.

Numerical results for surface albedo and water vapor feedbacks will now be presented. A surface albedo feedback can be introduced by making the surface albedo α_s (i.e., the sth component of α) a function of the surface air temperature averaged over the preceding 24 hours. For this albedo feedback, the only nonzero components of A is A_s; this component has the form

$$A_s(u) = F[\bar{T}_4(t)],$$

where

$$\bar{T}_4(t) = \int_{t-\tau}^{t} (1/\tau)\mathbf{d}\cdot\mathbf{u}(t')dt',$$

and F is a function of $\bar{T}_4$. Here, $\mathbf{d}\cdot\mathbf{u}$ is the surface air temperature and τ is 24 hours, so $\bar{T}_4$ is the average surface air temperature during the day before time t. The function F is defined as follows:

1) $|F| \leq 0.025$.

2) For $|F| < 0.025$, F causes the surface albedo to depend linearly on $\bar{T}_4$, so α_s changes from its nominal value of $\alpha_s^0 = 0.1$ at a rate β K^{-1}.

3) F (299.17 K) = 0, so when $\bar{T}_4$ = 299.17 K, the value of the average surface air temperature without feedback, the surface albedo takes on its nominal value.

Figure 1 illustrates the function F for $\beta = 0.005$.

Table 1 shows the effect of the surface albedo

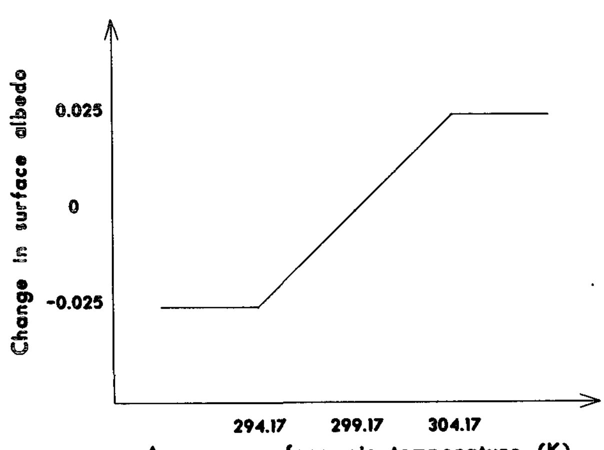

FIG. 1. Change in surface albedo as a function of average surface air temperature for feedback parameter $\beta = 0.005$ K^{-1}.

TABLE 1. Effects of surface albedo feedback on a CO_2-doubling experiment using the radiative-convective model. (CO_2 warming without feedback was 1.664 K.)

	Feedback parameter β			
	0.005	0.0005	−0.0005	−0.005
CO_2 warming with feedback (K)	1.27	1.614	1.717	2.389
Actual effect of feedback (K)	−0.39	−0.050	0.053	0.72
Sensitivity to feedback (K)	−0.51	−0.051	0.051	0.51

feedback on the results of a CO_2 doubling experiment using the radiative-convective model. As expected, negative values of β correspond to positive feedback which enhances the CO_2 warming. For the weaker feedbacks (i.e., $\beta = \pm 0.0005$), the actual effect agrees with the sensitivity to within 4%, indicating that numerical implementation of the adjoint method is correct. For the stronger positive feedback (i.e., $\beta = -0.005$), the sensitivity is about 30% smaller than the actual effect, while for the stronger negative feedback (i.e., $\beta = 0.005$), the sensitivity is about 30% larger than the actual effect. Thus, even for the severe effects of the stronger feedback, the sensitivities calculated using adjoint functions give useful estimates of the actual effects of the feedbacks.

A more complicated but also well-recognized feedback is that caused by water vapor. This feedback can be introduced in the radiative–convective model by allowing the water vapor mixing ratios α_{m1} and α_{m2} of the two atmospheric layers to be determined by a prescribed relative humidity profile. The water vapor mixing ratios thus become functions of temperature. For this feedback, the only nonzero components of the feedback operator A are A_{m1} and A_{m2}. These components are

$$A_{mk} = RH_k q_k^*[u_k(t)] - \alpha_{mk}^0,$$

where, for the kth atmospheric layer, RH_k is the prescribed relative humidity, q_k^* is the saturation mixing ratio, and u_k is the temperature.

Table 2 shows the effect of this water vapor feedback on the results of a CO_2-doubling experiment using the radiative-convective model. The actual effect of introducing water vapor feedback is to increase the CO_2 warming by 1.11 K. For this feedback, the sensitivity calculated using adjoint functions is 0.67

TABLE 2. Effects of water vapor feedback on a CO_2-doubling experiment using the radiative-convective model.

CO_2 warming without feedback (K)	1.66
CO_2 warming with feedback (K)	2.77
Actual effect of feedback (K)	1.11
Sensitivity to feedback (K)	0.67

K. Thus, although the sensitivity is about 40% less than the actual effect, it still provides a useful estimate of the effect of the water vapor feedback.

The sensitivities to feedback presented in this section were estimated using the same adjoint solution discussed by Hall and Cacuci (1984). To calculate this adjoint solution required about the same computing time as needed to obtain the nominal solution $\mathbf{u}^0$. Once this adjoint solution has been calculated, the additional computing time required to estimate the sensitivities to feedback [i.e., to evaluate (24)] is negligible.

4. Summary and conclusions

An efficient method for estimating the effect of feedback in a mathematical model has been presented. A feedback operator acting on the model's dependent variables defines a feedback mechanism by modifying the values of parameters or data in the model. Although the effect of prescribed variations in the parameters can be evaluated efficiently using the adjoint method, this method cannot be applied directly to estimate the effect of feedback; this is because the parameter variations are not prescribed but depend on the output of the model. Therefore, in this work we define a quantity called *sensitivity to feedback* that can be estimated using the adjoint method. It has been shown that the sensitivity to feedback is an estimate of the actual effect of feedback correct to first order in the strength of the feedback, and it has also been shown how the sensitivity can be estimated using the adjoint method. The principal advantage of this application of the adjoint method is that, once the adjoint solution has been calculated, the effect of a variety of different feedbacks can be estimated with minimal additional computing time.

Although the method has been presented for a general mathematical model, the results were obtained for a CO_2-doubling experiment using a radiative-convective model. Originally, in this model, the values of the surface albedo and specific humidity were independent of temperature. Two feedback mechanisms were introduced, the first by allowing the surface albedo to become a function of surface air temperature, and the second by allowing the specific humidity to be determined by a prescribed relative humidity profile. Both feedbacks significantly affect the result of the CO_2-doubling experiment. The sensitivities calculated using adjoint functions gave useful estimates of the feedback effects, with at least the correct sign and order of magnitude. The computing time required to calculate the adjoint solution was about the same as that required for the original calculation. The additional computing time to estimate the sensitivities to feedback (once the adjoint solution had been calculated) was negligible.

The efficient technique presented in this work for estimating the effect of feedback is likely to be useful for models where extensive recalculation with a variety of feedbacks is impractical. An approximate yet quantitative indication of the effects of a wide range of potentially important feedbacks will help identify sources of uncertainty in model predictions, and will indicate priorities for incorporating feedbacks rigorously.

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