Two-Dimensional Turbulence and Persistent Zonal Jets in a Global Barotropic Model

HUEI-PING HUANG* AND WALTER A. ROBINSON

Department of Atmospheric Sciences, University of Illinois at Urbana-Champaign, Urbana, Illinois

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ABSTRACT

The dynamics of two-dimensional turbulence on a rotating sphere are examined. The anisotropic Rhines scale is derived and verified in decaying turbulence simulations. Due to the anisotropic nature of the Rossby waves, the Rhines barrier is displaced toward small total wavenumber \( n \) with decreasing zonal wavenumber \( m \). Upscale energy transfer along the zonal axis (\( m = 0 \)) is not directly arrested by beta. A forced dissipative model with high-wavenumber forcing is used to investigate the dynamics of persistent zonal jets. Persistent jets form in the low energy (strong rotation) cases with the root-mean-square velocity \( V_{\text{rms}} \). Under a fixed rotation rate, the jet scale decreases with the energy. The equilibrated jets generally stay at fixed latitudes. The zonal bands are nearly uniformly distributed in latitude, except that bands in the high latitudes tend to be wider and weaker, as clearly affected by a decreasing beta with latitude. The time-mean zonal winds in the forced simulations appear to be stable, with their absolute vorticity gradient dominated by beta. The increase of the jet scale with energy as required by stability is consistent with the simulated results.

Diagnostic analysis shows that the persistent jets are primarily maintained by the shear-straining mechanism involving small-scale eddies and large-scale zonal jets, with a clear scale separation between them. Although large-scale eddies, those at scales near the Rhines scale, possess most of the eddy energy, in the time mean they contribute little to the maintenance of the zonal jets. Thus, despite the similarity between the Rhines scale and the jet scale, their dynamical link is not obvious in the time-mean statistics. The presence of persistent zonal jets modifies the normal modes of the system. Pure Rossby-Haurwitz modes at small and medium scales are severely modified and fall into the continuum. Large-scale modes, however, may remain discrete. The discreteness of the large-scale modes limits their ability to exchange energy with the zonal jets in the time mean.

1. Introduction

Because of its elegance in representing interactions among scales, two-dimensional turbulence is one of the most popular physical models of large-scale atmospheric flows. Based on a two-dimensional turbulence model, Lorenz (1969) derived the predictability of atmospheric flow as a function of spatial scale. Later refinement of Lorenz’s predictability estimate by Leith (1971) and Leith and Kraichnan (1972) further illustrate the application of a turbulence theoretical approach to important meteorological problems. In these classical studies the effect of planetary rotation (“beta”) was ignored, and the existence of long-lived coherent structures in two-dimensional turbulence was not widely known at that time. The effect of planetary rotation on two-dimensional turbulence was first investigated by Rhines (1975). He pointed out three important effects of beta. The first is the creation of the (now called) Rhines scale, above which upscale turbulent energy transfer ceases. The second is the anisotropization of the flow field (with a preference for zonally elongated eddies over meridionally elongated ones), especially at large scales. The third is the stabilization of the zonal flow. From the third effect, Rhines inferred that under strong rotation zonal anisotropy could persist in the asymptotic state of two-dimensional turbulence, a point later rigorously established by Shepherd (1987c). That the above three effects increase predictability was demonstrated by Basdevant et al. (1981).

Since strong zonal jets are commonly seen in planetary circulations (e.g., terrestrial and Jovian flows), the simulation of the zonal flow itself is of great interest. Based on Rhines’ thinking, attempts have been made to generate persistent zonal jets in models of forced two-dimensional turbulence. Using a sector model (e.g., on one-sixteenth of the surface of a sphere) with stochastic forcing at small scales, Williams (1978) simulated multiband zonal jets that broadly resemble the zonal flow of the Jovian atmosphere in a 2D turbulence model. Later studies of the generation of zonal flows in forced 2D or geostrophic turbulence include those by Basdevant et al. (1981), Maltrud and Vallis (1991), Vallis and Maltrud (1993), and Panetta (1993), most of them con-
sider beta-plane geometry. A review of these studies is given by Rhines (1994). The emergence of zonal jets in decaying shallow water turbulence on a sphere is recently demonstrated by Cho and Polvani (1996). In these studies, the jet scale is shown to decrease with decreasing energy (or increasing beta), and it is generally of the same order as the Rhines scale (Vallis and Maltrud, Panetta).

Rather than study the emergence of the zonal flow, Shepherd (1987a) investigated the dynamics of 2D turbulence on a β plane with an imposed zonal flow. His background shear flow had profound effects on the inhomogeneous turbulence. For example, when viewed in spectral space, the shear straining process induced a downscale enstrophy transfer that is not unlike the traditional downscale enstrophy cascade due to local eddy—eddy interaction. The energy exchange between the large-scale zonal flow and the medium- or small-scale eddies creates nonlocal energy transfer in the spectral domain. He applied these concepts to explain the observed atmospheric energy spectrum (Shepherd 1987b).

In this paper we examine several aspects of two-dimensional turbulence in spherical geometry, with emphasis on the persistent zonal jets in forced—dissipative simulations. Our study is in the spirit of the work described above, but it contributes several new points. In section 2 we show the spherical-geometric counterpart of the anisotropic Rhines scale that was demonstrated for the β plane by Vallis and Maltrud (1993). In section 3 a forced—dissipative model with a high-wavenumber forcing is used to investigate the characteristics of persistent zonal jets. Persistent zonal jets are found to form under a wide range of equilibrium energies. In the statistically equilibrium state the zonal jets stay at fixed latitudes for very long time, with their time-mean profiles generally stable. The jets have nearly constant widths, but jets at higher latitudes are somewhat wider (and weaker), presumably due to the decrease of beta with latitude. A rough equipartition between the zonal and eddy energy is seen in these equilibrium states.

With the forcing at small scales, two significant features appear at large scales in the energy spectrum. One is the concentration of energy in the large-scale zonal modes associated with the persistent jets. Another is a peak in the eddy (defined as the departure from zonal flow) energy at the Rhines scale. We find that in the time mean the zonal jets are maintained primarily by the vorticity fluxes of small-scale eddies, particularly those at the forcing scale or smaller, through the shear-straining mechanism. While eddies at the Rhines scale interact with the zonal jets on short timescales, statistically they contribute little to the net maintenance of the zonal flow. Thus, despite the closeness between the jet scale and the Rhines scale, the dynamical link between them is not obvious, at least in a barotropic model forced stochastically at high wavenumbers.

The presence of persistent zonal jets influences the spectral transfers of energy and enstrophy, as shown by Shepherd (1987a,b). In section 3 we show that in our simulations the shear-induced transfer by persistent zonal jets can be as important as the “classical” (eddy—eddy) transfer. The zonal flow also modifies the Rossby—Haurwitz frequencies, which we use in section 2 to derive the anisotropic Rhines scale. Under the influence of an energetic zonal flow most of the medium- and small-scale Rossby—Haurwitz modes are significantly modified and actually fall into the continuum. Large-scale Rossby—Haurwitz modes, however, are only mildly modified, with many of the modified modes remaining discrete. This limits the ability of the large-scale eddies to contribute to the net energy exchange with the zonal jets. Since the pure Rossby—Haurwitz frequencies are not severely modified in the large scale, the anisotropic Rhines barrier derived in section 2 can still be seen in the large-scale part of the spectrum.

A general discussion is given in section 4. We examine the impact of the choice of the forcing scale and type on the generation of zonal jets. We note that the dominance of the shear-straining mechanism in maintaining the jets may rely on the clear scale separation between the small-scale eddies (at the forcing scale) and the jets. When a stochastic forcing is applied at large scales, the zonal jets become relatively obscure. A brief analysis of the interaction coefficients is given in this section, with a potentially useful result that might explain why the jets do not go to the largest scale in the model. Conclusions follow in section 5.

2. The beta effect and the Rhines scale

For clarity of later discussion, a few conventions are given here. A global barotropic model is considered. The vorticity equation is

\[ \frac{\partial \zeta}{\partial t} = -J(\psi, \zeta + f) - \kappa \zeta - D \zeta + F, \tag{1} \]

with \(-\kappa \zeta\), \(-D\zeta\), and \(F\) the (optional) linear drag, higher-order diffusion, and forcing. Unless otherwise noted, terrestrial conditions are assumed. Vorticity and velocity are nondimensionalized by \(\Omega\) and \(a\Omega\), and time and length by \(\Omega^{-1}\) and \(a\), where \(a\) is the radius of the earth and \(\Omega\) is its rate of rotation. An “order one” nondimensional velocity relates to a dimensional velocity of order \(a\Omega = 463\ \text{ms}^{-1}\) (the tangential velocity of the earth at the equator). The observed terrestrial regime has, therefore, a nondimensional velocity much less than unity. In our numerical simulations the rotation rate is fixed at the terrestrial value, and properties of the flow are changed by changing the energy level. Lower energies imply a greater influence of planetary rotation.

