The Effect of Critical Levels on 3D Orographic Flows: Linear Regime

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ABSTRACT

The effect of a critical level on airflow past an isolated axially symmetric obstacle is investigated in the small-amplitude hydrostatic limit for mean flows with linear negative shear. Only flows with mean Richardson numbers (Ri) greater or equal to ¼ are considered. The authors examine the problem using the linear, steady-state, inviscid, dynamic equations, which are well known to exhibit a singular behavior at critical levels, as well as a numerical model that has the capability of capturing both nonlinear and dissipative effects where these are significant. Linear theory predicts the 3D wave pattern with individual waves that are confined to paraboloidal envelopes below the critical level and strongly attenuated and directionally filtered above it. Asymptotic solutions for the wave field far from the mountain and below the critical level show large shear-induced modifications in the proximity of the critical level, where wave envelopes quickly widen with height. Above the critical level, the perturbation field consists mainly of waves with wavefronts perpendicular to the mean flow direction. A closed-form analytic formula for the mountain-wave drag, which is equally valid for mean flows with positive and negative shear, predicts a drag that is smaller than in the uniform wind case. In the limit of Ri \( \leq \frac{1}{4} \), in which linear theory predicts zero drag for an infinite ridge, drag on an axisymmetric mountain is nonzero.

Numerical simulations with an anelastic, nonhydrostatic model confirm and qualify the analytic results. They indicate that the linear regime, in which analytic solutions are valid everywhere except in the vicinity of the critical level, exists for a range of mountain heights given Ri > 1. For Ri \( \leq \frac{1}{4} \) this regime is difficult to achieve, as the flow is extremely sensitive to nonlinearities introduced through the lower boundary forcing that induce strong nonlinear effects near the critical level. Even well within the linear regime, flow in the vicinity of a critical level is dissipative in nature as evidenced by the development of a potential vorticity doublet.

1. Introduction

The nature of topographically induced perturbations in stratified flows has been a widely studied subject due to its applicability to airflows over complex terrain and the associated mesoscale phenomena. The variety of the topographically induced perturbations reflects the sensitivity of the system to the nature of forcing as well as to the properties of the medium through which they propagate. In the atmosphere, where (in general) both wind and stability vary with height, the impact of this variation on the vertical structure of internal gravity waves can be profound. This is especially evident if the mean flow contains critical levels for mountain-generated internal gravity waves.

A critical level is one at which the intrinsic frequency \( \omega' = \mathbf{U} \cdot \mathbf{k} - \omega \) of a plane wave with horizontal wavenumber vector \( \mathbf{k} \) and frequency \( \omega \) becomes zero. In other words, it is a level where wave phase speed becomes equal to the component of the mean velocity parallel to the horizontal wavenumber vector. For mountain waves resulting from a steady forcing by the flow over topography, \( \omega = 0 \), and the critical levels arise wherever \( \mathbf{U} \cdot \mathbf{k} = 0 \). The name critical derives from the singular behavior of the linearized inviscid equations at such levels, recognized originally in the context of hydrodynamic stability of parallel shear flows [see Drazin and Reid (1971) for a review]. The seminal contributions by Bretherton (1966) and Booker and Bretherton (1967) were first to describe the effects of critical levels on the propagation of internal gravity waves, analyzing both the transient and steady-state aspects of the linearized inviscid 2D problem. Using the WKB approximation (assuming the mean-flow Ri is large), Bretherton (1966) showed that as a wave packet approaches a critical level, the vertical component of the group velocity and the vertical wavelength tend to zero. The wave packet cannot reach the critical level in finite time, and its energy becomes absorbed by the mean current without any reflection. Booker and Bretherton (1967) investigated the interaction of monochromatic internal waves generated by a flow over a wavy boundary and a critical level with Ri > \( \frac{3}{4} \). Their results showed that the vertical wavenumber and horizontal velocity become infinite at

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the critical level. Aloft, the amplitude of the waves as well as energy and momentum fluxes are strongly attenuated. The effective absorption of waves (i.e., the deposition of the wave momentum and energy into the mean flow) occurs below the critical level.

The singular behavior of the linear, inviscid, time-independent equations indicates that the transience, momentum, and heat diffusion, as well as nonlinear steepening and amplification of waves, are important in a vicinity of the critical level (i.e., within a critical layer). Different balances between these effects can lead to a variety of flow evolution scenarios, from linear viscous wave absorption to strongly nonlinear “cat’s eye” circulations within critical layers, as demonstrated in a number of studies (theoretical, numerical, and laboratory) that have examined the interaction of small- and large-amplitude waves with a critical level. For reviews see Kelly (1977), Maslowe (1986), and chapter 4.11 in Baines (1995). For $Ri > \frac{1}{4}$ at the critical level, two aspects of the gravity wave–critical level interaction are well established. First, in agreement with the linear prediction, there is no significant transmission of wave energy through the critical level. Second, for sufficiently large-amplitude incoming waves, nonlinearities dominate viscosity and diffusion within the critical layer, resulting in wave steepening and overturning. Thus, the critical level is a preferred location for internal wavebreaking (Winters and D’Asaro 1989; Winters and Riley 1992; Fritts et al. 1994). The resultant convective and shear instabilities lead to the formation of a homogeneous mixed layer below the critical level that acts as a perfect reflector to all incoming waves.

Such a scenario of the local flow evolution has been identified by Clark and Peltier in their numerical simulations of 2D finite-amplitude mountain waves (Clark and Peltier 1977). They have attributed the formation of the downslope windstorms (cf. Lilly and Zipser 1977) to the wave–critical level interaction. Here, the critical level and the associated critical layer are wave-induced local features corresponding to a quasi-stagnant homogeneous region of finite thickness and horizontal extent that forms as a result of mixing in the overturning mountain waves above the lee slopes. The reflective behavior of the critical level decouples the flow aloft from that at lower levels, whereupon the low-level stratified flow yields a sub- to supercritical transition over the obstacle for certain critical-level heights (Peltier and Clark 1979; Smith 1985). A similar flow evolution has been simulated by substituting the local, wave-induced critical level with a global critical level in the ambient flow (Bacmeister and Pierrehumbert 1988; Durran and Klemp 1987). By specifying an ambient critical level, the height of the wave reflecting level could be controlled, which has led to the numerical solutions consistent with the nonlinear analytic model of Smith (1985). Although nonlinearity seems to be an inherent part of the 2D gravity-wave–critical-level interaction, the recent study by Wurtele et al. (1996) documents that this does not necessarily extrapolate to all types of flows with critical levels. Their investigation addresses the propagation of small-amplitude 2D gravity-inertia waves below corresponding critical levels (Rossby singular levels) and provides an example of linear critical-level behavior. For a monochromatic gravity-inertia wave they found that the wave–critical-level interaction leads to a nonlinear reflection similar to the case of a gravity wave. However, when a prescribed forcing excites a continuous wave spectrum, with the Rossby critical levels different for every wave component, the problem is essentially linear as the critical levels produce no singular effects even in the absence of viscous dissipation.

