Wavelets, Period Doubling, and Time–Frequency Localization with Application to Organization of Convection over the Tropical Western Pacific

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ABSTRACT

In this paper, preliminary results in using orthogonal and continuous wavelet transform (WT) to identify period doubling and time–frequency localization in both synthetic and real data are presented. First, the Haar WT is applied to synthetic time series derived from a simple nonlinear dynamical system—a first-order quadratic difference equation. Second, the complex Morlet WT is used to study the time–frequency localization of tropical convection based on a high-resolution Japanese Geostationary Meteorological Satellite infrared (IR) radiance dataset.

The Haar WT of the synthetic time series indicates the presence and distinct separation of multiple frequencies in a period-doubling sequence. The period-doubling process generates a multiplicity of intermediate frequencies, which are manifested in the nonuniformity in time with respect to the phase of oscillations in the lower frequencies. Wavelet transform also enables the detection of extremely weak signals in higher-order subharmonics resulting from the period-doubling bifurcations. These signals are either undetected or considered statistically insignificant by traditional Fourier analysis.

The Morlet WT of the IR radiance dataset indicates the presence of multiple timescales, which are localized in both frequency and time. There are two regimes in the variation of IR radiance, corresponding to the wet and dry periods. Multiple timescales, ranging from semidiurnal, diurnal, synoptic, to intraseasonal with embedding structures, are active in the wet regime. In particular, synoptic variability is more prominent during the wet phase of an intensive intraseasonal cycle. These are not only consistent with, but also show more details than, previous findings by using other techniques. The phase-locking relationships among the oscillations with different timescales suggest that both synoptic and intraseasonal variations may be mixed oscillations due to the interaction of self-excited oscillations in the tropical atmosphere and external forcings such as annual and diurnal solar radiation variations.

Both examples show that WT is a powerful tool for analysis of phenomena involving multiscale interactions that exhibit localization in both frequency and time. A discussion on the caveats in the use of WT in geophysical data analysis is also presented.

1. Introduction

Multiple timescales are associated with almost all naturally occurring phenomena in the earth’s weather and climate system. In particular, tropical cloudiness is known to possess variability with timescales ranging from several hours to many years. Recently, using Japanese Geostationary Meteorological Satellite (GMS) infrared (IR) radiance data, Nakazawa (1988), Lau et al. (1991), and Sui and Lau (1992) found a complex hierarchy of periodicities ranging from diurnal, 2–3 day, to 10–15 day associated with supercloud cluster (SCC) organization over the equatorial western Pacific, with the shorter scales embedded in and interacting strongly with successive longer scales. The Lau et al. results suggest that there is a time–frequency localization in the organization of SCCs and that there may be some fundamental relationships governing the occurrence of the observed multiple timescales. Further understanding of these relationships demands special efforts in multiscale analysis. This is the motivation of the present paper.

In the present work, we will show examples of using orthogonal and continuous wavelet transform (WT) to analyze two different kinds of time series with multiple timescales. The first is synthetic time series deduced from a simple nonlinear dynamical system, a first-order quadratic difference equation, known as the logistic map. Although simplistic in form, this kind of first-order quadratic difference equation is known to possess highly complex behavior representative of generic nonlinear dynamical systems, including the earth’s climate (Lorenz 1964). The second is a time series taken from real observations from high-resolution Japanese GMS IR radiance data for 1987–88 over Java, an island in the tropical western Pacific. This location is chosen be-

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cause previous studies have suggested the presence of multiscale phenomena involving SCC development over this region. The analysis of the logistic map will demonstrate how period doubling and time–frequency localization occurring in an idealized time series can be delineated by WT in the unfolding of a two-dimensional manifold. The results of this analysis will provide a theoretical reference for the subsequent analysis of tropical convection using GMS IR radiance data.

Section 2 gives the background and a tutorial of continuous and orthogonal WT used in this study. The readers who have already had such background may skip this section and go directly to section 3, in which the Haar WT is applied to time series from the logistic map to illustrate the characteristics of multiscale oscillations, with doubling and non-doubling periods, resulted from period-doubling bifurcations. Section 4 applies the complex Morlet WT to the IR radiance data to detect the signals localized in both time and frequency, and shows some relationships among different timescales. Section 5 discusses the new scientific information on tropical SCC organization yielded by the WT of the IR radiance data. In section 6, we present the conclusions including a discussion on the caveats in the use of WT in geophysical data analysis.

2. Background

a. Wavelet transform versus Fourier transform

Fourier transform (FT) is one of the most commonly used tools to study frequency spectrum of geophysical time series. A spectral density distribution identifies the underlying frequencies and their relative contributions to the time series, but it shows no information regarding their temporal locality. The FT is a useful tool to extract global information of wave-like signals. However, if a signal is altered in a small neighborhood at certain time instant, the entire spectrum may be affected. A local information is thus split up into a large number of global bases. Nonstationary signals that appear only in a short time interval may not be detected by Fourier analysis because these signals are averaged out over the whole time domain. In the search for fundamental structures in the time–frequency domain, some frequency signals, albeit weak, may still be important if they are members of a frequency hierarchy. Therefore, in many applications, such as analysis of nonstationary and/or weak signals in geophysical data, FT alone is not adequate. In this regard, detection of nonstationary and/or weak signals may be crucial, and alternate time series analysis technique may be required.

The study of nonstationary time series by localizing signals in both frequency and time domains was first introduced by Gabor (1946) using a windowed Fourier transform (WFT) technique. The width of the time–frequency window, however, is fixed in WFT. For a given window width, as the frequency increases more cycles can be included; as the frequency decreases fewer cycles are included. Since a precise definition of high-frequency oscillations requires a narrow time window while a thorough description of low-frequency oscillations requires a wide time window, the WFT with fixed window width has limited application for simultaneously detecting high-frequency signals embedded within low-frequency phenomena. In this regard, WT is most appropriate.