The inclusion of planetary rotation (“beta”) in a model of 2D turbulence has several effects on the flow. The first of these is to establish a unique definition of a zonal flow (in a nonrotating model the definition of a “zonal direction” is arbitrary) and at the same time to stabilize it. Beta stabilization permits the formation
of persistent zonal jets in 2D turbulence. We will discuss this subject in section 3. [At the same time, strong beta usually (but not always) suppresses the localized, "circular" type coherent vortices, which otherwise can exist in models of nonrotating turbulence (Basdevant et al. 1981; McWilliams 1984). We do not consider this issue in the present work.] Another effect of beta is the creation of the Rhines scale, above which Rossby waves dominate and the turbulent upscale transfer of energy ceases. The Rhines scale is anisotropic because beta has a stronger effect on meridionally elongated than on zonally elongated eddies. The related physical concepts have been thoroughly discussed by Rhines (1975) and Vallis and Maltrud (1993), among others. Here we only give a few essential points specific to our model. Our approach closely follows Vallis and Maltrud.

a. Derivation of anisotropic Rhines curves on the sphere

For convenience one may write Eq. (1) in spectral form, with forcing and dissipation neglected,

\[ \frac{d \psi_m^n}{dt} = i \omega_m^n \psi_m^n + \frac{i}{2} \sum_m \sum_n \sum_n \sum_{m'} L_{mn,m'n}^{m,m'} \psi_m^n \psi_{m'}^{n'} \]

(2)

In (2) \( \psi_m^n \) is the expansion coefficient of the streamfunction \( \psi = \sum \psi_m^n Y_m^n \), where \( Y_m^n \) is the spherical harmonic with zonal wavenumber \( m \) and total wavenumber \( n \), \( \omega_m^n = -2m/[n(n+1)]^{1/2} \) the Rossby–Haurwitz frequency, and \( L_{mn,m'n}^{m,m'} \) the interaction coefficient. The timescale of the beta term is given by (the reciprocal of) the Rossby–Haurwitz frequency \( \omega_m^n \). The timescale of the nonlinear term can be estimated by the turbulent-eddy turnover time. Assuming that turbulent eddies are incoherent (i.e., large eddies have a longer turnover time, and small eddies have a shorter one), and assuming that the eddy field is nearly isotropic, we may estimate the turbulent eddy turnover as \( \sigma^*_m = KV_{rms} \), where \( V_{rms} \) is the eddy root-mean-square velocity and \( K \) is the isotropic spatial wavenumber associated with the mode \( \psi_m^n \). In plane geometry, \( K = (k_x^2 + k_y^2)^{1/2} \), where \( k_x \) and \( k_y \) are the zonal and meridional wavenumbers. In spherical geometry its analog is \( K = [n(n+1)]^{1/2} \) (Boer 1983). Thus, we have \( \sigma^*_m = [n(n+1)]^{1/2} V_{rms} \). For a given mode \( \psi_m^n \), if \( |\omega_m^n| > \sigma^*_m \), then its dynamics is predominately wavelike; otherwise, its behavior is dominated by turbulence. Thus, on the \( (m, n) \) plane the Rhines barrier may be defined by \( |\omega_m^n| = \sigma^*_m \), which gives

\[ \frac{2m}{[n(n+1)]^{1/2}} = V_{rms} \]

(3)

Note that the nondimensional velocity \( V_{rms} \) is related to the real velocity \( V_{rms} \) by \( V = a \Omega V_{rms} \). Since \( V_{rms} \) is defined by,

\[ V_{rms} = \left[ \frac{1}{2\pi} \int_{0}^{2\pi} \left| \begin{array}{c} \int_{0}^{\pi} \int_{0}^{2\pi} \left( V^2 \right) d\Omega \end{array} \right]^{1/2} \]

\[ = \left[ \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} \left| V^2 \right| d\Omega \right]^{1/2} \]

\[ = \left( \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} \left| V^2 \right| d\Omega \right)^{1/2} \]

(or more often the time mean or ensemble mean of the above quantity), where \( V \) is the velocity vector, \( \Omega \) is the solid angle, and the integral is over the globe, we can also replace \( V_{rms} \) by \( (2E)^{1/2} \), where \( E \) is the globally integrated energy.

Figure 1 shows the anisotropic Rhines barrier, based on (3), at several values of \( V_{rms} \). The analogy between the curves in Fig. 1 [we may call them “AR (anisotropic Rhines) curves” for short] and the “dumbbell curves” of Vallis and Maltrud (1993) is obvious. A few dimensional values of \( V_{rms} \) (in m s\(^{-1}\)) are labeled at the right. Since (3) is derived from scale analysis, these values should be regarded more qualitatively than quantitatively. For a given energy level or rms velocity, Rossby wave propagation dominates below the corresponding AR curve, while turbulent behavior is dominant above it. Because (3) is derived from the balance between the nonlinear term and the beta term in the spectral governing equation (2) and because the beta term vanishes for zonal modes, this derivation of the Rhines scale is not applicable to the zonal flow. This is reflected in Fig. 1, as the AR curves never intersect the zonal axis.

The above derivation of the AR curve is not unique because the turbulent eddy turnover time can be estimated in different ways. Vallis and Maltrud use three different estimates of the turbulent timescale and derive three different formulas for the AR curve in plane geometry. The three are, however, very similar qualita-
tively. Also, the above argument may be generalized or refined in several different ways. First, if there is a stationary component in the flow (particularly a persistent zonal flow), the Rossby–Haurwitz frequencies can be modified. Thus, instead of using the Rossby–Haurwitz frequencies, it may be necessary to use the eigenfrequencies of the background flow to derive the AR curve (§3b and 3c). Second, the presence of the nonlinear term itself modifies the Rossby wave frequencies (Legras 1980). The modification, however, is not significant in the large scales and under low energy (see Figs. 4 and 5 of Legras).

**b. Numerical simulations of decaying turbulence**

We next test the above idea by integrating a decaying turbulence model. The linear drag, $-\kappa \zeta$, and forcing, $F_z$, in (1) are set to zero, and the diffusion is given by $D_z = \gamma \nabla^m \zeta$. The nondimensional $\gamma$ is set to $4.55707 \times 10^{-30}$, which gives a one-half-day $e$-folding time at $n = 63$. Triangular 63 (T63) truncation is used, and the nonlinear term is calculated using the spectral transform method (Orszag 1970), employing a $192 \times 96$ Gaussian grid. Figure 2 shows the energy spectrum from an ensemble average of 10 samples of decaying turbulence at day 80. It suffices to show the positive half triangle with $m > 0$. We adopt the convention that both modes $\psi^n_k$ and $\psi^m_k$ are recorded at the point $(m, n)$ on the positive half triangle. Initially, for each sample the energy is concentrated in wavenumbers $34 < n < 46$, and centered at $n = 40$. As time advances the “center of mass” of the energy spectrum marches toward small $n$ until it is eventually arrested by the anisotropic Rhines barrier. Associated with the upscale transfer of energy, enstrophy is transferred downscale and is dissipated by the diffusion. This inevitably causes an energy loss for the system. After an initial stage, the rate of energy loss due to enstrophy dissipation becomes small, and the level of energy in the system can be more meaningfully defined. For the case shown in Fig. 2, the ensemble mean of the rms velocity is about $5.5 \text{ m s}^{-1}$ at day 80. Qualitatively, Fig. 2 compares well with Fig. 1. Moreover, the AR curves shift to larger (smaller) scales in
the numerical experiments with higher (lower) energy levels (not shown) as expected.

Several interesting features in the decaying simulations are noteworthy. In general, a trend of zonalization is seen in the flow field, as discussed by Rhines (1975). On the other hand, strong and persistent zonal jets do not commonly form in integrations up to a few hundred days (and with a typical rms velocity of several meters per second). This is consistent with recent results by Cho and Polvani (1996). Interestingly, in their simulations jets form in decaying shallow water turbulence.

In some long enough integrations, we see the formation of circumpolar vortices in the polar regions (not shown), a confirmation of Yoden and Yamada's (1993) discovery. Inside the polar vortex (but not elsewhere) the absolute vorticity is homogenized, as shown by Cho and Polvani. In contrast to this, we will show in section 3 that in the forced turbulence simulations the time-mean absolute vorticity gradient can be dominated by beta even in the high latitudes.