While the understanding of the gravity-wave–critical-level interaction in 2D orographic flows is still incomplete, recently, interests have shifted toward 3D flows (Nappo and Chimonas 1992; Shutts 1995; Broad 1995). In general, 3D gravity waves encounter their critical levels at different altitudes (similar to the 2D gravity-inertia waves), where the projection of the mean velocity $U(z)$ on a given wavenumber vector $\kappa = (k, l)$ vanishes. Using linear analysis, Shutts (1995) and Broad (1995) have examined the effects of the directional wind shear (and the associated continuous distribution of critical levels) on the vertical flux of horizontal momentum. In particular, they showed that the stress force due to wave absorption at any critical level is normal to the local wind direction—a finding important for parameterizing the gravity wave drag in large-scale models. A single critical level in a unidirectional mean flow has been addressed by Nappo and Chimonas (1992). Employing linear theory, they investigated the effects of wave absorption below the critical level within a stably stratified boundary layer. The surface wave drag and the stress due to the critical-level absorption were found to be of the same order of magnitude as the drag conventionally associated with the surface friction.

In this study, we address unidirectional steady flows with the critical level located where $U = 0$ and describe the 3D steady-state wave pattern forced by an isolated axisymmetric obstacle. We combine the approaches of Booker and Bretherton (1967) and Smith (1980) to provide linear hydrostatic solutions for a class of constant-shear flows past a bell-shaped mountain. In the zero-shear limit, our solutions reduce to those of Smith (1980) for the uniform-wind case. Similar to the equivalent 2D problem (Smith 1986), we find that the wave drag (as well as the solutions themselves) depends on the mean shear. However, in the limit of $Ri < \frac{1}{4}$, this dependence is essentially different from that in 2D, due to an inherently three-dimensional effect of lateral wave dispersion. Since linear inviscid solutions are singular at the critical level we use a numerical model to assess the impact of dissipation and local nonlinearities on the solutions. We determine the bounds, in the parameter space spanned by nondimensional mountain height $\bar{h}$...
and the global Richardson number \( R_i \), of a regime in which our linear solutions are uniformly valid except in the vicinity of the critical level where numerical solutions, as well as natural flows, are regularized by viscous dissipation (Kelly 1977; Worthington and Thomas 1996).

Numerical simulations also indicate that the wave breaking below a single critical level in 3D orographic flows can be induced for relatively small mountains, a behavior already familiar from the 2D orographic flows. Thus, by introducing a critical level in an ambient flow, the wave breaking aloft can be examined separately from other nonlinear aspects of the 3D flows. This provides an opportunity to clarify the role of wave breaking in fostering the transition from a linear flow regime dominated by mountain waves to a nonlinear regime characterized by splitting of the low-level upwind flow and the formation of lee eddies (Hunt and Snyder 1980; Smolarkiewicz and Rotunno 1989, 1990; Smith 1989; Crook et al. 1990; Miranda and James 1992; Smith and Grönas 1993; Schär and Durran 1997 and references therein). Such finite-amplitude effects in 3D flows with critical levels will be addressed in future studies.

The paper is organized as follows. In section 2, we present the linear equations and discuss their solutions. In section 3, we present the numerical model and compare the results of selected simulations with the linear theory predictions. Section 4 concludes the paper.

2. Linear model

a. Governing model

The linear model outlined in this section is used to obtain steady-state solutions for a stably stratified, inviscid, nonrotating flow past a small-amplitude, gently sloped 3D hill. The Euler equations in the incompressible Boussinesq approximation, cast in Cartesian framework \((x, y, z)\), are linearized with respect to a hydrostatically balanced reference state characterized by the velocity \( U(z) \), pressure \( p(z) \), and density \( \rho(z) = \rho_0(1 - N^2 z/g) \). Here, the Brunt–Väisälä frequency \( N \) and reference density \( \rho_0 \) are assumed constant, and \( g \) is the acceleration of gravity. With these assumptions the linear model equations are

\[
\begin{align*}
\rho_0 U u_x + \rho_0 w U z &= -p_x, \quad (1a) \\
\rho_0 U v_z &= -p_y, \quad (1b) \\
\rho_0 U w_z &= -p_z - \rho g, \quad (1c) \\
u_x + u_y + w_z &= 0, \quad (1d) \\
U p_x + w \bar{p}_z &= 0, \quad (1e)
\end{align*}
\]

where \((u, v, w)\) are the perturbation velocity components; and \( p = p(x, y, z) \) and \( \rho = \rho(x, y, z) \) are the perturbation pressure and density, respectively.

Equations (1a)–(1e) can be combined into a single PDE for the vertical velocity

\[
\left( \nabla^2 + \frac{N^2}{U^2} - \frac{U z}{U} \right) w_x + \frac{N^2}{U^2} w_y = 0,
\]

where \( \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \). Equation (2) is subject to the linearized impermeability boundary condition \( w = U h_x \) at \( z = 0 \) [assuming the lowest isopycnal conforming to the surface of the mountain \( z = h(x, y) = 0 \)] and the radiation condition aloft requiring exclusive upward energy propagation for \( z \to \infty \).

Using in (2) the Fourier transformation with the pair of transforms,

\[
\hat{w}(k, l, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x, y, z) \exp[-i(kx + ly)] \, dx \, dy,
\]

\[
w(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{w}(k, l, z) \exp[i(kx + ly)] \, dk \, dl,
\]

results in the equation for the vertical structure of the Fourier modes

\[
\hat{w}_z + \frac{\kappa^2}{k^2} \left( \frac{N^2}{U^2} - \frac{U z}{U} \frac{k^2}{\kappa^2} - k^2 \right) \hat{w} = 0.
\]

In (3a)–(3b), \((k, l)\) are the components of a horizontal wavenumber vector \( \mathbf{k} = ki + lj \); where \( \kappa^2 = k_x^2 + F \), and \( i \) and \( j \) are unit vectors in the \( x \) and \( y \) directions, respectively. In the remainder of this analysis we shall assume the hydrostatic approximation and, consequently, neglect the acceleration term on the lhs of (1c) and equivalently the last term in the parenthesis on the lhs of (4). The physical justification of such an approximation in a sheared environment is less straightforward than in an atmosphere with constant wind and stability (where it is valid if \( k \ll N/U \)); see Keller (1994) for a discussion. However, when the \( U_{ex} \) term in (4) is zero everywhere and the mean velocity decreases with height, the importance of non-hydrostatic effects also decreases with height. In such a case, the hydrostatic approximation can be justified given \( k \ll N/U(z = 0) \).

For the sake of convenience, we shall formulate our problem in terms of the vertical displacement of isopycnal surfaces instead of \( w \). For any given isopycnal of upstream height \( z_0 \), we define the displacement field \( \eta(x, y; z_0) = z(x, y; z_0) - z_0 \). The density perturbation \( \rho \) can be expressed in terms of \( \eta \) using the linearized kinematic condition \( w = \eta = U \eta_x \) in the entropy equation (1e). Assuming all perturbations vanish as \( x \to -\infty \), this leads to \( \rho = -\rho_0 \eta \) in lieu of (1e). With the hydrostatic approximation, (4) is then replaced by

\[
\hat{\eta}_z + \frac{2U}{U} \hat{\eta}_x + \frac{\kappa^2 N^2}{k^2 U^2} \hat{\eta} = 0,
\]

subject to
\[ \eta(k, l, z = 0) = \tilde{h}(k, l), \]  
and the radiation boundary condition aloft. In (6), \( \tilde{h}(k, l) \) denotes the Fourier transform of the mountain shape.