In WT, a given dataset is divided into components with different scales, which allows the investigation of each component with a resolution matched to its scale. Goupillaud et al. (1984) first used WT to represent a one-dimensional seismic dataset as a function of both time and frequency. Since then, WT has begun to find applications in many different fields, including meteorology (Mahrt 1991; Serrano et al. 1992; Gollmer et al. 1992; Kumar and Foufoula-Georgiou 1993; Myers et al. 1993; Gamage and Blumen 1993). Several books (Daubechies 1992; Chui 1992; Young 1993) and many papers and monographs (Strang 1989; Orbes et al. 1990, Ruskai et al. 1992; Farge 1992) have given detailed description of wavelets and their applications. In principle, a continuous WT provides a flexible time–frequency window that automatically narrows when focusing on high-frequency oscillations and widens on the low-frequency background, in a manner analogous to a “zoom lens.” As will be shown in this paper, some weak but important signals, which are usually ignored by FT, may be detected by WT.

b. A wavelet tutorial

We restrict the following description of basic concepts to one-dimensional WT only.

The term wavelets refers to a set of small waves formed by dilations [e.g., \( \psi(t) \rightarrow \psi(2t) \)] and translations [e.g., \( \psi(t) \rightarrow \psi(t + 1) \)] of a single function \( \psi(t) \), which is square integrable over the range of real time or space \( L^2(\mathbb{R}) \); that is, it has finite energy. The function \( \psi(t) \) is sometimes called the “mother wavelet,” “basic wavelet,” or “analyzing wavelet,” while the dilated and translated functions derived from the “mother wavelet” are called “daughter wavelets” or simply “wavelets.” These daughter wavelets have the same shape as that of the mother wavelet. Their amplitudes must rapidly decay away from the center of the wave in both time and frequency domains. Mathematically, a daughter wavelet at the scale \( a \) and the position \( b \) is expressed as

\[
\psi_{a,b}(t) = a^{-1/2} \psi \left( \frac{t - b}{a} \right),
\]

where \( a, b \) are real and \( a > 0 \). The dilation parameter \( a \) and the translation parameter \( b \) may range over either a continuous or a discrete set. Note that the wavelets expressed by Eq. (1) include an energy normalization.
$a^{-1/2}$, which keeps the energy of daughter wavelets the same as the energy of the mother wavelet.

For an oscillating function $\psi(t)$ to be a mother wavelet, it must have finite energy and a zero mean; that is, the first moment must be zero, or the zero frequency must vanish. This is the admission condition that a wavelet must comply. For some sophisticated wavelets, higher moments may also be zero.

1) Continuous WT

For most real-valued geophysical time series, it is suitable to choose a continuous WT with complex-valued wavelets. A complex-valued wavelet provides important pieces of information via (i) the $L^2$ modulus, which gives the energy density, (ii) the phase, which detects singularities and measures instantaneous frequencies, and (iii) the real part of the wavelet coefficients, which depicts both the intensity and phase of the signal variation, at particular scales and locations in the wavelet domain (the time–frequency domain).

A commonly used complex-valued wavelet is the Morlet wavelet, having the form

$$\psi(t) = e^{ik_0t}e^{-\left(\frac{t^2}{2}\right)},$$

which is a plane wave of wavevector $k_0$ modulated by a Gaussian envelope of unit width. Figure 1 shows the real part of the complex mother Morlet wavelet for
k_p = 5.4 (Fig. 1a) and some daughter wavelets with different dilations and translations (Figs. 1b–f). In Fourier space, the Morlet wavelet is given by
\[
\hat{\psi}(k) = (2\pi)^{-1/2}e^{-(k-k_p)^2/2} \quad \text{for} \ k > 0,
\]
\[
\hat{\psi}(k) = 0 \quad \text{for} \ k \leq 0.
\] (3)

The continuous WT of a square integrable function \(f\) is defined as an inner product of this function and the dilated and translated wavelets; that is,
\[
\mathcal{W}(a, b) = \langle f, \psi_{a,b} \rangle = \left\langle f, a^{-1/2} \int_{-a}^{a} f(t) \psi^*(\frac{t-b}{a}) \, dt, \right\rangle
\] (4)

where the asterisk denotes complex conjugate. This transform can also be performed in the Fourier domain; that is,
\[
2\pi \hat{\mathcal{W}}(a, b) = \langle \hat{f}, \hat{\psi}_{a,b} \rangle,
\] (5)

where
\[
\hat{\psi}_{a,b}(k) = a^{1/2} e^{-iak} \hat{\psi}(ak).
\] (6)

Here the caret denotes the FT.

Since the FT of a Morlet wavelet is a Gaussian function centered at its wavevector \(k_p\), it is easier to perform the Morlet WT in the Fourier domain than in the time domain. We will describe the algorithm for applying the Morlet WT to the IR radiance data in section 4.

2) ORTHOGONAL WT

The wavelets that form an orthogonal basis of \(L^2(\mathbb{R})\) are orthogonal wavelets; that is,
\[
\psi_{i,j}(t) = 2^{j/2} \psi(2^j t - i),
\] (7)
where \(i, j\) are integers. Equation (7) is a special case of Eq. (1) when the functions \(\psi(t)\) are orthogonal to their dyadic dilations by \(a = 2^{-j}\) and their translations by discrete steps \(b = 2^{-j} i\) and the family of Eq. (7) is complete in \(L^2(\mathbb{R})\).