Traditionally the arrest of upscale energy transfer by beta is viewed in a one-dimensional picture; that is, the energy cascade is considered to be blocked at a certain total wavenumber \( n_p \) without further distinction between different zonal wavenumber \( m \). The anisotropy of the Rhines barrier, however, allows energy to reach larger scales through the “funnel” to the left of the AR curve in Figs. 1 or 2. Moreover, the energy transfer along the zonal axis (\( m = 0 \)) is not directly blocked by beta (Vallis and Maltrud 1993; Frederiksen et al. 1996). If we take an average of Eq. (3), without any weighting, over all zonal wavenumbers from \( m = 1 \) to \( m = n \), we obtain the “one-dimensional” Rhines barrier as \( V_{\text{rms, eddy}} \approx n^{-1/2}[n(n + 1)]^{-1/2} \) (the subscripts “eddy” indicates that the zonal flow is excluded). For not too small \( n \), approximating \( n \) by \([n(n + 1)]^{-1} \) gives

\[
V_{\text{rms, eddy}} \approx [n(n + 1)]^{-1},
\]

the spherical-geometric version of the familiar formula for the isotropic Rhines barrier. Observing the energy spectrum in Fig. 2, we see that a higher weighting should have been given to a mode with smaller zonal wavenumber in the “averaging” process in deriving (4). Thus, the relationship in (4) usually overestimates the mean total wavenumber (or underestimates the scale) at which the energy spectrum peaks. The bias becomes significant in highly rotating cases, as has been shown in Vallis and Maltrud (1993) and Cho and Polvani (1996).

Since strong and persistent zonal jets are not seen in the decaying turbulence, in the next section we turn to a forced turbulence model to investigate the dynamics of the persistent zonal jets.

3. Persistent zonal jets in forced–dissipative turbulence

A forced–dissipative model is created by using all terms in (1). The model is truncated at T85, with a 256 \( \times 128 \) Gaussian grid. The diffusion is given by \( D = \gamma V^{1/2} \). In most cases the diffusion coefficient is fixed at \( 3.90401 \times 10^{-32} \), which gives a one-half-day \( e \)-folding time at \( n = 85 \). A few experiments with a time-dependent diffusion coefficient estimated from the enstrophy are also performed. A small linear drag \( -k \zeta \) is included. As guided by Vallis and Maltrud (1993), the linear drag coefficient is uniform over spectral space. In most cases its value is fixed to \( k = 3.18309 \times 10^{-4} \), which gives an \( e \)-folding time of 500 days. The forcing is generated by a first-order Markov process, following the standard procedure of Lilly (1969):

\[
F_{t+1} = RF_t + (1 - R^2)^{1/2} \nu,
\]

with \( F_t \) and \( F_{t-1} \) the forcing at two consecutive time steps, \( R = (1 - \Delta t/2\tau_e)/(1 + \Delta t/2\tau_e) \), \( \Delta t \) the time step size, \( \tau_e \) the forcing decorrelation time, and \( \nu \) a random number. This approach is used in previous work, e.g., Williams (1978) and Maltrud and Vallis (1991). The decorrelation time of the forcing, \( \tau_e \), is usually set to one day, much longer than a numerical time step and much shorter than the timescale of the phenomena of interest.

For the results presented herein, the forcing is applied at total wavenumber \( n = 55 \) and is statistically uniform over zonal wavenumbers, \( m \), with the zonal (\( m = 0 \)) mode excluded. We are able to produce similar results with the forcing applied over a band of total wavenumbers \( n_1 < n < n_2 \). We focus on the single wavenumber forcing cases because of their simplicity in the interpretation of later analyses. It should be noted that when the forcing is put in a single odd total wavenumber and the model starts with a zero-field initial condition, only modes with odd \( n \) can be generated (see §4). Thus, an initial condition containing both odd and even modes is preferred. Such an initial condition is created from an ensemble average of nonrotating, decaying turbulence experiments. Unless otherwise noted all zonal modes are set to zero in the initial conditions. Further discussion regarding the model truncation and the use of forcing are given in section 4.

a. Persistent zonal jets

A series of numerical integrations are performed under different forcing magnitudes. We focus on the low energy regime with the nondimensional rms velocity much less than unity, in which persistent zonal jets form. Figures 3a–c summarize the basic properties of nine long-term integrations. Each point represents the statistics of the last 1000 days of a long-term integration, typically of several thousand days. The initial condition used in the integration contains small amplitude eddies and zero zonal components. Persistent zonal jets appear in the statistical equilibrium states in all cases. In Fig. 3 the horizontal axis is the time mean of the mean total wavenumber of the persistent zonal flow, \([n(n + 1)]^{1/2}_{\text{CM}}\), defined as the square root of the ratio between
Fig. 3. Energy versus zonal jet scale for nine major cases in the forced experiments. The abscissas are the mean total wavenumber of the zonal flow, defined by the square root of the ratio between zonal enstrophy and zonal energy. The ordinates are (a) rms velocity of the total field, (b) rms velocity of the zonal component (both in m s\(^{-1}\)), and (c) ratio between zonal and total energy. The eight open circles correspond to the eight cases listed in Table 1. The open triangles are for a case with a time-dependent diffusion coefficient (see text).

The zonal enstrophy and the zonal energy (CM stands for center of mass). This is a measure of the scale of the zonal jets. Figure 3a shows the rms velocity of the total field versus the time mean of \([n(n+1)]^{1/2}_{\text{CM},Z}\). The scale of the jets generally decreases with decreasing energy, consistent with the phenomenology described by Vallis and Maltrud (1993) and by Panetta (1993). Figure 3b is similar to Fig. 3a, but the vertical axis is now the rms velocity of the zonal flow. It shows a similar trend to that in Fig. 3a. As shown in Fig. 3c, the ratio of zonal to total energy does not vary greatly from case to case, with a typical value of around 0.5, thus a rough equipartition between the eddy and zonal energy. The numerical values for the quantities shown in Fig. 3 are listed in Table 1. Three other quantities are listed. The \(E_{\text{SZ}}\), \(E_{\text{TZ}}\), and \(E_{\text{TE}}\) in Table 1 are the time mean of the stationary zonal energy, transient zonal energy, and transient eddy energy. Note that the time-mean total energy can be decomposed by \(E = E_{\text{SZ}} + E_{\text{TZ}} + E_{\text{SE}} + E_{\text{TE}}\). The stationary eddy energy \(E_{\text{SE}}\) is generally very small and is not listed (but can be inferred from the table). In all of the cases, the most important components in the flow field are the stationary zonal flow and the transient eddies.

Figure 4 shows the equilibrated time-mean zonal-mean zonal wind profiles, taken from 1000-day statistics, for the eight cases corresponding to the open circles in Fig. 3. The profiles are arranged, from left to right, with descending energy. Each horizontal grid in the figure represents 1 m s\(^{-1}\). These profiles are generally stable (see below). The number of jets increases with decreasing energy. The equilibrated zonal flows show common characteristics. Westerly jets are generally stronger and narrower than easterly ones (as in Vallis and Maltrud 1993; Panetta 1993) except for those westerly bands closest to the poles. The zonal bands closest to the poles are usually westerly. The most energetic case in Fig. 4 (case I) is not too far from the observed terrestrial regime, in which the zonal-mean wind has one major easterly and two major westerly jets, and the rms velocity is of order 10 m s\(^{-1}\). The zonal wind profile of case I can be made to more closely resemble a terrestrial profile by superimposing a westerly solid-body-rotation component. Caution is given, however, that the linear damping and the forcing used in our model are idealized and may not fit the observed values from the real atmosphere.

The temporal evolution of the zonally averaged zonal winds for cases IV and VII in Fig. 4 are shown in Figs. 5 and 6. As each integration starts with little energy, there are more zonal bands early than in the final equilibrium state. The reduction of the number of bands with increasing energy and time is generally achieved by the merging of westerlies and the disappearance of easterlies, possibly reflecting the fact that westerlies are generally more stable than easterlies (see below). Upon reaching equilibrium, the widths of the individual jets tend to be nearly equal except that the jets at higher
As has been shown in Figs. 5–7, the adjustment of the zonal jets to their equilibrium configuration can take a very long time. For example, in Fig. 5 there is an early stage with six easterly jets that lasts for almost 1000 days before changing into the four-easterly jet configuration. In the following we show a case that exhibits an extreme of this behavior. Knowing the “five-easterly” results of case VII [the five easterly jets remain robustly up to 9000 days (not shown)], in a new integration we keep the parameter setting for that case but deliberately use an initial condition containing weak six-easterly zonal jets. The first 6000 days of the case is shown in Fig. 8. The six-easterly profile lasts for almost 3000 days before changing into a five-easterly configuration. (Before extending the integration to this length, we mistakenly identified the result as an indication of the existence of multiple equilibrium states.) We integrate the case up to 15 000 days. In general, a five-easterly configuration dominates the later-time evolution (not shown), a few outbursts of six-easterly jets are also seen (e.g., the episode between days 5100 and 5500 in Fig. 8). The results in this case recall Panetta’s (1993) “quantization” argument. Basically, zonal wind profiles with integral numbers (e.g., five or six) of jets fit better scales set at one-half day. To test the impact of the diffusion coefficient on the model results, a few simulations with a time-dependent diffusion coefficient were performed. Following many previous studies (e.g., Basdevant et al. 1981; Sadourny and Basdevant 1985), the $e$-folding time of the diffusion term for the highest wavenumber ($n = 85$), $\tau_{s_8}$, is set to $Z^{-1/2}$, where $Z$ is the enstrophy. The quantity $Z^{-1/2}$ gives an estimate of the eddy turnover time for the smallest scale. The diffusion coefficient $\gamma$ is related to $\tau_{s_8}$ by $\gamma = \left(\tau_{s_8}/85(85 + 1)\right)^{-1}$. Integrations with variable $\gamma$ generally behave similarly to those with a fixed diffusivity. The statistics of a long integration with variable $\gamma$ are recorded in Figs. 3a–c as the open triangles. In this case, $\tau_{s_8}$ is almost a constant in the equilibrium state and has a dimensional value of about 3 days, longer than the value of one-half day that was used in the experiments with fixed diffusivity. In the higher energy cases the fixed value is very close to that estimated from the enstrophy.