In the next section we discuss solutions of (5)–(6) for negatively sheared mean flows with critical levels.

**b. Integral solutions**

We consider a class of the mean velocity profiles,

\[ U(z) = U_0 \left(1 - \frac{z}{z_c}\right), \]  

For such profiles, (5) reduces to

\[ \xi^2 \eta_{,\xi} + 2 \xi \eta_{,\xi} + \frac{\kappa^2}{k^2} \eta = 0, \]  

where \( \xi = z - z_c \). The general solution of (8) for \( \xi \neq 0 \) is

\[ \eta(k, l, \xi) = \xi^{-1/2} [A(k, l) \xi^\mu + B(k, l) \xi^{-\mu}], \]  

\[ \mu(k, l) = \left[ \text{Re} \left( \frac{2}{k} \right) \right]^{1/2}, \]  

\[ \text{Ri} = \left( \frac{N}{U_c} \right)^2 = \frac{\langle N_0 \rangle}{U_0^2}, \]  

where (9c) defines a global Richardson number [identical to the local Richardson number at the critical level \( z = z_c \) for this special case of the linear velocity profile (7)]. Our analysis will be limited to \( \text{Ri} > \frac{1}{4} \), which is sufficient for the stability of a parallel, stratified, inviscid, incompressible flow with respect to the Kelvin–Helmholtz waves (section 44.3, Drazin and Reid 1971).\(^1\)

At the critical level, (8) has a removable singularity; the nontrivial solutions on the two sides of this singularity can be connected by expanding \( U(z) \) in a Taylor series in the neighborhood of \( z_c \) and obtaining a solution there by the method of Frobenius (Miles 1961; Booker and Bretherton 1967). In general, for any \( U_0 < 0 \) at the critical level, the term in (9a) that satisfies the radiation condition aloft is

\[ \eta(k, l, \xi) = B(k, l) \xi^{-\mu(k, l)}, \]  

\[ \nu(k, l) = 1/2 + i \mu(k, l), \]  

(Booker and Bretherton 1967), implying \( A(k, l) = 0 \). Imposing the lower boundary condition (6), and using the same branch of the logarithmic function \( \ln(z - z_c) = \ln|z - z_c| + i \delta, \delta \in [0, \pi], \) employed in derivation of (10a), leads to

\[ B(k, l) = i \tilde{h}(k, l) \exp[-\pi \mu(k, l)] e^{i k x}. \]  

Then,

\[ \tilde{\eta}(k, l, z) = \begin{cases} h z^{1/2}, & 0 < z < z_c, \\ i e^{-\pi \kappa} h z^{-1/2}, & z > z_c. \end{cases} \]  

and the general solution in integral form is

\[ \eta(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{h}(k, l) \exp[i \Phi_+(k, l; x, y, z_\ldots)] dk dl, \]  

\[ \Phi_+(k, l; x, y, z_\ldots) = \frac{k x + i y - \text{sgn}(k) \mu(k, l) \ln z_\ldots}{1}, \]  

where \( \text{Ri} > \frac{1}{4} \), \( \text{Ri} < \frac{1}{4} \) and \( \text{Ri} = \frac{1}{4} \).

The lower sign in (14b) applies for \( z_c < z < 2z_c \) and the upper sign for \( z > 2z_c \). In both (13b) and (14b), \( \text{sgn}(k) \) has been included to select only wave modes that propagate energy upward. According to (14a), above the critical level, wave components emerge attenuated by the factor \( \exp(-\pi |\text{Ric|cos}\varphi|^{1/2}) \), where \( \varphi \) represents the angle between the horizontal wavenumber vector \( \mathbf{k} \) and the direction of the mean flow. In this directional filtering, the wave components with wave fronts parallel to the mean flow (\( \varphi = \pi/2 \)) will be entirely removed, and the wave solutions will contain only those components whose wave fronts are perpendicular or nearly perpendicular to the mean flow. As in the 2D case, the latter will be attenuated by their passage through the critical level by the factor \( \exp(-\pi (\text{Ric} - \frac{1}{4})^{1/2}) \) (Booker and Bretherton 1967). For \( \text{Ri} = 1 \) and \( \varphi = 0 \), the amplitude of the transmitted wave is only 6.5% of the amplitude of the incoming wave. Thus, transmission of waves through the critical level is extremely weak. For an axisymmetric bell-shaped mountain,

\[ h(r) = \frac{h_0}{1 + r^2/a^2}^{1/2}, \quad r = \sqrt{x^2 + y^2}, \]  

with the Fourier transform

\[ \tilde{h}(k, l) = \frac{1}{2\pi} h_0 a^2 e^{-ak}, \]  

the displacement field computed from (13a,b) and
FIG. 1. FFT representations of the displacement fields of the isopycnals with the undisturbed heights $z_1 (1 \pm 0.15)$ below (a) and above (b) the critical level, respectively. The mean flow (with $\text{Ri} = 1$ and $\hat{h} = 0.1$) is from left to right. The axisymmetric mountain (15) is displayed using thick solid lines with the contour interval 0.025. 

FIG. 2. FFT representations of isopycnal surfaces in the central $x-z$ plane for the same flow as in Fig. 1.

Using fast Fourier transforms (FFT) is displayed in Figs. 1a, b at $Z = Z_2 = 0.15$.

Infinitesimally below the critical level, solution (13a) oscillates rapidly and the predicted displacements approach infinity. The “local” vertical wavenumber

$$m(z) = \frac{\partial}{\partial z} (\mu(k, l) \ln Z_{\ast})$$

$$= \frac{N}{U(z)} \left( \frac{k^2}{4\text{Ri}} \right)^{1/2}$$

(17)
tends to infinity as $z \to z_{\ast}$. From (17), it is also evident that the decrease of the mean wind with height has the effect equivalent to an increase of the mean stability, making it more likely for the waves to steepen and break aloft. As in the 2D case (Booker and Bretherton 1967; Smith 1986), the vertical velocity is proportional to $(z_{\ast} - z)^{1/2}$ and becomes infinitesimally small in the vicinity of the critical level. At the same time, both horizontal velocity components, which are proportional to $(z_{\ast} - z)^{-1/2}$, tend to infinity as $z \to z_{\ast}$. This indicates that, at least in the linear approximation, motions become progressively more horizontal as the critical level is approached. The singular nature of the linear solution is illustrated in Fig. 2.

In the limit of zero shear, $z_c \to \infty$, $Z_c \to 1$, $-\mu(k, l)$ $\times \ln Z_c \to (\kappa N z/kU_0)$, and (13a)–(13b) reduce to the integral representation of the solution describing mountain waves in an atmosphere with constant wind and stability (Smith 1980).