The simplest orthogonal wavelet is the Haar wavelet, which comes from the box function (the scaling function of the Haar wavelet) and is defined as
\[
H(t) = \begin{cases} 
1 & \text{for } 0 \leq t < 1/2 \\
-1 & \text{for } 1/2 \leq t < 1 \\
0 & \text{otherwise}.
\end{cases}
\] (8)

The Haar wavelet has only a vanishing first moment. The Haar WT may be derived from the Haar matrix.

As examples, the \(2 \times 2\) and \(4 \times 4\) orthonormal Haar matrices are, respectively,
\[
\mathcal{H}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\] (9)

and
\[
\mathcal{H}_4 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \end{pmatrix}.
\] (10)

Extensions to higher-order orthonormal Haar matrices follow the structure indicated by Eqs. (9) and (10).

The Haar WT is analogous to the sampling process in which rows of the transform matrix sample an input data sequence with finer and finer resolution increasing with powers of two. It is not difficult to show the relationship between Haar matrices and Haar wavelets. In a Haar matrix, the first row represents the scaling function, which measures the mean of the dataset with equal weight to each data point. The lower rows represent the Haar wavelets at finer and finer dyadic scales and discrete positions. For a dataset with \(N = 2^j\) points, the Haar wavelets at the scale \(j\) and position \(i\) are
\[
H_{j,i}(t) = 2^{j/2} H(2^j t - i), \quad j = 0, 1, \ldots, J - 1;
\]
\[
i = 0, 1, \ldots, 2^j - 1.
\] (11)

There is a total of \(2^j\) Haar base functions, with one for the scaling function and \(2^j - 1\) for the Haar wavelets. To transform a dataset with \(4 (= 2^2)\) points, that is, \(J = 2\), we need the scaling function and three Haar wavelets at the level \(j = 0\) with position \(i = 0\) and the level \(j = 1\) with positions \(i = 0, 1\). These Haar wavelets are
\[
j = 0, i = 0: \quad H_{0,0} = H(t),
\] (12)
\[
j = 1, i = 0: \quad H_{1,0} = \begin{cases} 
\sqrt{2} & \text{for } 0 \leq t < 1/4 \\
-\sqrt{2} & \text{for } 1/2 \leq t < 1/2 \\
0 & \text{otherwise}.
\end{cases}
\] (13)
\[
j = 1, i = 1: \quad H_{1,1} = \begin{cases} 
\sqrt{2} & \text{for } 1/2 \leq t < 3/4; \\
-\sqrt{2} & \text{for } 3/4 \leq t < 1; \\
0 & \text{otherwise}.
\end{cases}
\] (14)

The values of \(H_{0,0}, H_{1,0},\) and \(H_{1,1}\) correspond to the elements in the rows 2, 3, and 4 of the Haar matrix \(\mathcal{H}_4\), given by (10), aside a normalization constant.

Figure 2 illustrates the 16 Haar base functions, corresponding to the elements of \(\mathcal{H}_{16}\), for a time series with 16 data points. In general, for a dataset with \(N = 2^j\),
the signal into a minimal number of independent coefficients. The orthogonal WT is also suited for analyzing a discrete series that contains dyadic scales.

3. Period doubling in a dynamical system

In this section, we use the simplest WT, the Haar WT, to show the merit of using WT compared with FT to a simple multiscale dynamical system before applying a more complicated WT to a real dataset in next section. The concept developed in this section may be used, as a guidance, in the interpretation of the real data. The dynamical system analyzed here is the well-known logistic map, which is a one-dimensional first-order quadratic difference equation defined by

$$X_{n+1} = rX_n(1 - X_n),$$

where $r$ is an external parameter and the range of $X_n$ is the interval $[0, 1]$. The logistic map has many practical applications. May (1976) found that, as a function of the external parameter $r$, the iterates $X_1, X_2, \ldots$ of Eq. (16) display an extremely complex behavior, containing a route to chaos by period-doubling bifurcations. It is well known that a period-doubling bifurcation results in a new scale that is doubled from the original scale. It is not so well known, however, that during a period-doubling bifurcation, the bifurcation actually results in not only a doubling scale but also the scales that are nondenoting but are rational fractions of the fundamental scale. It is the coexistence of these doubling and nondenoting scales that leads to chaos after infinite period-doubling bifurcations. This kind of simple dynamical system has also been applied to climate predictability studies. Lorenz (1964) used a similar one-dimensional first-order quadratic difference equation as a governing equation to illustrate the procedures to deduce the climate or the long-term statistical properties of a system. He related the solution of the difference equation to certain hydrodynamic systems, for example, the laboratory experiments of Hide (1958) and Fultz (1959). Lorenz felt “that this resemblance is no mere accident, but that the difference equation captures much of the mathematics, even if not the physics, of the transitions from one regime of flow to another, and, indeed, of the whole phenomenon of instability.” Therefore, the period-doubling bifurcations discussed in this simple dynamical system may have relevance to many nonlinear dynamical systems, including the earth’s weather and climate system, and may help us to understand some of the basic mechanisms leading to the variety of timescales observed in the atmosphere.

To show the time–frequency representation of WT, we iterate Eq. (16) from the same initial value of $X_0 = 0.01$ at seven different values of the parameter $r$. These values of $r$ are chosen such that the system is locked in several periodic states and one chaotic state in a “period-doubling tree.” The selected values of $r$ and the hierarchy of the dynamical equilibrium states

![Diagram of Haar wavelet bases](image-url)
Table 1. Values of $r$ and corresponding period with $N$ fixed points.