Long integrations of high energy cases, which require energies one order of magnitude greater than case I, are costly, so we do not attempt to perform. Following many previous studies (e.g., Basdevant et al. 1981; Sadourny and Basdevant 1985), the $e$-folding time of the diffusion term for the highest wavenumber ($n = 85$), $\tau_{s_8}$, is set to $Z^{-1/2}$, where $Z$ is the enstrophy. The quantity $Z^{-1/2}$ gives an estimate of the eddy turnover time for the smallest scale. The diffusion coefficient $\gamma$ is related to $\tau_{s_8}$ by $\gamma = \left(\tau_{s_8}/85(85 + 1)\right)^{-1}$. Integrations with variable $\gamma$ generally behave similarly to those with a fixed diffusivity. The statistics of a long integration with variable $\gamma$ are recorded in Figs. 3a–c as the open triangles. In this case, $\tau_{s_8}$ is almost a constant in the equilibrium state and has a dimensional value of about 3 days, longer than the value of one-half day that was used in the experiments with fixed diffusivity. In the higher energy cases the fixed value is very close to that estimated from the enstrophy.

Although the zonal flow is dominated by the stationary component (see Table 1), greater variability in the zonal mean flow is found near the poles, where the stabilizing effect of beta is weak. The fluctuations are more significant in the higher energy cases. Figure 7 shows the time versus zonal wind cross section for case II. Despite fluctuations in the two westerly bands in higher latitudes, the two easterly jets and one westerly jet in the middle and low latitudes are unaffected and show remarkable persistence. The case with the highest energy, case I, shows a similar picture but with only one robust easterly jet in low latitudes and two westerly jet with greater fluctuations in higher latitudes. With a further increase of the equilibrium energy, persistent jets may disappear. We do not see well-developed zonal jets in a few short (several hundred day) integrations with energies one order of magnitude greater than case I. Long integrations of high energy cases, which require short time steps, are costly, so we do not attempt to define the “boundary” at which persistent jets cease to exist.

For the cases discussed above, the diffusion coefficient is fixed with its $e$-folding time for the smallest latitudes appear to be wider and weaker, consistent with the smaller beta in the high latitudes. Moreover, each jet stays at a fixed latitude, as predicted by Panetta (1993). In his doubly periodic $\beta$-plane model, the equilibrated jets are allowed to drift in meridional direction due to the Galilean invariance in that direction. This does not hold for our rotating spherical model.

Table 1. Statistics for the eight cases shown in Fig. 3 as the open circles. Cases I–VIII are arranged in descending order of the equilibrium energy. All values are nondimensional except for those in the columns 4 and 5. The total and zonal root-mean-square velocities are listed in their dimensional values, $V_{\text{rms}} = a\Omega V_{\text{rms}}$ (in m s$^{-1}$).

<table>
<thead>
<tr>
<th>Case</th>
<th>Number of easterly jets</th>
<th>$[a(n + 1)]_{CM,Z}$</th>
<th>$V_{\text{rms}}^n$</th>
<th>$\langle V_{\text{rms}}^n \rangle_Z$</th>
<th>$\langle F_{\text{zer}} \rangle_{CM}$</th>
<th>$\langle F_{\text{ye}} \rangle_{Z}$</th>
<th>$\langle F_{\text{ye}} \rangle_{Z}$</th>
<th>$\langle F_{\text{ye}}/\langle F_{\text{ye}} \rangle_{CM} \rangle_{CM}$</th>
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<td>5.81</td>
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<tr>
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Fig. 4. Time-mean zonal-mean zonal wind profiles for cases I–VIII in Table 1 (the eight open circles in Fig. 3). Each grid on the abscissa represents 1 m s$^{-1}$.
FIG. 5. Time–zonal wind cross section for case IV. Contour interval is 2 m s$^{-1}$. Easterly regions are shaded.

FIG. 6. Time–zonal wind cross section for case VII. Contour interval is 1 m s$^{-1}$.
into the boundary condition than those with fractional numbers (e.g., “five and a half”). If the energy level of a case is close to the boundary between, for example, the five- and six-easterly jet regimes, then long-term variability of the zonal flow might appear, which exhibits a struggle between the five-jet and six-jet configurations. The case shown in Fig. 8 might have reflected a similar situation. We have not found the same behavior in other cases discussed earlier, although the possibility is not ruled out that this kind of variability could emerge in some of the cases I–VIII in extremely long-term integrations.

Figure 9a shows the meridional gradient of the absolute vorticity and planetary vorticity associated with the time-mean zonal flow for case VIII, taken from the last 1000 days of a 7500-day integration. This case shows a robust six-easterly jet configuration in its last 4500 days (not shown). For comparison Fig. 9b shows the meridional profile of the zonal wind. The structures shown in Figs. 9a and 9b are typical of other cases. In general, an inflection point, where the meridional gradient of absolute vorticity changes sign, does not appear in the time-mean zonal winds. With the absence of an inflection point, the nonlinear stability of a zonal flow on a rotating sphere is proved by Dikiy (1965), following the general variational approach of Arnol’d (1965). (Here the Rayleigh–Kuo criterion for linear stability is the same as the Arnol’d–Dikiy criterion, the notion of the latter is more useful since our turbulent eddies are not small amplitude.) The stability of these zonal flows, a consequence of the northward increase in planetary vorticity, permits their persistence. From Figs. 9a and 9b, large positive values of absolute vorticity gradient coincide with westerly jets. Since westerlies are generally associated with a positive meridional gradient of relative vorticity, they are favored over easterly jets to appear in the high latitudes where beta is small. Note that, in general, the time-mean state of a forced turbulence system need not be stable. An example is the earth’s (three-dimensional) atmosphere, in which the
time-mean zonal flow could be baroclinically unstable. Since a stable time-mean zonal flow is generally seen in our model, and in Vallis and Maltrud’s similar model on a beta plane, we may consider it as a common feature for 2D turbulence randomly forced at small scales.

Since the time-mean zonal winds in our simulations appear to be stable, it is useful to further examine the relationship between the stability of a zonal flow and its scale. As a simple example consider a zonal flow vorticity \( \zeta \) that contains only one spherical harmonic; that is, \( \zeta = c_n Y_n^m \). Following Shepherd (1987c, appendix A), the maximum rms zonal velocity \( V_{\text{rms,zonal}} \)\(^2 \) that satisfies \( d(\zeta + f) \text{d} \theta > 0 \) is \( V_{\text{rms,zonal}}^{\text{max}} = 4 [n(n + 1)]^{-1/2} (2n + 1)^{-1/2} \). When \( n \) is not too small, approximating \( (2n + 1) \) by \( 2[n(n + 1)]^{1/2} \) leads to

\[
V_{\text{rms,zonal}}^{\text{max}} \approx 2^{1/2} [n(n + 1)]^{-1/4}. \tag{5}
\]

Thus, with increasing energy one expects the jets to broaden in order to remain stable. This is consistent with our simulated results. Due to the severe simplification in deriving (5) (e.g., the jets in our model generally contain many spherical harmonic components), we use it more for a qualitative understanding than for a quantitative prediction of the jet scale.

Qualitatively, the relationships of energy to scale in Eqs. (4) and (5) are not dramatically different when \( n \) is not too large. [Also, in our simulations there is a rough equipartition between eddy and zonal energy (see Fig. 3c), \( V_{\text{rms,eddy}}^{2} = V_{\text{rms,zonal}}^{2} \). We have yet to determine if it is a coincidence or has a deeper implication.] This is consistent with simulated results that the jet scale and the Rhines scale are of the same order of magnitude (e.g., see Figs. 12–13). On the other hand, since the upscale energy transfer along the zonal axis is not blocked by beta, the jet scale is expected to be somewhat larger than the Rhines scale. The increase of the difference between the two scales with increasing rotation
has been shown by Vallis and Maltrud (1993, their Fig. 8). Note that if the jet scale is determined solely by the energy and beta, \( n_J = n_J(E, \beta) \), then dimensional analysis would lead to an energy-scale relationship the same as that of the Rhines scale (see Panetta 1993). In general, \( n_J \) may depend on other parameters, and this dependence could modify the energy-scale relationship. In the context of thermal equilibrium theory, the recent work of Frederiksen et al. (1996) hints that zonalization in rotating 2D turbulence generally depends not only on the energy and beta, but also on the enstrophy. In anisotropic and inhomogeneous turbulence one may also expect the higher-order invariants to influence its statistics (Shepherd 1987c). These may provide the phenomenological basis for the jet scale to differ from the Rhines scale. The interaction between the jets and the eddies at different scales will be further analyzed in sections 3b and 3c.