Unlike the 3D linear mountain waves in an atmosphere with constant wind and stability, the linear solution of (1a)–(1e) contains nonzero vertical vorticity. The vertical component of vorticity is generated by the tilting of the mean shear vector $U \hat{\gamma}$ in the linear wave field while the total vorticity vector remains tangential to the isopycnal surfaces. This mechanism is already captured in the linearized vorticity equation implied by (1a)–(1c). Figure 3 shows the vertical vorticity field corresponding to the wave solution in Fig. 1a.

In the following two sections we examine the asymptotic solutions describing the displacement of isopycnal surfaces at large distances from the mountain and close to the critical level, and the wave drag, a force exerted on the mountain by the flow.

c. Asymptotic solutions

In order to derive the asymptotic solutions, we start by introducing nondimensional variables

$$\hat{x} = x/a, \quad \hat{y} = y/a, \quad \hat{z} = zN/U_0,$$

$$\hat{k} = ka, \quad \hat{l} = la, \quad \hat{\text{Ri}} = ka,$$

(18)

(cf. Smith 1980). In (18), horizontal and vertical distances are scaled differently, so as to conform with the
hydrostatic approximation, with the ratio of the vertical to the horizontal scale equal $U_0/aN \ll 1$.

In nondimensional form, the integral solution (13a,b) for $0 \leq z < z_e$ and $\hat{h}$ given by (16) is

$$\hat{h}(\hat{x}, \hat{y}, \hat{z}; c) = z_e^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\xi} \exp[i\Phi_- (k, \hat{h}, \hat{x}, \hat{y}, \hat{z}; c)] \, dk \, dl,$$  

(19a)

where

$$\Phi_- (k, \hat{h}, \hat{x}, \hat{y}, \hat{z}; c) = k\hat{x} + l\hat{y} + \frac{\sqrt{c}}{2} \left[ (1 - c^{-1})\hat{k}^2 + \hat{l}^2 \right]^{1/2} \ln \frac{\hat{z}}{c},$$

(19b)

$$\hat{z}_e = \left( 1 - \frac{2z_e}{\sqrt{c}} \right),$$

(19c)

$c = 4Ri > 1,$

(19d)

$$\hat{h} = \frac{hN}{U_0}.$$  

(19e)

For large radial distances from the mountain, the integral in (19a) can be evaluated using the method of stationary phase (appendix A). The resulting far-field solution is

$x < 0$, $\hat{h}(\hat{x}, \hat{y}, \hat{z}; c) = 0,$  

(20a)

$x > 0$, $\hat{h}(\hat{x}, \hat{y}, \hat{z}; c)$

$$= \frac{2\hat{h}}{\hat{\rho}} z_e^{-1/2} B^* e^{-\rho c} \cos \left( \frac{\sqrt{c}}{\hat{\rho}} \frac{\ln \hat{z}_e}{2 \sin \theta} \right),$$  

(20b)

where

$$\hat{\rho} = [\hat{x}^2 + (1 - c^{-1}\hat{y}^2)]^{1/2},$$

(20c)

$$\hat{\rho} = [\hat{x}^2 + \hat{y}^2]^{1/2},$$

(20d)

$$\beta^* = \frac{\hat{\rho} \sqrt{c}}{\hat{\rho} \hat{z}} \left[ \ln \hat{z}_e \right], \quad \sin \theta = \frac{\hat{y}}{\hat{\rho}}.$$  

(20e)
Figure 4 displays the asymptotic solution (20b) at a particular height below the critical level. In any horizontal plane, the solution has a wavelike character, governed by the cosine factor in (20b). The phase lines of individual modes are radii originating at the center of the mountain. Their horizontal tilt \((\phi, \lambda = -k\phi_0)\); cf. appendix A) is a function of both the wavenumber vector and the mean shear. The wave envelopes, defined as locus of points where the amplitude factor \(\beta^* e^{-\eta z}\) is maximal, are paraboloidal surfaces

\[
\eta^2 = \frac{1}{2} \frac{\beta^*}{\beta_0} \ln |Z|, \tag{21}
\]

whose vertices coincide with the center of the mountain.\(^2\) Except for the immediate vicinity of the obstacle, the asymptotic solution compares well with the FFT representation (Fig. 1a).

Figure 5 illustrates one effect of negative shear on the far-field solutions. It compares the wave envelopes for \(Ri = 1\) and \(\infty\). The widening of the envelopes with height is apparent in both cases. However, it occurs much more rapidly for the sheared flow case. In the limit \(z \rightarrow z_c\), (19a)–(19e) can again be evaluated using the method of stationary phase (appendix A). The asymptotic solution in this limit is

\[
\eta(\xi, \zeta, \zeta'; c) = \sqrt{\frac{2}{\pi}} \frac{\beta^*}{\beta_0} \ln |Z| \alpha^2 \left(1 - 2\xi - \xi^2\right) \cos\left(\frac{\sqrt{c} - 1}{2} \ln |Z|\right) + \left(1 + 2\xi - \xi^2\right) \sin\left(\frac{\sqrt{c} - 1}{2} \ln |Z|\right), \tag{22}
\]

demonstrating that the wave fronts become perpendicular to the direction of the mean flow as \(z \rightarrow z_c\), similar to the unsheared case at \(z \rightarrow \infty\). In the zero shear limit, (22) simplifies to the corresponding expression [Eq. (34)] in Smith (1980).

d. Surface pressure and mountain wave drag

As a consequence of the hydrostatic approximation, the pressure perturbation at any height \(z\) can be related to the vertical integral of density perturbation aloft through

\[
p(x, y, z) = -g \frac{d\rho}{dz} \int_z^\infty \eta(x, y, z') dz'. \tag{23}
\]

For the case of negatively sheared mean flow with a critical level, the upper limit of the integration in (23) can be approximated by \(z_c\). The error introduced by neglecting the rest of the integral is small since the amplitude of the waves transmitted through the critical level is, in general, negligible compared to that of the incoming waves. In spite of infinitely large displacements at the critical level, the integral is still finite, because the vertical wavenumber tends to infinity as \(z \rightarrow z_c\) and the positive and negative displacements cancel out. In the analysis that follows, we shall also include the case of positively sheared mean flow, where the vertical integration in (23) extends to infinity. Using stationary phase arguments (cf. appendix A in Reisner and Smolarkiewicz 1994), it can be shown that the contribution to the total integral of the vertically oscillating term vanishes at the upper limit.