<table>
<thead>
<tr>
<th>$r$</th>
<th>3.2</th>
<th>3.5</th>
<th>3.56</th>
<th>$\ldots$</th>
<th>3.56979</th>
<th>3.56991</th>
<th>3.5699341</th>
<th>$\ldots$</th>
<th>3.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period $N$</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>$\ldots$</td>
<td>64</td>
<td>128</td>
<td>256</td>
<td>$\ldots$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

of Eq. (16) with period $N = 2^j$, where $j = 1, 2, 3, 6, 7, 8$ and $\infty$, are given in Table 1. Here period $\infty$ corresponds to the chaotic regime. The iterated steps for each case are 2000000, long enough to ensure that the system has reached its equilibrium. The last 4096 values of $X_n$ in the iteration process for each value of $r$ are taken as the time series for that case to be analyzed by both FT and the Haar WT. For efficiency, here we use Mallat’s pyramid algorithm to perform the Haar WT. For the details of this algorithm, readers are referred to some wavelet literature (e.g., Strang 1989).

The six time series for the cases given in Table 1 are shown in Fig. 3; the corresponding Fourier spectral densities are shown in Fig. 4; and their Haar wavelet representations are shown in Fig. 5. Figure 5 is a two-dimensional plot with a linear scale in time ($n$) and a

![Fig. 3. Time series from Eq. (16) for (a) $r = 3.2$ (period 2), (b) $r = 3.5$ (period 4), (c) $r = 3.56$ (period 8), (d) $r = 3.56979$ (period 64), (e) $r = 3.56991$ (period 128), and (f) $r = 3.569938$ (period 256).](image-url)
dyadic logarithmic scale in frequency (octave), in which the scale is doubled in the downward direction.

At $r = 3.2$ (Fig. 3a), the time series shows a stable period-2 oscillation. This is revealed by the Fourier spectral density of this time series with a single spectral spike at period 2 (Fig. 4a). The signal in the wavelet domain is represented by a single row at the top of the plot, with a uniform oscillation between two fixed points that is captured by the Haar bases with the smallest scale (Fig. 5a). It is clear that only one single scale exists in this case, that is, locality in frequency but not in time.

At $r = 3.5$ (Fig. 3b), in addition to the period-2 oscillation, a stable period-4 oscillation is obvious. In each period-4 cycle, there is a primary and a secondary oscillation, representing two nonuniform period-2 cycles. The distinct period 2 and period 4 manifest themselves in the two spectral spikes at the corresponding periods in the Fourier domain (Fig. 4b). However, the Fourier analysis is not able to show such a nonuniformity in time in the period-2 oscillation. In fact, due to the limitation of the FT, any scales shorter than that of the period-2 are not recognizable. Thus, a frequency peak for a nondoubling period $4/3$ may have been missed in Fig. 4b. The corresponding Haar wavelet coefficients show two rows at the top, with both periods of 2 and 4 (Fig. 5b). The period-4 oscillation (the second row from the top) is uniform in time, while the period 2 is nonuniform. The period-2 cycle related to the negative phase of a period-4 cycle is stronger than that to the positive phase. This nonuniformity is characterized by uneven lengths in time of period-2 cycles.

At $r = 3.56$ (Fig. 3c), the time series contains a period-8 oscillation. The Fourier spectral density in Fig. 4c shows a more complicated spectrum than Fig. 4b. Beside the spikes for the three doubling scales, there is also a spike for a nondoubling scale of $8/3$. At

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2 The frequency of the octave $m$ is doubled that of the octave $(m + 1)$.
Fig. 5. Time–frequency representation of the Haar wavelet coefficients of the time series from Eq. (16) for the same six values of r as for Fig. 3. The WT is based on 4096 data points. The frequency is represented by octave, which represents doubled periods downward or doubled frequencies upward. The row 0 (top row) indicates the octave 0, representing period-2 cycle; the row 1 indicates the octave 1, representing period 4; and so on.
the period-doubling bifurcation from period 4 (frequency 1/4) to period 8 (frequency 1/8), the frequency 1/4 actually splits to two new frequencies:

$$\frac{1}{4} \pm \frac{1}{8} = \left\{ \begin{array}{l} 3/8 \\ 1/8. \end{array} \right.$$

(17)

Thus, the period 8 and period 8/3 are a pair of new scales at this level. The wavelet coefficients in Fig. 5c now show three rows at the top of the plot, representing the coexistence of three doubling scales: period 2, period 4, and period 8. The period-8 cycle is uniform in time, but both the period-2 and period-4 cycles are not. The uniformity of the period-4 and period-2 oscillations reflects the existence of the nondenning periods of 4/3 and 8/3. The appearance of the nondenning periods, which are rational fractions of the fundamental period in period-doubling bifurcation phenomena, may be explained by a frequency splitting mechanism in nonlinear dynamical systems. (Weng 1993).

Based on the results discussed above for the first two period-doubling bifurcations, we may infer the results for bifurcations at higher levels. The plots in Fourier domain would show more spikes at the periods of 2^j and some nondenning periods associated with these doubling periods. The plots in wavelet domain for the period 2^j as j > 3 would show one more row downward each time as j increases by 1, that is, as the period is doubled from the previous one. When this dynamical system is in a state characterized by a period 2^j, it actually oscillates with multiple scales having the doubling periods of 2^j, 2^(j-1), and the nondenning periods at 2^j, 2^(j-2), . . . , 2^(-1). The nondenning periods, accompanied by the doubling periods, are reflected by nonuniformity in time of the oscillations at higher frequencies with respect to the phase of lower frequencies.