**b. Eddy–mean flow interaction**

In addition to planetary rotation, the presence of persistent zonal jets in our simulations render the turbulent fields inhomogeneous and anisotropic. Some insight into the dynamics of such turbulence may be obtained by decomposing the flow field into a “coherent structure” part and an incoherent, “classical turbulence” part and then studying their interaction, as suggested by McWilliams (1984). For rapidly rotating turbulence in which the major “coherent structures” are nearly stationary zonal jets, a simple and useful decomposition is the stationary–transient decomposition (see Shepherd 1987a,b). This decomposition is employed in the following analyses.

As shown in Table 1, the equilibrated flows in our simulations are dominated by their stationary–zonal and transient–eddy components. Thus, the transient–stationary interaction can also be regarded as the eddy–zonal flow interaction. In the cases with higher energy, although the stationary zonal flow still dominates the zonal component, the zonal flow exhibits more temporal fluctuation (which makes its “time mean” less representative) in the higher latitudes (e.g., Fig. 7). In the following we focus on an analysis of a low-energy case, case VIII, which has weak fluctuations in its zonal flow in the higher latitudes (not shown). This allows us to focus on the dominant interactions between the transient eddies and the steady zonal flow. The results for this case are typical of several other cases we have analyzed.

Figures 10a and 10b show the meridional profiles of some eddy quantities and the time-mean zonally averaged zonal wind for case VIII. The time-mean zonal-mean eddy enstrophy, the solid curve in Fig. 10a, has a quite uniform amplitude over the globe. A clear meridional variation, however, is superimposed upon the uniform distribution, with local maxima that coincide with the centers of the zonal jets, and local minima in the shear zones between the westerly and easterly jets.
The distribution of eddy energy (not shown) is similar. This picture, which is consistent with Shepherd’s (1987a) results (e.g., his Figs. 22c and 16b), is a signature of the shear straining of the eddies by the background zonal flow. In Shepherd’s simulations, a very large scale (wavenumber one) zonal flow is superimposed on a flow field of relatively small-scale eddies to ensure scale separation between the two. In our cases, most of the eddy energy is contained in large-scale eddies whose scale is similar to that of the persistent zonal flow. It is shown below, however, that in our simulations small-scale eddies, particularly in the forcing scale and smaller, are the main contributors to the maintenance of zonal jets, and there is a clear scale separation between these eddies and the zonal flow.

The time-mean zonal-mean eddy vorticity flux \( \frac{1}{2} \frac{\partial \zeta}{\partial z} \) is also shown in Fig. 10a. (We use \( X, \bar{X}, \tilde{X} \), and \( X' \) to denote the zonal-mean, eddy, stationary, and transient components of the flow variable \( X \).) The vorticity flux is positively correlated with the zonal wind profile. Figure 10b shows the time mean of \( -\frac{\partial \zeta}{\partial z} d\left( \frac{\zeta}{\rho} + f \right) / d\theta \), the negative covariance between vorticity flux and mean absolute vorticity gradient. An upgradient vorticity flux dominates westerly regions. Since the gradient of planetary vorticity dominates the zonally averaged absolute vorticity, potential vorticity homogenization (Rhines and Young 1982) has clearly not occurred. The persistent jets shown by Vallis and Maltrud (1993) and Panetta (1993) show a similar structure in their absolute vorticity.

The two-dimensional energy spectrum \( E(m, n) \) for the same case is shown in Fig. 11. (For clarity, only the part with \( n \leq 63 \) is shown.) Some of the highest energies occur in zonal components. An anisotropic Rhines curve appears in the low wavenumber eddies, for \( n < 20 \). Because this case has a lower energy than that shown in Fig. 2 for decaying turbulence, a larger AR curve is expected in Fig. 11. This apparent discrepancy is explained in section 3c as a consequence of the persistent zonal flow, which modifies the pure Rossby–Haurwitz modes so that the AR curve is no longer governed simply by (3). In Fig. 11, below the AR curve the eddy energy is, as expected, very small. On the high wavenumber side, far away from the Rhines scale, the energy...
Fig. 11. Energy spectrum of case VIII in the forced experiments. The magnitude of the spectrum is normalized by the maximum value on the map. Contour levels are 0.000 01, 0.0001, 0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 0.1, 0.2, and 0.5. Area with $E(m, n) > 0.005$ is lightly shaded, $E(m, n) > 0.05$ heavily shaded. For clarity only modes with $n \leq 63$ (out of a T85 truncation) are shown.

Contours become more parallel and depend only on total wavenumber $n$, indicating greater isotropy for eddies at small scales. The local maximum at $n = 55$ is a result of the forcing at that wavenumber.

The one-dimensional energy and enstrophy spectra for the same case are shown in Fig. 12. The spectra are the summation of the two-dimensional spectra over all zonal wavenumbers $m$ at total wavenumber, $n$. Spectra are shown for the eddies and for the total flow. Most eddy energy is in the large-scale eddies, with a maximum at around $n = 10–15$. This peak is related to the Rhines barrier. The zonal flow energy, the difference between the total and eddy spectra in Figs. 12a, has maximum at $n = 16$, and the scale is close to the Rhines peak for the eddies. While the eddy energy drops sharply below $n = 10$, the zonal energy does less so, as better seen in Fig. 13, a log–log plot for the eddy and zonal flow energy spectrum. (The zonal energy spectrum is shown only in the large scales. Medium and small scales are dominated by the eddies.) This is expected since the upscale energy transfer along the zonal axis is not directly arrested by beta (Vallis and Maltrud 1993). The enstrophy spectrum in Fig. 12b shows two peaks, one at the forcing scale in the eddies and the other at the large scale in the persistent zonal jets. In a snapshot of the vorticity field (not shown) these two scales are plainly visible, with small-scale eddies embedded within large-scale zonal jets. Similar pictures can be found in some recent work (e.g., Maltrud and Vallis 1991; Nozawa and Yoden 1995).

As the forcing maintains a spectral peak at $n = 55$, we expect a downscale enstrophy transfer and an upscale energy transfer from it (Fjørtoft 1953). Ideally, in the absence of dissipation there would be an energy inertial
range on the upscale side of $n = 55$ and an enstrophy inertial range on its downscale side, with slopes in the energy spectrum (in a log–log diagram) of $-5/3$ and $-3$ (Kraichnan 1967). [In spherical geometry the appropriate isotropic total wavenumber is $[n(n + 1)]^{1/2}$ (Boer 1983), so the $K^{-3}$ in plane geometry should be replaced by $[n(n + 1)]^{-3/2}$. At large $n$, however, this is indistinguishable from $n^{-3}$.] Two lines indicating the $-5/3$ and $-3$ slopes are shown in Fig. 13 for comparison. Despite the influence of the dissipation by linear drag, the slope of the upscale branch of the energy spectrum above the forcing scale is close to $-5/3$. The slope on the downscale side is much steeper than $-3$, however. For $n > 70$ the energy spectrum drops sharply due to the high-order diffusion, but even in the range $55 < n < 70$ where diffusion is relatively weak, the slope is still steeper than $-3$. The narrowness of the “inertial” range may contribute to this discrepancy. Recently many other numerical studies have found, however, that the spectral slope of the enstrophy inertial range is much steeper than $-3$ (Basdevant et al. 1981; Maltrud and Vallis 1991). It has also been argued that the $-3$ slope is the shallowest possible that can be achieved by an enstrophy inertial range [see Vallis (1992) for a review].

To further explore the spectral properties of the flow, the spectral energy flux is calculated based on the stationary–transient decomposition. A more detailed background can be found in Shepherd (1987a,b). As previously discussed, the stationary–transient interaction in our case can be regarded as the interaction between stationary zonal flow and transient eddies. The spectral energy equations for the transient (“$T$”) and stationary (“$S$”) components can be written by

$$\frac{\partial}{\partial t} E_T(m, n) = I_T(m, n) + C_T(m, n) + \{\text{forcing and dissipation}\} \quad (6)$$

$$\frac{\partial}{\partial t} E_S(m, n) = I_S(m, n) + C_S(m, n) + \{\text{forcing and dissipation}\} \quad (7)$$

where $E_T(m, n)$ and $E_S(m, n)$ are the energy of the spherical harmonic $Y_m^0$ component. In the time mean the terms in the right-hand side in (6) or (7) are in balance. The time mean of $I_T$, $C_T$, and $C_S$ (we hereafter use the same symbols to denote the time-mean quantities) are defined by

\[ I_T = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} E_T(m, n) \sin(n \theta) d\theta \, dn \]

\[ C_T = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{\partial}{\partial t} E_T(m, n) \sin(n \theta) d\theta \, dn \]

\[ I_S = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} E_S(m, n) \sin(n \theta) d\theta \, dn \]