The surface pressure can be obtained by substituting (19a)–(19d) in (23) and performing the integration in the vertical. Applying the transformations (A11) and integrating with respect to \(\xi\) (cf. appendix A) we arrive at the result

\[
\frac{p(\rho, \theta, \phi = 0)}{\rho_0 U_0^2} = \pm \frac{\sqrt{Ri}}{\pi} \frac{\cos\phi}{\left(\cos\phi + i \sqrt{c - \cos^2\phi} \left[1 - i \rho \cos(\phi - \theta)\right]\right)^2} d\phi, \tag{24}
\]

where the lower sign applies to positively sheared mean flow. For \(\rho = 0\), the integral (24) is elementary, and the normalized surface pressure at the center of the obstacle is

\[
\frac{p(\rho = 0, z = 0)}{\rho_0 U_0^2} = \pm \frac{\beta^*}{4\sqrt{\pi}}. \tag{25}
\]

Figures 6a,b show the FFT representation of the surface pressure, for negatively sheared mean flow with \(Ri = 9\) and \(Ri = 0.49\), respectively.\(^3\) Unlike in the uniform wind case where the pressure field has a fore-and-aft antisymmetry, here the high pressure anomaly is shifted toward the mountain top. For a sufficiently strong mean shear, it would cover the entire central portion of the

\(^3\) The pressure field for the positively sheared mean flow can be obtained from Figs. 6a, b by rotating the mean flow by \(\pi\) and reversing the signs of the pressure anomalies.
obstacle. In the zero shear limit, $\text{Ri} \to \infty$, and the surface pressure regains the fore-and-aft antisymmetry with $\bar{p}(\bar{r} = 0, z = 0) = 0$ in (25).

In order to evaluate the total force exerted on the mountain by the flow, we employ (23) and (13a)–(13b) in

$$
\mathbf{F} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y, z = 0) \nabla h \, dx \, dy,
$$

and integrate the resulting expression by parts in $(x, y)$ space (cf. Phillips 1984). This leads to

$$
\kappa \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{1 + i\sqrt{k}h^{-1}\sqrt{(1 - c^{-1})k^2 + l^2}} \hat{h}(k, l) \, dk \, dl,
$$

whose component parallel to the surface mean flow will be, hereafter, referred to as drag $D$. For axially symmetric obstacles, such as (15), the normal component always vanishes, since the mountain-induced pressure perturbation has a left–right symmetry with respect to the mean flow direction. Using the coordinate transformation (A11) in (27) and integrating the resulting expression with respect to $\kappa$ results in

$$
D = \frac{1}{4} \bar{\rho} U_0 a h_0^2 \int_{0}^{2\pi} \frac{\cos^2\varphi}{\sqrt{1 - c^{-1}\cos^2\varphi}} \, d\varphi.
$$

The integral on the rhs of (28) can be expressed in terms of the complete elliptic integrals

$$
D = \frac{1}{4} \bar{\rho} U_0 a h_0^2 \left[ (c - 1)\mathbf{K}(c^{-1/2}) + (2 - c)\mathbf{E}(c^{-1/2}) \right],
$$

where $\mathbf{K}$ and $\mathbf{E}$ denote the elliptic integrals of the first and second kind, respectively (appendix B). The result (29) is valid for $c = 4\text{Ri} > 1$ and for both negative and positive shear.

In the zero shear limit, $\text{Ri} \to \infty$ and (29) recovers the result derived by Smith (1988)

Both (30) and (31) can be obtained directly by evaluating (28) or alternately from (29) using power series expansions of $\mathbf{K}$ and $\mathbf{E}$ (Grabišić 1995). The latter approach provides an explicit form of $D(\text{Ri})$ in the vicinity of the limits, apparent in Fig. 7.
Figure 7 displays $D$ [computed numerically from (29) and normalized by (30)] as a function of $\text{Ri}$. It shows that $D$ is always smaller than in the constant wind case, a consequence of a shear-reduced positive correlation between the pressure anomalies and the mountain slopes. As $\text{Ri} \to \frac{1}{4}$,

$$
\lim_{\text{Ri} \to \frac{1}{4}} D = \frac{\pi}{4} \rho_0 N U_0 a h_i^2,
$$

and, in contrast to equivalent 2D flows (section 3a in Smith 1986), the drag does not vanish. In 2D, as $\text{Ri} \to \frac{1}{4}$, the wave solution achieves a fore-and-aft symmetry that generates symmetric surface pressure perturbations and no drag. In 3D, in the same limit, each wave component with $l \neq 0$ has a nonvanishing vertical structure [see Eq. (9b)], whereupon surface pressure anomaly remains asymmetric. Consequently, as $c \to 1$, the contribution to $D$ [Eq. (28)] from each Fourier mode $(k, \varphi)$—proportional to the factor $(1 - c^{-1} \cos^2 \varphi)^{1/2}$—vanishes for the components with wavenumber vector parallel to the mean flow, which are the only modes present in a 2D flow. Integrating over all other non-vanishing contributions in (28) then yields (31).

In the remainder of this paper we compare numerical and analytical solutions in order to assess the realizability of the linear solutions as well as the impact of local nonlinearities and dissipation on the singularities predicted by the linear model.

### 3. Numerical model

#### a. Model description and experimental design

The numerical model used in this study has been described in Smolarkiewicz and Margolin (1997). It is representative of a class of nonhydrostatic atmospheric models (Clark 1977; Kapitza and Eppel 1992) that solve the anelastic equations of motion (Lipps 1990; Lipps and Hemler 1982) in nonorthogonal terrain-following coordinates (Gal-Chen and Somerville 1975). The distinctive aspect of our model is its numerical design, which incorporates two-time-level, either semi-Lagrangian (Smolarkiewicz and Pudykiewicz 1992) or Eulerian (Smolarkiewicz and Margolin 1993), nonoscillatory forward-in-time (NFT), at least second-order-accurate finite-difference approximations to, respectively, pointwise and trajectory-wise integrals of the governing fluid dynamic equations. Also, the model admits two optional lattice structures: a co-located A grid, and a staggered B grid (section 4 in Smolarkiewicz and Margolin 1994).

All results reported in this paper have been generated with the semi-Lagrangian, A-grid variant of the model; however, selected experiments were repeated in other model configurations as well, to verify the overall accuracy of numerical results.

The numerically simulated problem is that posed in section 2. The mean velocity profile and obstacle shape

![Table 1](https://example.com/table1.png)

<table>
<thead>
<tr>
<th>Expt</th>
<th>Ri</th>
<th>$\tilde{h}$</th>
<th>$\tilde{h}/a$</th>
<th>$D_{\text{ana}}/D_{\text{num}}$</th>
<th>$D_{\text{ana}}/D_{\text{hydro}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LS1</td>
<td>9.0</td>
<td>0.1</td>
<td>18</td>
<td>0.99</td>
<td>0.97</td>
</tr>
<tr>
<td>LS2</td>
<td>1.0</td>
<td>0.05</td>
<td>18</td>
<td>0.90</td>
<td>0.89</td>
</tr>
<tr>
<td>LS3</td>
<td>0.7</td>
<td>0.9</td>
<td>18</td>
<td>0.90</td>
<td>0.89</td>
</tr>
<tr>
<td>LS4</td>
<td>1.0</td>
<td>0.3</td>
<td>18</td>
<td>0.90</td>
<td>0.89</td>
</tr>
<tr>
<td>LS6</td>
<td>0.64</td>
<td>0.025</td>
<td>18</td>
<td>0.84</td>
<td>0.82</td>
</tr>
<tr>
<td>LS7</td>
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<td>0.01</td>
<td>30</td>
<td>0.78</td>
<td>0.79</td>
</tr>
<tr>
<td>LS8</td>
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<td>0.0001</td>
<td>30</td>
<td>0.68</td>
<td>0.72</td>
</tr>
<tr>
<td>LS9</td>
<td>0.25</td>
<td>0.0001</td>
<td>30</td>
<td>0.43</td>
<td>0.66</td>
</tr>
</tbody>
</table>