The wavelet transform also has the ability to detect extremely weak signals resulting from higher-level period-doubling bifurcations as the low-frequency oscillations have weaker and weaker amplitudes. This is illustrated in Figs. 3d–f, Figs. 4d–f, and Figs. 5d–f, which are the time series, spectral densities, and wavelet coefficients of Eq. (16) for the cases with period 64, period 128, and period 256, respectively. The time series in Figs. 3d–f appear identical, and so do the Fourier spectra in Figs. 4d–f, where the largest scale in all the three cases is represented by period 64 with hardly any noticeable differences in magnitude. Moreover, the magnitudes of the spectral density of the new low-frequency oscillations shown in Figs. 4d–f are so small that the spectral analysis fails to separate them from the noise background. However, in Figs. 5d–f, we clearly see one more row appearing in the wavelet domain as r increases, showing that the dynamical system has undergone two period-doubling bifurcations from Fig. 3d to Fig. 3f.

After infinite period-doubling bifurcations, the system may exhibit chaos. As an example, a chaotic time series at r = 3.8 and its Haar WT representations are shown in Figs. 6a,b. A striking feature in Fig. 6b is that there are different dominant scales at different times and there is not any regularity of variation at each scale where each cycle is different from its previous or following one. That is, the evolution of the time series does not repeat itself at any timescale, exhibiting a chaotic behavior. This chaotic, yet deterministic, behavior may have more relevance with the real world.

The existence of multiple scales with doubling and nondenning periods and their relationships among them shown by this simple dynamical system may provide a theoretical background needed to understand the results of WT applied to much complicated meteorological data containing nonlinear interactions among multiple timescales. The next section will give an example of WT application to the IR radiance data to study multiple timescales involved in tropical SCC organization.

4. Multiple timescales in IR radiance data

As mentioned in the introduction, previous studies have investigated multiple timescales in tropical convection by using GMS IR radiance data. Because of relative short length of their records, the representativeness of their findings for longer records need to be established. In addition, since the timescales noted in their studies may be related to even longer timescales, datasets with longer records are required to unveil the entire hierarchies of the SCC timescales.

Previous studies have also shown that the SCC organization involves both multiple temporal and spatial scales. In this paper we concentrate only on the temporal scales of the SCCs at a single location. As will be seen below, the convective activity at that location associated with different timescales is not uniformly distributed over the whole data length, that is, the SCCs exhibit the characteristics of time–frequency localization, which is best studied by wavelet analysis.

a. Data and methods

The data we use here are high-resolution, 1° x 1° and 3-hourly, GMS IR radiance data, which is equivalent blackbody temperature \( T_{BB} \), named IRTBB in our discussion) at the location 7°S, 108°E, in the western mountain area of Java. The data period is from 1 January 1987 to 31 December 1988. A total of 5848 data points for the two years are used for both the complex Morlet WT and the FT. The Java region is chosen because it possesses rich timescales; it has distinct di-

\[ \text{For a description of the GMS IR data, the readers are refer to Murakami (1983) and the references therein.} \]
urnal variation in different seasons and is in the location affected by the synoptic-scale features, Madden–Julian Oscillation, and Australian monsoon, as well as El Niño–Southern Oscillation. Thus, this time series may serve as a good illustration of the usefulness of WT in geophysical data.

In section 3, we demonstrated an application of the Haar WT to time series of the logistic map with dyadic scales. However, the same WT may not be of the most use in the appropriate analysis of the IRTBB data. This is because that the Haar WT detects the signal at the dyadic scales of power of 2, which is not the case in the real atmospheric oscillations where the timescales could be continuous. Although a continuous WT is, in practice, also performed discretely, the scale intervals can be specified as small as the resolution allows, so that a frequency signal with a scale that is not a power of 2 may be detected more accurately. When the intervals between scales are very small, the scales may be considered as continuous. The complex Morlet WT is continuous and is able to localize the signal in both time and frequency continuously. Moreover, the complex Morlet WT provides information of a signal on both amplitude and phase, while the Haar WT only yields information on amplitude with a plus or minus sign. Hence, an aliasing problem may occur. The drawback of using Haar WT for scale analysis for the IR data is discussed in more detail in the appendix.

The algorithm for the complex Morlet WT used is as follows. The WT is performed in the Fourier domain, as shown by Eqs. (3), (5), and (6) with \( k = 5.4 \). First, we specify the scales at which the wavelet coefficients will be calculated. The scale at the octave \( m \) and the voice \( n \) is

\[
a_{m,n} = 2^{m+n/2},
\]

where \( m = 10, 9, \ldots, -1, -2 \), \( n = 0, 1, \ldots, v - 1 \), and \( v = 8 \), which is the total number of voices per octave. The scale at \( m = 0 \) may be set at one's disposal; here it corresponds to 0.78125 day. Next, we calculate the dilated and translated wavelets for the octave 10 to -2, and for the voices 0 to 7 in each octave at all temporal positions. The complex Morlet wavelet coefficients are obtained by taking inner product of the FT of the time series and the wavelets expressed in the frequency domain, and then performing the inverse FT. The Morlet WT has edge effect at the beginning and the end portions of the time domain. To reduce such an edge effect, we need to pad some data points at the two ends. For efficiency, we have padded 1172 data points at both sides of the dataset by assuming a periodic condition in time. The new dataset with total 8192 (=2^13) points is then used for the complex Morlet WT. After the WT is performed, we discard the first and the last 1172 wavelet coefficients in time domain for each scale calculated, keeping the wavelet coefficients for the original 2-yr time interval only.