\[ C_S = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{\partial}{\partial t} E_S(m, n) \sin(n \theta) d\theta \, dn \]
\[ I_t(m, n) = \frac{1}{4} \psi_n^{m^*} \{ J(\psi', \zeta') \} \text{c.c.} \]

\[ C_t(m, n) = \frac{1}{4} \psi_n^{m^*} \{ J(\psi, \zeta) \} \text{c.c.} \]

\[ C_s(m, n) = \frac{1}{4} \psi_n^{m^*} \{ J(\psi', \zeta') \} \text{c.c.} \]

where ( )\text{c.c.} means the spherical harmonic $Y^m_n$ component of the quantity ( ) and an asterisk indicates complex conjugate. $I_s(m, n)$ is similarly defined as $I_t(m, n)$, with the $(\psi' \zeta')$ replaced by $(\bar{\psi}, \bar{\zeta})$. It is unimportant here since the stationary component is almost zonal and the Jacobian of a zonal flow vanishes. A positive $I_t(m, n)$ indicates a gain of the transient energy at the mode $\psi_n^m$ through nonlinear eddy-eddy interaction. A positive $C_t(m, n)$ ($C_s(m, n)$) indicates a gain of the energy for the transient (stationary) component at $\psi_n^m$. Locally $C_s(m, n)$ and $C_t(m, n)$ do not have to be equal and opposite in sign, but their summations over all wavenumbers have to be so. In other words, the loss of eddy energy at one wavenumber must appear as a gain of the mean flow energy at some wavenumber. When considering the two together, we obtain an apparent picture of energy “flows” in the spectral domain due to energy conversions. One can define $I_{ST}(m, n) = C_s(m, n) + C_t(m, n)$ as the spectral energy “transfer” due to stationary-transient interactions. Since the summation of $I_t(m, n)$ or $I_s(m, n)$ over all wavenumbers is zero, we can define the following one-dimensional spectral energy fluxes

\[ F_T(n) = -\sum_{\alpha=0}^{n} \sum_{m=-\alpha}^{\alpha} I_t(m, \alpha), \quad (9) \]

\[ F_{ST}(n) = -\sum_{\alpha=0}^{n} \sum_{m=-\alpha}^{\alpha} I_{ST}(m, \alpha), \quad (10) \]

or write $F_{ST}(n)$ as

\[ F_{ST}(n) = Q_s(n) + Q_t(n), \]

\[ Q_t(n) = -\sum_{\alpha=0}^{n} \sum_{m=-\alpha}^{\alpha} C_t(m, \alpha), \]

\[ Q_s(n) = -\sum_{\alpha=0}^{n} \sum_{m=-\alpha}^{\alpha} C_s(m, \alpha). \quad (11) \]

Unlike $F_{ST}$, $Q_t$ or $Q_s$ alone is not a flux. They nevertheless can be used to identify the major interacting scales of the transient and stationary components if there is a clear scale separation between the two.

Figure 14a shows the two fluxes $F_T(n)$ and $F_{ST}(n)$ for the same 1000-day statistics from case VIII. A positive value corresponds to a downscale flux as indicated by the arrows in the figure. Note that for classical turbulence, that is, in the absence of persistent zonal jets, $F_T(n)$ alone is responsible for the spectral transfer of energy. In our case, however, the eddy-eddy interaction

\[ F_T \] and the eddy-mean flow interaction $F_{ST}$ are equally important. For the $F_T(n)$ curve, the upscale side (with respect to $n = 55$) between $n = 40$ and 55 appears to be flat. The constant spectral energy flux is consistent with an energy inertial range. Toward the large scales the eddy energy flux gradually converges below $n = 35$. The flux convergence falls to nearly zero for $n < 10$, showing the effect of the Rhines barrier. The downscale energy flux of $F_T(n)$ for $n > 55$ is associated with the classical downscale transfer of enstrophy from the forcing scale.

For the $F_{ST}(n)$ curve, the upscale flux does not change...
greatly until reaching about \( n = 16 \), where a sharp convergence occurs. To interpret this, it is helpful to use the decomposition (11). Figure 14b shows the \( Q_{\xi}(n) \) and \( Q_{\eta}(n) \) that constitute \( F_{st}(n) \). Clearly, the loss of eddy energy and the gain of mean flow energy occur at different scales. The major loss of eddy energy due to eddy–mean flow interactions occurs at \( n = 55 \), the forcing scale, where \( Q_{\xi} \) has a sharp divergence. This energy reappears in a narrow band around \( n = 10–16 \), the scale of the persistent zonal jets, as a gain of the mean flow energy, where \( Q_{\eta} \) shows a sharp convergence. This demonstrates the nonlocal interaction between small-scale eddies and large-scale zonal jets. Also, note that the downscale energy flux \( F_{st}(n) \) for \( n > 55 \) can be associated with the upscale eddy enstrophy transfer due to shear straining by the zonal flow. Shepherd (1987b) pointed out that, in the absence of forcing and dissipation, the eddy enstrophy should be nearly conservative through the shear straining process, which would lead to the relationship \( \Sigma \Sigma n(n+1)C_{m}(m,n) = 0 \). This is not quite satisfied in our simulation due to the strong forcing and dissipation. Finally, \( Q_{\xi}(n) \) is almost zero for \( n < 30 \), indicating that the interaction between the large-scale eddies and the zonal-mean flow produces little net energy exchange in the long-term mean.

The analysis shown in Fig. 14 is based on the energy budget of each individual mode. From an interaction coefficient perspective, a pair of interacting modes is needed to generate a third one. Thus, one might obtain an even clearer view about the energy transfer by looking at the statistics of each triad in the spectral space. An analysis of this type would be very complicated. In the following, we adapt a relatively simple diagnostic analysis to partially address the issue.

The energy equation for the zonal flow is

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \left| u \right|^2 \right) = \left[ \tilde{\nu} \tilde{\nu} \right][u] - \kappa[u]^2 - [u][D_n],
\]

where \( D_n \) is a momentum diffusion term that can be calculated from \( D_c \). A forcing term does not explicitly appear in (12) since forcing is excluded from zonal modes. The global integrals of the three terms \( \tilde{\nu} \tilde{\nu} [u] \), \( -\kappa[u]^2 \), and \( -[u][D_n] \) are denoted as \( M_e \), \( M_s \), and \( M_p \), respectively. For example,

\[
M_e = \frac{1}{2} \int_{-n/2}^{n/2} \left[ \tilde{\nu} \tilde{\nu} \right][u] \cos \theta \, d\theta.
\]

In the time mean of a statistical equilibrium state, 0 = \( M_e + M_s + M_p \), the net loss of zonal energy due to dissipation \( M_s + M_p \) (\( M_e \) dominates) is balanced by a net gain of energy through eddy–mean flow interaction, \( M_e \). To see what scales in the eddy field contribute to the maintenance of the zonal flow, we define \( M_e(N) \) by the same global integral as in (13) but with the eddy vorticity field truncated at total wavenumber \( N \). (Thus, \( M_e(85) = M_e \).) A reversely truncated function \( M_e(N) \) is similarly defined as \( M_e(N) \), but with only the modes with \( n \geq N \) retained for the eddy field. Figure 15 shows the time mean of \( M_e(N) \) and \( M_e(-N) \) as functions of \( N \) for the previously discussed case VIII. Each point in the figure represents a 1000-day statistic. With decreasing truncation of the eddy field, \( M_e(N) \) decreases significantly from \( n = 75 \) to \( n = 55 \), indicating that vertical pairs of nonzonal modes (modes with different values of \( n \) but with the same \(|m|\)) within the band \( 55 \leq n \leq 75 \) are important in maintaining the zonal flow. While \( n = 55 \) is the location of the forcing, the value of \( M_e(55) \) is small. In other words, the interaction of the forcing-scale eddies with eddies of greater scales does little to maintain the zonal flow. This is expected from Fjørtoft’s theorem. The value of \( M_e(N) \) for \( N < 55 \) is generally small. To estimate the contribution of the eddies at \( n = N \) to the shear straining process, it is perhaps more natural to look at \( M_e(-N) \), as it contains the eddies at \( n = N \) and smaller scales. The \( M_e(-N) \) curve rises from \( n = 75 \) to \( n = 45 \) (with a sharp rise when \( n = 55 \) is included), indicating that mainly this range of the eddies are involved in the shear straining process. From either \( M_e(N) \) or \( M_e(-N) \), in the time mean the large-scale (e.g., \( n < 30 \)) eddies have little net contribution to the maintenance of the zonal flow, consistent with our previous discussions based on \( F_{st}(n) \). When looking at the transient evolution of \( M_e(N) \), large values and large temporal fluctuation of \( M_e(N) \) are seen for small \( N \). Thus, large-scale eddies actively interact with the zonal jets on a short timescale, but this interaction is statistically incoherent and sums to nearly zero in the long-term mean. The role of the large-scale eddies in the short-term dynamics deserves further examination, but an analysis in that aspect would be very complicated as the turbulent flow is not in equilibrium on a short timescale.
c. A linear model simulation

Since the eddy–mean flow interaction plays an important role in the spectral energy transfer in the forced turbulence model, it is of interest to isolate this process from the fully nonlinear system. As shown by Shepherd (1987a), some insight may be gained from a linear model with a fixed zonal basic state. A forced linear model is created by linearizing Eq. (1) with respect to a basic zonal flow and assuming that the basic state itself is not subject to linear damping and diffusion (thus fixed in time). In our case, the basic state of the linear model is defined as the time-mean zonal flow from a nonlinear simulation, we choose case VIII for the following discussion.