![Figure 8](https://example.com/figure8.png)

**Figure 8.** Regime diagram of the critical-level flow past an axisymmetric obstacle for linearly sheared ambient wind. The parameter space is spanned by the nondimensional mountain height $\tilde{h}$ and the inverse Richardson number $1/\text{Ri}$. The shaded region represents the “linear regime” where numerical solutions achieve steady states and agree closely with the theoretical predictions. The labels correspond to the experiments listed in Table 1.
are specified by (7) and (15), respectively. Given all the assumptions adopted, the parameter space of this problem is spanned by only two parameters: the nondimensional mountain height $\hat{h}$, and Richardson number $\text{Ri}$ [the latter is equal to the square of the nondimensional height of the critical-level $\hat{z}_c = N z_c / U_0$, see Eq. (9c)]. Here, we explore the portion of the parameter space with $\hat{h} \leq 0.3$ and $\text{Ri} \in [0.25, 9]$, that is, that relevant to the linear model results. Our strategy is to vary the mountain and critical level heights while keeping the ground-level wind $U_0$, Brunt–Väisälä frequency $N$, and half-width of the obstacle $a$ fixed such that the ratio $U_0 / Na = 0.2$ in all experiments. This assures that the degree to which the mountain generated perturbations are hydrostatic is always the same. In the runs reported, $U_0 = 10 \text{ m s}^{-1}$, $N = 0.01 \text{ s}^{-1}$, and $a = 5000 \text{ m}$.

The physical domain, $20a \times 20a \times 3z_c$, with the mountain centered in $z = 0$ plane, is covered with $81 \times 81 \times 91$ grid points. Note that the actual vertical grid increment depends on the height of the critical level itself. In the vertical, the upper third of the domain is occupied by a gravity wave absorber (cf. Smolarkiewicz and Margolin 1997), while the critical level is placed in the middle of the remaining portion of the domain. Periodic boundaries are assumed in the $y$ direction, whereas open boundaries in $x$ are simulated using 10$\Delta x$-wide lateral absorbers (Davies 1983; Kosloff and Kosloff 1986). The model was initialized by elevating the
mountain gradually over the first $T_0 = t_0 U_0/a = 8$ time units. Since the model formulation admits a generalized time-dependent "terrain-following" coordinate transformation (Prusa et al. 1996), this initialization is physically realizable (i.e., free of zeroth-order errors due to neglect of time-dependency of the geometry in the governing fluid equations). All experiments were continued until $T = tU_0/a = 18$—this was sufficient to reach the steady state (if such existed)—and selected experiments were carried out to $T = 30$. The integration time step was chosen such that the Lipschitz number $\| \Delta \omega / \partial x \|$, a semi-Lagrangian counterpart of the Courant number familiar from Eulerian computations, was always less than 0.5 [see Smolarkiewicz and Pudykiewicz (1992) for a discussion].

The accuracy of the model in reproducing linear hydrostatic steady-state solutions for flows past an axisymmetric obstacle has been tested using the reference problem of a constant mean wind and stability (Smith 1980, 1988). The experiments were conducted for $h = 0.1, 0.05$ and $U/Na = 0.2, 0.1$. The numerical and analytical results were in good agreement; for instance, the discrepancies between the two predictions of the wave drag were 4% and 7% for $U/Na = 0.1$ and 0.2, respectively. We attribute larger discrepancy in the latter case to a stronger excitation of the nonhydrostatic wave components.

b. Results and discussion

In order to assess realizability of the linear solutions, that is, to establish a region of the $(h, Ri)$ parameter space where the linear theory is adequate, we have conducted a series of over 30 numerical experiments. The experiments summarized in Table 1 are representative of the entire series. They illustrate well the extent to which linear theory provides an adequate description of the flow, as well as the sensitivity of solutions to local nonlinearities and dissipation.

Figure 8 is a regime diagram of the $(h, Ri)$ parameter space. The shaded region represents the "linear regime" where numerical solutions achieve steady state and agree closely with the linear predictions. As measured
Fig. 13. Isentropic surfaces in the central x–z plane for (a) numerical experiment LS2 (the linear regime, Ri = 1.0, ̃h = 0.05) and (b) the corresponding linear solution.

Fig. 14. Vertical vorticity ($\times 10^{-4}$ s$^{-1}$) from numerical experiment LS2 (Ri = 1.0, ̃h = 0.05) [(a) and (b)] and the corresponding linear solution (c). Vertical vorticity is shown (a) in the y–z plane at $x = a/4$; (b) in the horizontal plane at $z = 0.94z_c$; and (c) the same as in (b).
by the mountain wave drag (29), within this region the discrepancies between the numerical and analytic solutions do not exceed 7% of the linear predictions (cf. Fig. 7). Outside this region, the solutions diverge rapidly (we shall return to this point later in this section) and, more importantly, the numerical results fail to confirm the existence of steady-state solutions. The latter is illustrated in Fig. 9, which displays drag histories (starting from the model initialization) for the experiments in Table 1. Experiments LS1, LS2, and LS6 to LS9 evince fairly well-defined steady-state solutions, whereas experiments LS3 to LS5 depart steadily from the linear results.

In the linear regime, the numerical and analytic solutions agree uniformly well except in the vicinity of the critical level. Figure 10 shows the surface pressure fields for the weak and strong shear cases [runs LS1 (Fig. 10a) and LS7 (Fig. 10b), respectively], whose analytic counterparts are displayed in Fig. 6. Figure 11 displays u-velocity perturbations in the central x-z plane for these same two runs (corresponding analytic results are not shown as they are singular at the critical level). It illustrates yet another aspect of the agreement between the linear and numerical predictions. As the wave approaches the critical level, the vertical wavelength diminishes and the wave motions become increasingly more horizontal; furthermore, this change is more gradual in a weakly sheared environment [see Eq. (17) and the accompanying discussion]. Figure 12 shows the profile of the vertical flux of the horizontal momentum for the strong shear case (run LS7). When the mean-flow direction is constant with height, the Eliassen-Palm theorem (Eliassen and Palm 1960) for 3D flows (Broad 1995) assures that the vertical flux of the horizontal momentum,

\[
\langle uw \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} uw \, dx \, dy,
\]

is the same (and equal to the drag) at all heights, except at the level where \( U(z) = 0 \), where it is undetermined. In the numerical simulation, the flux profile is nearly constant below \( z = 0.75z_c \). Above this altitude, there is a considerable divergence of the flux up to the critical level, and a strong attenuation above it. The latter is consistent with the linear prediction. The positive momentum flux right above the critical level indicates the presence of upward propagating waves in the reversed mean flow, a residual transmission through the critical level.