For comparison, we also performed FT. The Fourier spectral density is calculated based on the 2-yr data (5848 points). We use IMSL subroutine SSWD, with a Bartlett–Priestley window having the window parameter \( M = 2400 \).

b. Results

Figure 7a presents the 2-yr daily averaged time series, and Figs. 7b–d present the complex Morlet wavelet coefficients in time–frequency domain for modulus, phase, and real part, respectively, based on 3-hourly data. In Figs. 7b–d, the abscissa is time (days), which is labeled sequentially, that is, the day 1 denotes 1 January 1987 while the day 731 denotes 31 December 1988. The left ordinate is frequency in octave, and the right ordinate is period in day. The time series shows two annual cycles with the minima in Dec–Jan–Feb (DJF) and maxima in Jun–Jul–Aug (JJA). There are obvious “slow” intraseasonal oscillations with 50–60-day periodicity, especially in the DJF of 1987/88. Also noticeable are “fast” synoptic-scale oscillations, whose scales may be seen from Fig. 7b but are not easily discernible from Figs. 7c and 7d. This is due to the practical difficulty of plotting a large number of data points on the same chart, and not due to any inherent ability of the WT to resolve high frequency scales. Thus, the ranges of the ordinate in Figs. 7b–d are different for different needs. Figure 7b includes the full range of the scales from 0.5 to 2 yr, while Figs. 7c and 7d show only the scales from 2 day to 1 yr. Since the variations at the scales shorter than 2 days are hardly recognized in the plots with phase, the high-frequency oscillations will be shown later by using the “zoom” feature of the WT.

In Fig. 7b, the shading density represents the intensity of the signal in the wavelet domain. Several dominant scales or scale ranges are found.

(i) Annual and semiannual variability: The annual signal is represented by a rather evenly shaded dark horizontal bar near the bottom, which is very pronounced but does not have any obvious locality in time. The semiannual variation is a relatively weak signal, appearing as a harmonic of the annual variation. It has slightly stronger amplitude in the last four months of 1987. There is very little signal at the quasi-2-yr scale, which will not be discussed in this study.

(ii) Diurnal and semiidiurnal variability: These are shown near the top of the figure between octave interval (-1, 1). The intensity exhibits obvious seasonal vari-
ability. Two completely different regimes are observed for austral winters and summers. Java is in the equatorial trough in austral summer (DJF), which corresponds to the wet season. The dry season, which is identified by values of the IRTBB exceeding 270 K in most days, corresponds to the austral winter condition. Moreover, asymmetry exists within an annual cycle with a longer wet regime (about 8 months from Nov to Jun) and a shorter dry regime (about four months from Jul to Oct).

(iii) Synoptic variability: These are pronounced oscillations with periods of 2~6 days in octave interval (2, 3). These oscillations are closely related to diurnal and semidiurnal cycles, as well as intraseasonal scales, and mostly active in the wet season.

(iv) Intraseasonal variability: These are oscillations in the range of 10~90 day period in octave interval (4, 7). The most pronounced intraseasonal timescale is 50~60 days in DJF. The timescale and the amplitude of the intraseasonal variation is strongly modulated by the annual variability. The scale decreases toward shorter periods in both directions in time; that is, the intraseasonal timescales in falls and springs are shorter compared to those in DJF. The dominant timescales during the transition seasons are found to be 14~30 days. During austral winters (JJA), the dominant timescale is about 10 days.

The phase plot of the wavelet coefficients (Fig. 7c) is used to detect the singularities or the sudden changes by examining the convergence of phase lines in the time-frequency domain. The local phase is given by shading density on the plot. At each scale, from left to right within a cycle, one can follow a decrease in density, corresponding to a decrease of the phase from $\pi$ to $-\pi$. When the phase reaches to $-\pi$, it is wrapped around to the value of $\pi$. The distinct division between $\pi$ and $-\pi$ clearly shows constant phase line for each cycle at each scale. Several locations of convergence of constant phase lines are seen in this plot, corresponding to sudden variations in the time series. For example, the phase lines between day 220 and day 275 around octave 5 converge to a small location in the time-frequency domain around day 255 and octave 2. This cone-shaped area is known as the “influence cone” where the wavelet
coefficients herein have most influence on such an abrupt change in the signal.

The real parts of the wavelet coefficients provide a combined depiction of both the intensity and phase of a signal at a given time and scale relative to other times and scales. Figure 7d is a combination of Figs. 7b and 7c for real wavelet coefficients. Near the bottom of the plot, there are two prominent annual cycles, and four weak semiannual cycles with slightly stronger intensity in the second semiannual cycle of the first annual cycle. The semiannual signal is more clear in Fig. 7d than that in Fig. 7b because of the phase information included.

In the middle, there are two pronounced intraseasonal cycles at the scale of 50–60 days during DJF 1987/88 and shorter scales before and after. Mapes and Houze (1993) used 3-hourly infrared satellite images from the same Japanese GMS satellite to study cloud clusters and superclusters over the oceanic warm pool. One of the data periods they examined was from 25 November 1987 to 29 February 1988, which is covered by our data period. Their GMS imagery showed two eastward-moving cloudiness enhancements in that period. At 7°S, 108°E these eastward-moving cloudiness enhancements match quite well with the two intense cycles of the 50–60-day oscillation shown in Fig. 7d. It is also clear that the locations of convergence of phase lines correspond to sudden changes in signal. The details of the high-frequency oscillations in wet and dry seasons may be better discerned using the “zoom” feature discussed below.

Figures 8a and 8b show the time series and wavelet coefficients during 1 January–29 February 1988 (days 366–425), which is a “zoom in” picture of Figs. 7a and 7d, respectively, showing the high-frequency oscillations up to semidiurnal scale. This period is in the negative phase of an annual cycle, implying a wet period, as shown by the dark bar at the bottom across the whole time domain. During this period, the diurnal signal is generally strong and so is the semidiurnal signal. This period covers a negative phase and two positive phases of the pronounced 50–60 day oscillation in DJF. Relative to the negative phase (18 Jan–7 Feb 1988), there are several pronounced cycles at synoptic scales with 2–6 days. During the positive phases of the
50~60 day oscillation, that is, in the first half of January and the second half of February, the synoptic variability is relatively weak and the active scales are relatively shorter compared with those during the intensive negative phase. Thus, a phase-locking relationship between the synoptic variability and the intraseasonal with 50~60 day period is obvious during this time interval.