Before describing the results of the forced linear simulation, a few characteristics of the basic flow are noteworthy. In section 2 the Rossby–Haurwitz frequencies are used to derive the anisotropic Rhines curve, as in (3). The presence of persistent zonal jets causes the frequencies and structures of the normal modes to deviate from the pure Rossby–Haurwitz frequencies and structures. To see the extent of this modification, we calculate the normal modes of the zonal basic state. For simplicity we perform an inviscid calculation, with the linear drag and diffusion ignored. The eigenvalue problem and the technique for its solution are described in Dikiy and Katayev (1971). Since we have 85 zonal wavenumbers, we solve 85 one-dimensional eigenvalue problems (for \( m = 1, 2, \ldots , 85 \)) to construct Fig. 16. In Fig. 16, \( C_r \) is the angular phase speed, related to the frequency \( \omega \) by \( \omega = m C_r \). Modes with \( C_r < -0.08 \) are not shown. In the absence of an inflection point for the basic state, all modes are neutral with \( C_r = 0 \). To be consistent with our T85 truncation, for each zonal wavenumber \( m \) we truncate the eigenvalue problem at \( n = 85 \). In the absence of a basic zonal flow the eigenfrequencies are just the Rossby–Haurwitz frequencies under T85 truncation. The angular phase velocity \( C_r \) of a neutral mode should not exceed \( W_{\text{max}} \), the maximum angular velocity of the basic flow (Dikiy and Katayev 1971); \( W \) is defined by \( W(\theta) = U(\theta)/\cos \theta \), where \( U(\theta) \) is the velocity of the basic state. This condition is satisfied in Fig. 16.

In the absence of a basic state, the normal modes, pure Rossby–Haurwitz modes, are all westward propagating, with frequencies \( \omega_n = -2m[n(n + 1)] \) and phase speeds \( C_r = -2[n(n + 1)] \), independent of \( m \). In Fig. 16 many of these modes have become eastward propagating. In fact, most of the “modes” are within the range \( W_{\text{min}} < C_r < W_{\text{max}} \) and therefore fall into the continuum. The only modes that remain discrete are those below the line labeled \( W_{\text{min}} \) in Fig. 16. In the absence of unstable modes, these discrete modes can be unambiguously tracked back to pure Rossby–Haurwitz modes by gradually reducing the basic flow to zero (Dikiy and Katayev 1971; Branstator and Held 1995). The Rossby–Haurwitz modes thus obtained are shown as circles in the inserted triangle in Fig. 16. Generally, these large-scale modes are not severely modified by the zonal flow. We may, therefore, think of the discrete modes in Fig. 16 as approximately the Rossby–Haurwitz modes particularly for small \( n \). This is why the anisotropic Rhines curve can be clearly seen in the large-scale part of the energy spectrum (e.g., Fig. 11), even in the presence of a persistent zonal jet. In regard to the eddy–mean flow interaction, the discrete modes evolve periodically on the background zonal flow and contribute little to a net exchange of energy with the zonal flow (see Held 1985). We will see this in the forced linear simulation below. This could also be one of the reasons that (as shown in §3b) in the nonlinear simulations the large-scale, energy-containing eddies make little contribution to the maintenance of the persistent zonal jets in the time mean.

We now turn to the forced linear simulation. Except for the fixed zonal basic flow and the omission of eddy–eddy interactions, the other aspects of the model are the same as the nonlinear model. The governing equation for the eddy vorticity \( \zeta \) is
\[ \frac{\partial \zeta}{\partial t} = -J(\Psi, \zeta) - J(\psi, Q + f) - \kappa \zeta - D_{\zeta} + F_{\zeta}, \]

with the linear drag \(-\kappa \zeta\) diffusion \(-D_{\zeta}\), and forcing \(F_{\zeta}\) identical to those in the nonlinear model; \(F_{\zeta}\) is set to the value that, in the nonlinear simulation, produces the persistent jets used as the basic state. The \(\Psi(t)\) and \(Q(t)\) are the basic-state streamfunction and vorticity. A zero eddy field is used as the initial condition. The balance between forcing and dissipation allows the model to reach a statistical equilibrium. As discussed before, in a classical turbulence model without persistent zonal jets the spectral energy transfer is due to the eddy–eddy spectral energy flux \(F_{ST}\), with negligible \(F_{TT}\). In the linear model it is the opposite. The spectral energy transfer is solely due to the eddy–mean flow interaction \(F_{ST}\), while \(F_{TT}\) is zero. Figures 17a and 17b show \(F_{ST}\), \(Q_S\), and \(Q_T\) from the statistics of the last 1000 days of a 2000-day linear simulation. The downscale (with respect to \(n = 55\)) side of the \(F_{ST}\) curve is similar to Fig. 14. In general, the shear straining by the zonal flow is reproduced by the forced linear simulation. The upscale side of \(F_{ST}\), however, is different. In the linear model the zonal flow is an energy source for large-scale eddies with \(20 < n < 55\). The basic features of the eddy energy spectrum in the large scales is not recovered by the linear model. This is expected because in the fully nonlinear model the large-scale eddies are maintained by a classical upscale energy transfer by the eddy–eddy interactions, which is missing in the linear model. Finally, \(Q_T(n)\) is small in the large scales and is almost zero for \(n < 20\), consistent with the fact that eddies in the large scales are dominated by discrete modes that are immune to a net energy exchange with the zonal-mean flow. We conclude that in our model the interactions between the large-scale zonal flow and the small-scale eddies can be approximately simulated by a linear model representing the shear straining mechanism. The classical eddy–eddy interaction, however, is essential to producing a realistic eddy energy spectrum particularly for the large scales.

4. Discussion

In section 3 we have focused on the T85 simulations with stochastic forcing at \(n = 55\). Here we discuss more general cases and the impact of the forcing scale and other model parameters on the simulated zonal jets. As pointed out by Vallis and Maltrud (1993) and Panetta (1993), in a stochastically forced barotropic model, the forcing scale and the jet scale have to be separated to reliably obtain persistent zonal jets. In our case VIII, for example, the jet scale \(n_J\) is around 15, clearly separated from the forcing scale \(n_F = 55\). In some cases (not shown) when the stochastic forcing is placed at the large scales close to \(n_F\), the jets become more poorly defined and less persistent. We have previously discussed that the shear-straining mechanism involving the small-scale eddies and the zonal-mean flow plays an important role in maintaining the jets. As our result is consistent with Shepherd’s (1987a) investigation of inhomogeneous 2D turbulence, we have to point out that the models used in both studies have been artificially designed such that a wide separation exists between the scales of the forcing (or random initial condition) and the zonal jets. Without such a scale separation, the shear straining process might not efficiently take place. This could be one of the reasons that the jets in our model, and similar barotropic models in other work, become relatively obscure when \(n_F\) approaches \(n_J\).

Note that when the “forcing” is not random, a clear scale separation discussed above might not be necessary for the jets to survive. In Panetta’s (1993) two-layer QG
model simulations, the energy injection scale (close to the most energetic scale for baroclinic instability) is not widely separated from the jet scale, yet the jets are strong and persistent. In the QG model, however, the baroclinic waves may not be random, and their characteristics (growth rate, momentum, and heat fluxes, etc.) depend on the jets they help to maintain. Not surprisingly, some of the eddy statistics in Panetta’s work show more complicated structures than would be expected from a simple shear straining mechanism. [For example, subtle differences exist between Panetta’s eddy variance fields (his Figs. 8 and 9) and ours (see Fig. 10a) or Shepherd’s. In the latter, the variance maxima are located at the zonal wind maxima and minima; in the former, variance peaks are found on the westerly jet flanks.] Thus, although the shear straining picture explains the maintenance of the jets in our model, it might not be regarded as a universal mechanism for more complicated jet–turbulence systems.

Keeping the stochastic forcing at the small scales, we have tried to put the forcing in a narrow band \( n_1 \leq n \leq n_2 \) (instead of a single total wavenumber \( n_0 \)), or use a random-phased forcing with constant energy generation rate [similar to Eq. (4.6) of Shepherd (1987a)]. Persistent jets are seen in both cases. Thus, as long as the forcing is used to represent small-scale random stirring, its details may not be crucial to the jet formation. Note that in the case with the forcing placed in a band \([a, b]\), two modes inside the band, \( \phi_{m}^{n}, \phi_{m}^{-n} \) \( (n_1 \leq p, \quad q \leq n_2) \), could interact to generate large-scale zonal modes. When the forcing band is narrow, however, this kind of interaction is generally weak due to its small interaction coefficient (e.g., see Fig. 19).