Even well within the linear regime, the analytic and numerical solutions differ considerably in the vicinity of the critical level. Figure 13 shows the isentropes in the central x-z plane for the LS2 together with the corresponding linear result. Respective regularity and singularity of the numerical and analytic results are apparent. Guided by the 2D viscous linear theory of Hazel (1967), we anticipate that the difference between the two solutions is primarily due to the implicit viscosity of the numerical schemes becoming nonnegligible in the vicinity of the critical level. There, our linear theory predicts sharp gradients of the primary variables, and the nonoscillatory finite-difference schemes employed in the model activate minimized \( O(\Delta x) \) nonlinear dissipation, sufficient to preserve monotonicity of the

\( ^{5} \) For \( \text{Ri} < 0.5 \) (experiments LS7 to LS9), an accurate steady state is difficult to define as the drag oscillates slowly, with amplitude less than 4%, around the values listed in Table 1.
transported variables [consistent with analytic properties of fundamental conservation laws; Smolarkiewicz and Pudykiewicz (1992), see also section 2d in Prusa et al. (1996)]. Effectively, this mimics dissipation that is inevitable in turbulent critical-level flows.

The dissipative nature of the wave–mean-flow interaction near the critical level leads to a deposition of wave momentum into the mean flow consistent with a strongly divergent flux profile evident in Fig. 12. This results in a tendency for generating a pair of counterrotating eddies in a layer of fluid within which the wave absorption occurs (Nappo and Chimonas 1992; Booker and Bretherton 1967). Such a vortex pair should form directly above the mountain with a cyclonic and anticyclonic eddy to the right and left, respectively, of the flow symmetry axis. However, the model predicted vortical vorticity is nonzero in the entire fluid depth below the critical level, corresponding closely to the linear prediction (Fig 14). In order to diagnose dissipatively generated eddies, we employ potential vorticity,

\[ PV = \frac{1}{\rho_0} \nabla \theta \cdot \Omega, \]

where \( \Omega \) denotes the total vorticity vector (including the ambient component \( U \hat{z} \)) and \( \theta \) is the potential temperature. Figure 15 shows the model predicted normalized potential vorticity \( PV' = PV \rho_0 / V \theta_0 \) for the LS2 experiment discussed above. It reveals the dissipatively generated eddies [recall that no PV can be created in inviscid adiabatic flows, Ertel (1942)] confined within \( 0.8z_c < z < z_c \) and spanning approximately \(-2a < y < 2a\) in the cross-wind direction. In the linear flow
Fig. 17. Normalized potential vorticity ($10^{-4}$ s$^{-1}$) from numerical experiment LS5 ($R_i = 1.0$, $h = 0.3$): (a) in the $y-z$ plane at $x = a/4$; (b) in the horizontal plane at $z = 0.8z_c$.

regime, these eddies are relatively weak (in Fig. 15, the vorticity component normal to isentropes is only $\sim 0.1\% \left| \Omega \right|$) and have no apparent signature in the local flow.

Experiments LS3-LS5 ($R_i = 1; h = 0.1, 0.2, 0.3$) illustrate the transition to a nonlinear regime. For experiment LS3 (located near the regimes’ interface in Fig. 8) nonlinear effects are weak and linear theory still provides a qualitatively meaningful description of the flow (compare Fig. 16a with Fig. 2 and Fig. 16b with Fig. 1a). Farther into the nonlinear regime (runs LS4, LS5), the analytic predictions are useless. Large-amplitude perturbations are not confined anymore to the vicinity of the critical level but affect the flow everywhere above the lee slopes (Figs. 16c, d). Here, PV anomalies (with amplitude $\sim 10\% \left| \Omega \right|$) surround the stagnant region in the lee of the mountain (Fig. 17) reflecting the complexity of the flow due to wave overturning, viscous momentum deposition, and nonhydrostatic effects. For $R_i$ close to $1/4$, the nonlinear regime extends to infinitesimal mountain heights (Fig. 8). Even though the numerical solutions reach quasi steady states there, the discrepancy between the model prediction and the analytic value of drag remains large (cf. Fig. 7 showing the 53% discrepancy for $R_i = 1/4$ at $h = 0.0001$). This is due to the increased importance of nonlinearities compared to that of the dissipation at very small Richardson numbers.

4. Summary

Using linear theory and numerical simulations, we have examined small-amplitude 3D gravity waves in a vertically sheared, unidirectional flow past an isolated axisymmetric mountain. We have assumed linear ambient wind profiles and constant environmental stability. This is perhaps the simplest scenario that provides a critical level for all wave components at the height where the mean flow vanishes. Our linear theory, formally valid for Richardson numbers greater than one quarter, is hydrostatic, Boussinesq, irrotational, and inviscid. In order to verify realizability of the analytical results, we have conducted a series of experiments with a numerical model suitable for simulating natural stratified flows.

Below the critical level, the linear theory predicts 3D wave pattern similar, in some respect, to that characteristic of a constant mean flow. The asymptotic far-field solutions yield waves confined to paraboloidal envelopes [Eq. (21)] that widen quickly with height. As a result, the wave fronts become normal to the mean flow as $z \rightarrow z_c$ (rather than as $z \rightarrow \infty$, a behavior familiar from the constant mean flow case). At $z = z_c$, the linear solutions are singular; the vertical wavenumber, isopycnal displacements, and horizontal velocity become infinite. Above the critical level, the waves are strongly attenuated, as in the equivalent 2D problem. In 3D, however, the attenuation factor depends on the horizontal wavenumber. Modes with wave fronts parallel to the mean flow are filtered more effectively than those with the fronts perpendicular to the mean flow, resulting in a solution above the critical level reminiscent of the flow past a ridge. Similar to the equivalent 2D problem, 

Radical departures from the linear theory for small mountain heights are not limited to critical-level flows. For a model atmosphere with uniform wind and a two-layer stability profile, Durran (1992) found a similar behavior for $h$ as small as $0.2-0.3$. 


mountain wave drag [Eq. (29)] decreases with the Richardson number but, in contrast to the 2D case, it does not vanish as Ri \( \leq \frac{1}{4} \). Zero drag in 2D flows is a consequence of a fore-and-aft symmetry of the linear solutions at strong shears. In 3D, this symmetry is maintained only for the modes with wave fronts normal to the mean flow, whereas all other modes induce asymmetric pressure perturbations leading to a nonzero drag. The linear theory prediction of drag is the same for positively and negatively sheared flows.

The numerical simulations with a nonhydrostatic nonlinear model confirm the theoretical predictions for a range of \( \tilde{h} \) values that rapidly diminishes with decreasing Ri (Fig. 8). The confirmation of the realizability of the linear theory is important because critical-level flows exhibit local nonlinearities of potentially dominant global effect. Within the linear regime [i.e., within a portion of the \((\tilde{h}, \text{Ri})\) parameter space where the linear theory applies] a good agreement between the theoretical and numerical solutions is found everywhere, except in the vicinity of the critical level where numerical solutions are regularized by implicit viscosity of finite-difference approximations. As a result, the waves approaching the critical level are smoothly absorbed as they deposit momentum into the mean flow. The dissipative wave–mean-flow interaction leads to the generation of the potential vorticity (McIntyre and Norton 1990). In the linear regime, the potential vorticity doublet forms above the mountain right below the critical level (Fig. 15), where it is confined to a fluid layer with a strongly divergent wave–momentum-flux profile. Beyond the linear regime, the utility of the theoretical predictions ceases abruptly with increasing \( \tilde{h} \) as large amplitude perturbations develop within the entire depth below the critical level.