Figures 9a and 9b show a further “zoom in” picture of Figs. 8a and 8b, respectively, for the period of 18 January~7 February 1988 (days 383~403). Figure 9b enlarges the four prominent synoptic cycles with timescale of about 5 days. Embedded in this timescale are the oscillations with shorter timescales ranging from 1.5 to 3 day, which were referred to as the quasi-2-day oscillation by Lau et al. (1991). The semidiurnal signal is prominent during this period. In Fig. 9b, where the semidiurnal oscillation is strong, so are the 2~3-day and/or 4~6-day oscillation, suggesting the existence of doubling periodicities within the synoptic scales.

Figures 10a and 10b present a “zoom in” picture of Figs. 7a and 7d, respectively, for the period of 5 July~2 September 1987 (days 186~245). This period is in the positive phase of the first annual cycle, implying a dry period, as shown by the lightly shaded bar at the bottom across the whole time domain. There is no significant intraseasonal variability except for a few weak 10~12-day variations associated with somewhat spo-
radic abrupt reduction in IRTBB. These abrupt changes are represented in Fig. 10b by the convergence of the phase lines toward the time–frequency locations where the sudden changes occur. Diurnal variability is much weaker than those shown in Figs. 8a and 8b. Since there is almost no convective activity during daytime in austral winters, the variation in IRTBB is likely due to the variation in surface temperature or moisture caused by diurnal variation of surface radiation. The synoptic-scale activity during this period is also weak, which is mainly limited to the first 14 days. The semidiurnal signals are also relatively clear during the first 14 days when the synoptic-scale activity is relatively strong.

As a comparison, Fig. 11 shows the spectral density of the 2-yr data. This is basically a red spectrum with isolated spikes at periods of 1 day and one-half day. The energy is mainly concentrated at the annual scale, with several peaks in the descending spectrum at 45–60 days, 25 days, 14 days, and 3.5–5 days. Because of their nonstationary nature, some of these periodicities may not pass traditional statistical significance tests. In particular, there is no indication of the 2–3-day variability in the spectral density. The intensity of the annual variability is about an order larger than that at 45–60-day intraseasonal variability, with the diurnal variability in between. The Fourier spectrum does yield a distribution of multiple frequencies, but clearly it is unable to provide information on time–frequency localization provided by the wavelet analysis.

5. Discussion

In this section, we offer some plausible physical interpretation of the multiscale structure found in the IRTBB data, in the context of nonlinear dynamical systems. We speculate that the multiscale oscillations revealed by the wavelet analysis may be the manifestation of two basic dynamical mechanisms: forced and self-excited oscillations. The forced oscillations are the response of the tropical convection to changes in the external forcing while the self-excited variations are due to internal instabilities and nonlinear interactions among different frequency components of tropical con-

Fig. 8. A "zoom in" picture from Figs. 7a and 7d for the time interval of days 366–425 (1 Jan–29 Feb 1988) based on 3-hourly data (480 points). This period is within a wet season.
Convection and the large-scale environment. Clearly, the most important external forcing is the solar radiation. The annual and diurnal variations of tropical convection are due to such external forcings. The semidiurnal variations in IRTBB may also be a forced mode due to both atmospheric solar tide oscillation, known as \( S_2 \) wave (the semidiurnal solar tidal component) and the harmonic resonance of the diurnal forcing (Bjerknes 1948; Brier 1965, 1966; Brier and Simpson 1969). Brier and Simpson (1969) established a statistical relationship between tropical cloudiness and rainfall and the semidiurnal solar \( (S_2) \) atmospheric tide. The possible relationship between the semidiurnal variation and synoptic-scale variation was discussed by Brier and Carpenter (1967) using tidal theory. They suggested that the tropical atmosphere may be more prone to the development of synoptic-scale disturbances when the \( S_2 \) wave has a large amplitude.

Murakami (1983) analyzed the IR radiance data observed by GMS-1 geostationary satellite for the period during northern winter (December 1978–January 1979) and summer (July–August 1979) to study the deep convective activity over the western Pacific and Southeast Asia. He found that during northern winter, the amplitude of the diurnal variation is large over the Indonesian region and the northern Australia. Our wavelet analysis (Figs. 8b and 10b) shows that the amplitude of the diurnal cycle is also much larger during northern winter than summer over Java. Our result is not only consistent with Murakami’s result but also reveals more details about the locality of the diurnal and semidiurnal variations in time.

Synoptic variability in the Tropics is due to a combination of latent heat induced circulation instability as well as instability in the zonal flow in the tropical atmosphere. This variability has been considered as arising from a self-excited, internal mode, independent of external astronomical forcing. The phase-locking relationship between synoptic and diurnal/semidiurnal and that between synoptic and intraseasonal/annual variations suggests that synoptic variation may be mixed oscillations of self-excited modes and forced modes due to harmonic and subharmonic resonance of diurnal and annual solar radiation variations.

For intraseasonal variations, many mechanisms have been proposed. For review, the readers are referred to
Wang (1993). The intraseasonal oscillation is a relatively broadband phenomenon that exhibits continuous as well as episodic development with mean period of about 40–50 days. While these oscillations can also be influenced by synoptic-scale disturbances, the intraseasonal oscillations have been considered essentially as self-excited and are not caused by external astronomical forcing. However, the phase-locking relationship between intraseasonal and annual variations suggests that intraseasonal variation may also be partially influenced by harmonic resonance of annual solar forcing.

Results in section 3 show that even a simple nonlinear dynamical system may have very complicated oscillation behavior containing many timescales due to period-doubling bifurcations. The period-doubling process, as indicated by the logistic map, provides a source of a rich variety of oscillations with nonuniformity in time. However, because of strong influence by external forcings and the weak low-frequency signals in multiply doubled periodicities, high-level period doublings are extremely difficult, if not impossible, to be detected in real geophysical data. The small amplitude low-frequency modes in a dynamical system would be obscured by random noise. If the forcing is periodic with a period that coincides with one of the natural periods, the nonlinear interaction of the self-excited mode and the forcing may result in excitation of that period and its harmonic and subharmonic modes by nonlinear interactions. The magnitude of the forcing can also significantly alter the dynamic phenomena exhibited by the simple unforced nonlinear system. Therefore, in order to understand the multiscale quasi-periodic variation such as the one we discussed here, we must not ignore the role of external forcing.

The hierarchy of multiple scales found in IRTBB (1/2 day, 1 day, 2–3 day, 4–6 day, 10–15 day, 20–30 day, 50–60 day, etc.) and their phase relationships appear to conform to the major periodicities in an approximate period-doubling tree, modified by external forcing and internal frequency splitting. It is conjectured that the above synoptic and intraseasonal periodicities may represent a system of mixed oscillations caused by the interaction internal dynamics with one or more time-dependent energy sources.
there are sporadic outbreaks of weak convection activities at the dominant timescale of about 10–12 days. The intraseasonal timescales in transition seasons, spring and fall, appear to be shorter than those in winter seasons but longer than those in summer seasons during 1987–88. The phase-locking relationships among the oscillations with different timescales suggest that the multiple timescales in tropical convection appear to conform to an approximate period-doubling tree, possibly arising as a result of mixed oscillations due to the interaction of self-excited oscillations in the tropical atmosphere and the external forcings, such as the annual and diurnal solar cycle. Further theoretical studies on these mixed modes will be helpful to understand the observed timescales and predictability of these variations.

It is emphasized that the above results may not be generalized. They are representative of the kind of information that can be obtained by WT. Finally, we point out certain precautions in applying WT. While we have demonstrated its important applications to geophysical data analysis, WT should not be considered as a replacement for FT. For global wave–like features and stationary time series, there is no need to use WT. It is mostly useful for detecting signals in non-stationary time series. Since the wavelet coefficients are inner products of a time series with the dilated and translated wavelets and contain information of both the signal and the wavelets themselves, the magnitudes of the coefficients are wavelet dependent. Therefore, a judicious choice of wavelets is required. Some prior knowledge of the characteristics of the data is desirable. Because of the lack of a standard test for statistical significance, WT is most effective when used as a complement to other time series techniques.

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APPENDIX

Haar Wavelet Coefficients of an IRTBB Series

The Haar WT may not be suitable for scale analysis of data that contain continuous phase information and
the scales that may not be dyadic (discrete as power of 2). Figure A1 shows an example of how a phase shift in a dataset may affect the scale analyzed by the Haar WT.

In Fig. A1a, the coefficients at the semidiurnal scale are in general larger than those at the diurnal scale, which does not represent the true situation, where the diurnal variation of IRTBB is in general stronger than the semidiurnal variation, which is found by both Morlet WT (Fig. 7b) and FT (Fig. 11). The aliased scale selection is caused by the fact that the Haar wavelets are orthogonal with only dyadic scales. For example, for a digital “cosine-type” signal with unit amplitude and a period of four time units, that is, 1, -1, -1, 1, the main scale detected by the Haar WT is two time units, not four, with the coefficients 0, 0, $\sqrt{2}$, $-\sqrt{2}$ based on Eqs. (10) and (15). However, for a signal in quadrature with the previous signal at the same period of four time units but phase shifted 90°, that is, 1, 1, -1, -1, we will find that the main scale detected by the same Haar WT is four time units with the coefficients 0, 2, 0, 0. Thus, for cosine-type signals, a pair of positive and negative Haar WT coefficients at the scale $a/2$ may improperly represent the true scale $a$ that this signal has. Since the IR radiance dataset we used here starts at midnight, which resembles a cosine-type signal rather than a sine-type signal regarding the diurnal variation, a pair of positive and negative (or vice versa) coefficients at the semidiurnal scale may represent a signal with the diurnal variation. After we skipped the first two records, that is, made the dataset starting early morning so that the new dataset contains a sine- rather than a cosine-type signal relative to diurnal variation, the most significant Haar wavelet coefficients in this case appear at the diurnal scale, as shown in Fig. A1b.

The problem of using the Haar WT to this IR radiance dataset may be partially corrected using prior information about the diurnal variation. But the remaining problem is that we still do not know what the initial phases should be for the remaining low-frequency variations in the real data in order to get a proper scale representation by the Haar WT. This kind of problem does not occur when using the Morlet wavelet. Thus, although the Haar WT is very useful for some analytic time series with dyadic scales, such as those from the logistic map, it may not be suitable for analyzing a dataset with more continuous scales and phases.

The advantage of using the Haar WT or other orthogonal WT is that we get the minimum number of wavelet coefficients, which is the same number as that of sampling data points of the signal, and may reconstruct the data using fewer coefficients without loss of significance; this is very useful in data compression. Since the orthogonal Haar WT is performed at dyadic

**Fig. A1.** The Haar wavelet coefficients of IRTBB for the first 64 days of 1987 at different scales. (a) The data begin at 0200 local time (the first record), and (b) 0800 local time (the third record).
scales and discrete positions, some important signal information, which exists between these dyadic scales and discrete positions, may not be resolved properly by the Haar WT.

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