While one might argue that a low-wavenumber forcing is more “realistic” since it represents the energy injection by baroclinic eddies, a high-wavenumber forcing used here is not necessarily unphysical. When comparing the observed atmospheric turbulence spectrum with theories, Lilly (1989) suggests that large-scale, 2D turbulence is forced at two scales. One is the baroclinic energy injection scale; another is the mesoscale, whose energy may originate from convective or shear instability. The complexity of a two-forcing turbulence is demonstrated by Maltrud and Vallis (1991), who establish in a \( \beta \)-plane model the coexistence of an upscale energy inertial range and a downscale enstrophy inertial range between the forcings, without the assistance of dissipation. This shows that there is still much that is unknown about 2D turbulence forced by a general forcing \( F(n, \omega) \) (with \( n \) and \( \omega \) wavenumber and frequency). Since there are also many unknowns regarding the observed atmospheric forcing \( F_{\text{obs}}(n, \omega) \), it is worthwhile trying a wider variety of forcings in future studies of forced 2D turbulence.

We have discussed in sections 2 and 3 that the upscale energy transfer along the zonal axis is not directly arrested by beta, which leaves a question as to why the energy does not accumulate at the largest zonal scales [up to \( n = 2 \); energy accumulation at \( n = 1 \) violates the conservation of angular momentum (Tang and Orszag 1978)]. A possible reason, suggested by Vallis and Maltrud (1993) for \( \beta \)-plane geometry, is that the interaction coefficient for a triad including one very large-scale zonal mode and a pair of nonzonal modes is generally small. Here we take a look at the interaction coefficient in spherical geometry. Classical work (Silberman 1954; Platzman 1960) described that the interaction coefficient \( L_{\phi_{m}^{n}, \phi_{m}^{-n}}^{n, m, m, m} \) in (2) is nonzero when all of the following criteria are satisfied: (I) \( m = m' + m'' \), (II) \( |n' - n''| < n < n' + n'' \), (III) \( n + n' + n'' \) is odd, (IV) at least one of \( m' \) and \( m'' \) is nonzero, (V) \( n' \neq n'' \), and (VI) \( (m', n') \neq (-m, n), (m'', n'') \neq (-m, n) \). The forms of these criteria may vary depending on the definitions of the interaction coefficient equation and the spherical harmonics. We define \( Y_{n}^{m} \) as in Boer (1983). Strictly speaking, the above criteria are not complete. For example, the triad \( \{\phi_{m}^{n}, \phi_{m}^{-n}, \phi_{m}^{0}\} \) satisfies all of the criteria (I)–(VI), but its interaction coefficient is identically zero. Exceptions of this kind, which are presumably not abundant, can be covered by a new criterion. As the criterion is somewhat complicated and is not crucial to our present discussion, we will report its detail in a separate note\(^1\) (manuscript available from the authors).

From criteria (I) and (IV), to generate a zonal mode \( (m = 0) \), the other two modes in the triad must satisfy \( m'' = -m' \neq 0 \). Criterion (II) further restricts the “vertical separation” (in total wavenumber \( n \)) of the two modes that give nonzero interaction coefficients. This is illustrated in Fig. 18a by considering a given zonal mode (the cross labeled \( n \)), a given nonzonal mode (the filled circle, \( n' \)), and a third mode (an open circle, \( n'' \)). For given \( n \) and \( n' \), the third mode \( n'' \) must fall within the range \( n' - n + 1 \leq n'' \leq n' + n - 1 \) for a nonzero interaction coefficient to be possible. We call this range the “nontrivial range.” Criterion (III) further eliminates about a half of the modes \( n'' \) in the nontrivial domain. Figures 18b and 18c show two cases, with the zonal mode in (c) having a larger scale (smaller wavenumber \( n_{0} \)) than that in (b). For a given second mode \( n'' \) (e.g., consider one at the forcing scale), the nontrivial range for the third mode \( n'' \), the shaded area, shrinks with decreasing \( n_{0} \). Thus, it is apparently more difficult to maintain a very large scale zonal jet than a medium-scale one, as fewer modes can be useful in the nonlinear interaction involving the former. It is also interesting to point out that within the nontrivial range the size of the interaction coefficient is not uniform but tends to be larger with larger separation between \( n' \) and \( n'' \). Figure 19 shows an example, the absolute value of \( L_{\phi_{m}^{n}, \phi_{m}^{-n}}^{n, m, m, m} \) as a function of \( m' \) and \( n'' \). The size of the circles, shown

\(^1\) Briefly, the criterion requires that the sextuplet \((-m, m', m'', n, n', n'') \) is not a polynomial zero of the Wigner's 3-j symbol \((\phi_{m}^{n}, \phi_{m}^{-n})\) [Wigner (1959) and the polynomial zero defined as in, e.g., Srinivasa Rao and Rajeswari (1993)].
only for nonzero modes, indicates the magnitude of the interaction coefficient. The smallness of the interaction coefficient with \( n'' = n' \), which is seen in many of our numerical calculations, might also contribute to the difficulty of maintaining the very large scale zonal modes.

5. Conclusions

The dynamics of two-dimensional turbulence on a rotating sphere are examined in this work. The anisotropic Rhines scale is derived for the two-dimensional energy spectrum in spherical geometry. A simple formula for the anisotropic Rhines curve is verified in simulations of decaying turbulence. Forced–dissipative model simulations are performed to investigate persistent zonal jets in 2D turbulence. Persistent zonal jets form in the low energy (strong rotation) cases with \( V_{rms} \ll 1 \) (or \( V_{rms} \ll \Omega \)). The scale of the jets decreases with the energy. After reaching equilibrium, the zonal jets generally stay at fixed latitude for very long time. Long-term variability of the zonal jet configuration is shown in one case. It may reflect Panetta’s argument that, a case with its energy level close to the boundary of two integral-jet-number regimes may not possess “fractional number” of jets but instead show long-term variability characterized by the struggle between the two integral regimes. A more detailed analysis of this type of variability is left for future study, as it requires extremely long-term integrations of many nearby cases in the parameter space.

The time-mean zonal-mean flow in our forced simulations appear to be stable, with the absolute vorticity gradient dominated by beta. A similar feature was found by Vallis and Maltrud (1993), who also used a barotropic model randomly forced at small scales. Based on a simple “single mode” model of the zonal flow, a relationship between the energy and the zonal jet scale is derived to demonstrate the increase of the jet scale with increasing energy under the stability requirement. Qualitatively, at small \( n \), it is not very different from the dependence of the Rhines scale on the energy. Both show a similar trend of increasing scale with increasing energy. This is consistent with the similarity of the jet scale and the Rhines scale shown in the simulations.

The spectral properties of the inhomogeneous turbulence is investigated using a transient–stationary (or eddy-zonal flow) decomposition. In the presence of persistent zonal jets, the shear-induced spectral transfer of energy can be as important as the classical energy transfer by eddy–eddy interactions. Nonlocal interactions between small-scale eddies and the large-scale zonal flow is demonstrated. In the time mean, the shear straining mechanism involving small-scale eddies and the large-scale zonal flow plays an important role in maintaining the jets. While large-scale, energy-containing eddies (around the Rhines scale) interact with the zonal jets on short timescales, in the long term the interaction is statistically incoherent and averages to nearly zero in the time mean. Thus, despite the similarity between the jet scale and the Rhines scale, the dynamical link between them is not obvious in the time-mean statistics.

The pure Rossby–Haurwitz modes, which we use to derive a simple formula for the anisotropic Rhines curve (2), are modified by the persistent zonal flow. Most modes actually fall into the continuum after the modification by the zonal flow. Large-scale modes, however, may remain discrete. The discreteness of these modes limits the ability of large-scale eddies to make a net exchange of energy with the zonal jets. The modification of the spectral properties of the large-scale modes is less severe than for the medium and small-scale modes. As a result, the idealized formula for the anisotropic Rhines curve (derived from pure Rossby–Haurwitz frequencies) is still useful for defining the wave and turbulence boundary in the large-scale part of the energy spectrum.

In this work we have focused on the equilibrated zonal jets and the interaction between the jets and the eddies in the equilibrium statistics. The transient evolution of the zonal flow during its formative stage, particularly the merging and widening (or narrowing) of zonal jets as seen in Figs. 5–8, appears to be interesting. As most turbulence theories are based on equilibrium statistics, an investigation of the nonequilibrium dy-
namics is not trivial and is not pursued here. Like many of the previous studies, we have mainly focused on the analysis of the energy and enstrophy. In an inhomogeneous and anisotropic system higher-order invariants may also be important (e.g., Shepherd 1987c). Can analyses based on higher-order invariants shed new light on the dynamics of the jets? We have not answered these fundamental questions, but our work may provide a useful reference point for future research on the dynamics of zonal flows generated by turbulence.

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Fig. 19. The magnitude of the interaction coefficient $L_{m,n}^{m',n'}$ as a function of $m'$ and $n'$, represented by the size of the circles.