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**APPENDIX A**

**Asymptotic Solutions: Further Details**

In order to evaluate the integral in (19a), we employ the multidimensional variant of the method of stationary phase (e.g., Lighthill 1980, 351–360). The method of stationary phase is commonly used for evaluating integrals of the form

\[
I(t) = \int F(k) \exp \{i\psi(k)\} \, dk
\]

where \( t \) is large and positive, and a mapping \( \psi: \mathbb{C} \to \mathbb{R} \) is smooth along a contour \( C \). Unless \( \psi'(=d\psi/dk) = 0 \) somewhere along \( C \), contributions from different sections of the contour cancel due to a rapidly oscillating character of \( \exp \{i\psi(k)\} \). The only nonvanishing contribution comes from the vicinity of \( k_0 \), a stationary-phase point, where \( \psi'(k_0) = 0 \). Given \( \psi''(k_0) \neq 0 \), \( \psi(k) \) is expanded in the second-order Taylor series around \( k_0 \), and (A1) is approximated [with \( O(t^{-1}) \) error] as

\[
I(t) = F(k_0) \exp \{i\psi(k_0)\} \sqrt{\frac{2\pi}{\|\psi''(k_0)\|}} \times \exp \left\{ \pi \frac{1}{4} \text{sgn}[\psi''(k_0)] \right\}. \tag{A2}
\]

For multiple stationary phase points, \( I(t) \) is approximated with the sum of contributions around each point.

For a multidimensional problem \( \psi: \mathbb{C}^n \to \mathbb{R} \), the equivalent of (A1),

\[
I(t) = \int F(k_1, k_2, \ldots, k_n) \exp \{i\psi(k_1, k_2, \ldots, k_n)\} \, dk_1 \, dk_2 \cdots \, dk_n
\]

is approximated as

\[
I(t) = F^{(0)} \exp \{i\psi^{(0)}\} \left(\frac{2\pi}{t}\right)^{n/2} |J^{(0)}|^{-1/2} \exp \{i\theta\}, \tag{A4}
\]

where superscript \( (0) \) refers to a stationary phase point \((k_0, k_0, \ldots, k_0)\); \( J^{(0)} \) is the Jacobian of the second derivative of \( \psi \) at this point; and \( \theta = (\pi/4) \text{sgn}[\psi''^{(0)}] \), with \( \psi'' \) denoting elements of the diagonalized tensor of the second derivative. For \( n = 2, \theta = \pi/2, -\pi/2, \text{and } 0 \), at a minimum, maximum, and a saddle point of \( \psi(k_1, k_2) \), respectively.

We employ (A4) to find a far-field solution for the displacement field of density surfaces \( \tilde{\eta} \) given by (19a). This solution is valid at large radial distances from the mountain. Stationary phase points \((k_0, \ell_0)\) must satisfy \((\partial \tilde{\phi}/\partial k)^{(0)} = (\partial \tilde{\phi}/\partial \ell)^{(0)} = 0\), resulting in

\[
\tilde{\phi}(k, \ell) \sim \frac{1}{2\pi} \text{Re} \left\{ \int F^{(0)} \exp \{i\psi^{(0)}\} \left(\frac{2\pi}{t}\right)^{n/2} |J^{(0)}|^{-1/2} \exp \{i\theta\} \right\} \left[ \frac{1}{(k-k_0)^2 + (\ell-\ell_0)^2} \right]^{1/2} \exp \left\{ -\frac{1}{2} \left[ \frac{(k-k_0)^2}{(k-k_0)^2} + \frac{(\ell-\ell_0)^2}{(\ell-\ell_0)^2} \right] \right\}.
\]
For every stationary phase point \((\xi, \eta, \zeta; c)\), there is an integral [2.583.4, Gradshteyn and Ryzhik (1980)]:

\[
\int \frac{\sqrt{c}}{2} \ln|\dot{Z}_.| \, d\phi = 0,
\]

which implies all stationary phase points are at \(\dot{\phi}_. = 0\), \(\dot{\zeta}_. < 0\). (A7)

For every stationary phase point \((\dot{k}_0, \dot{l}_0)\), there is an image point \((-\dot{k}_0, -\dot{l}_0)\), with

\[
\dot{k}_0 = \frac{\dot{\xi} \sqrt{c}}{2} \ln|\dot{Z}_.|,
\]

\[
\dot{l}_0 = -\frac{\dot{\eta} \sqrt{c}}{2} \ln|\dot{Z}_.|,
\]

where

\[
\dot{c} = \frac{\dot{k}_0 \sqrt{c}}{\dot{k}_0^2 + \dot{l}_0^2} \ln|\dot{Z}_.|,
\]

(A5)

\[
\dot{\eta} = -\frac{\dot{l}_0 \sqrt{c}}{\dot{k}_0^2 + \dot{l}_0^2} \ln|\dot{Z}_.|,
\]

(A6)

where

\[
\dot{k}_0^{(0)} = [(1 - c^{-1})\dot{k}_0^2 + \dot{l}_0^2]^{1/2}.
\]

According to (A5), all stationary phase points are at \(\dot{\phi}_. = 0\); therefore,

\[
\dot{\eta}(\dot{\xi}, \dot{\eta}, \dot{\zeta}; c) = 0, \dot{\xi} < 0.
\]

(A7)

Using elementary transformations, the integral on the rhs of (28) is expressed as

\[
I = 4 \int_0^{\pi/2} \sin^2 x \sqrt{1 - p^2 \sin^2 x} \, dx,
\]

(B1)

whose indefinite counterpart is representable via elliptic integrals [2.583.4, Gradshteyn and Ryzhik (1980)]:

\[
\int \sin^2 x \sqrt{1 - p^2 \sin^2 x} \, dx = -\frac{\Delta}{3} \sin x \cos x + \frac{1 - p^2}{3p^2} F(x, p)
\]

\[+ 2p^2 \frac{1}{3p^2} E(x, p),\]

(B2)

where

\[\Delta = \sqrt{1 - p^2 \sin^2 x}, \quad 0 < p^2 < 1,\]

(B3)

\[F(x, p) = \int_0^x \frac{1}{\sqrt{1 - p^2 \sin^2 x'}} \, dx',\]

(B4)

\[E(x, p) = \int_0^x \sqrt{1 - p^2 \sin^2 x'} \, dx'.\]

(B5)

Applying the integration limits to (B2) and substituting \(c^{-1}\) for \(p^2\) results in

\[
I = \frac{4}{3} [(c - 1)K(c^{-1/2}) + (2 - c)E(c^{-1/2})],
\]

(B6)

where \(K(p) = F(\pi/2, p)\), and \(E(p) = E(\pi/2, p)\) are the complete elliptic integrals of the first and second kind, respectively. Substituting (B6) for the integral on the rhs of (28) results in (29).

REFERENCES


