Finite-Time Evolution of Small Perturbations Superposed on a Chaotic Solution: Experiment with an Idealized Barotropic Model

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ABSTRACT

Fundamental principles of finite-time evolution of small perturbations in chaotic systems are examined by using an idealized barotropic model on a rotating sphere, which is a forced-dissipative system of 1848 real variables.

A time-dependent solution that is investigated is a chaotic solution with four nonnegative Lyapunov exponents. Attention is focused on the subspace spanned by the first four backward Lyapunov vectors. It is found that the time variations of the subspace Lorenz index, which is the mean amplification rate of perturbations defined in the subspace, are highly correlative with those of the Lorenz index, which is the mean amplification rate defined in the whole phase space, when the time interval of the Lorenz index is several days longer than that of the subspace Lorenz index. The first forward singular vector in the subspace has a property that its amplification rate is insensitive to the measuring norm, like the first backward Lyapunov vector, and has a tendency that its evolved pattern becomes similar to that of the first forward singular vector in the whole phase space.

Application of the method introduced in this study to construct initial members in ensemble forecasts is discussed.

1. Introduction

Lorenz (1963) indicated with a three-variable model that there is a limit in extended-range weather forecasts by any numerical model, even if the model is perfect, because an inevitable error in the initial value increases with time exponentially due to the nature of chaos. His indication became an actual problem in medium-range weather forecasts a few decades later through the developments of numerical modeling techniques and computing facilities; it appeared that the traditional deterministic forecasts with only one member, which work well for short-range forecasts, are not always useful for medium-range forecasts. In recent years, ensemble forecasts with multiple members have been in operation at some centers for medium-range forecasts (Toth and Kalnay 1993, 1997; Mureau et al. 1993; Molteni et al. 1996). The ensemble forecasts provide probabilistic forecasts based on the statistics of the ensemble members at forecasting time, not deterministic forecasts. Each member forecasts a possible future weather and the spread of ensemble members gives information on the reliability of the forecasts, or the predictability.

Ensemble forecasts with enormously large members cannot be applied to highly sophisticated models such as operational numerical weather prediction models since there is still a limit in computing facilities. It is necessary in practice to choose a limited small number of initial members that produce as efficient ensemble forecasts as possible. There exist two major methods of choosing initial members, which are both based on the theory of the evolution of small perturbations. One is breeding vectors, which are operationally used at NCEP (Toth and Kalnay 1993, 1997); the vectors that have been bred for a long time in the analysis cycle are used as the initial perturbations to construct the initial ensemble members. The breeding vectors correspond to the backward Lyapunov vectors in the linear limit. The terminology in this paper follows Legras and Vautard (1996). The other is forward singular vectors, which are operationally used at ECMWF (Mureau et al. 1993; Molteni et al. 1996); the vectors that will grow rapidly for a prescribed finite time interval are used as the initial perturbations to construct the initial ensemble members. The problem of which method is better or not is still in controversy.

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Theoretical studies on the finite-time evolution of small perturbations have been made since Lorenz (1965) pointed out with a 28-variable atmospheric model that the amplification rate of perturbations for a prescribed time interval depends on the flow field. He showed that such a finite-time linear evolution of perturbations can be analyzed by the singular value decomposition of the error matrix that expresses the mapping of perturbations for the time interval. This analysis corresponds to the ordinary Lyapunov stability analysis when the time interval approaches infinity (Goldhirsch et al. 1987). Some new concepts and relationships on the perturbation evolution have been proposed by the studies with low-order systems (e.g., Mukougawa et al. 1991; Trevisan and Legnani 1995; Anderson 1996; Trevisan and Pancotti 1998) and simple atmospheric models (e.g., Lacarra and Talagrand 1988; Houtekamer 1991; Yoden and Nomura 1993; Yamane and Yoden 1997). However, the dynamical properties of the perturbation evolution have not been thoroughly clarified yet.

The purpose of this paper is to clarify the fundamental dynamics of the finite-time evolution of small perturbations. We focus our attention on the subspace spanned by leading backward Lyapunov vectors, and examine the properties of the root-mean-square amplification rate of perturbations and the first forward singular vector in the subspace by comparison with those in the whole phase space, using an idealized barotropic model. The definitions of the used quantities are given in section 2. Section 3 shows the numerical results in the idealized barotropic model. Discussion is given in section 4, and conclusions are in section 5.

2. Definitions

a. System and perturbation

Let us consider a nonlinear dynamical system of dimension $n$:

$$\frac{d}{dt}x(t) = f[x(t)],$$

$$x_0 = [x_1(t), \ldots, x_n(t)]^\top \in \mathbb{R}^n,$$  \hspace{1cm} (1)

where $\top$ denotes transpose. A solution $x(t)$ of the system is uniquely determined by setting an initial state.

An infinitesimally small perturbation $y(t) \in \mathbb{R}^n$ superposed on the solution $x(t)$ obeys the tangent linear equation of Eq. (1):

$$\frac{d}{dt}y(t) = J[x(t)]y(t),$$

where $J[x(t)]$ is the Jacobian matrix, $J[x(t)] = (\partial f/\partial x)_{x=x_0}$. The solution of Eq. (2) can be written in the form of a linear transformation with an $n \times n$ matrix $M(t_2, t_1)$ as

$$y(t_2) = M(t_2, t_1)y(t_1).$$

The matrix $M(t_2, t_1)$ is referred to as the error matrix (Lorenz 1965).

b. Singular value analysis and Lorenz index

The amplification rate of the perturbation for a time interval from $t_1$ to $t_2$ is defined as

$$\gamma(t_2, t_1, y(t_1)) = \frac{\|M(t_2, t_1)y(t_1)\|}{\|y(t_1)\|},$$

(4)

where $\| \|$ represents a norm. Here we use the norm defined by the Euclidean inner product, that is, $\langle a, b \rangle = a^\top b$ and $\|a\| = \sqrt{\langle a, a \rangle}$. The amplification rate $\gamma$ is rewritten as

$$\gamma = \sqrt{\frac{\langle My(t_1), My(t_1) \rangle}{\langle y(t_1), y(t_1) \rangle}} = \sqrt{\frac{\langle y(t_1), M^\top My(t_1) \rangle}{\langle y(t_1), y(t_1) \rangle}}.$$  \hspace{1cm} (5)

The square roots of $n$ eigenvalues of the semipositive definite symmetric matrix $M^\top M$ are referred to as singular values, and the corresponding eigenvectors of $M^\top M$ are referred to as forward singular vectors (Legras and Vautard 1996). In general, the singular values are arranged in decreasing order. Equation (5) describes that the amplification rate of the perturbation that is parallel to a forward singular vector at $t_1$ is given by the corresponding singular value. The first forward singular vector has a maximum amplification rate for the time interval, which is equal to the first singular value.

The root-mean-square amplification rate of the perturbations distributed equally in $\mathbb{R}^n$ at $t = t_1$ is given by the root-mean-square of singular values (Lorenz 1965). This quantity is referred to as Lorenz index and has been used as a standard measure of the perturbation growth in the analysis of simple atmospheric models (Mukougawa et al. 1991; Yoden and Nomura 1993; Yamane and Yoden 1997). Since the sum of eigenvalues of a matrix is equal to the trace of the matrix, the Lorenz index $\alpha(t_2, t_1)$ is given by

$$\alpha(t_2, t_1) = \frac{\text{tr}(M^\top M)}{n}.$$  \hspace{1cm} (6)

c. Lyapunov exponents and Lyapunov vectors

If the time interval from $t_1$ to $t_2$ is infinitely long, the evolution of the perturbations is characterized by Lyapunov exponents and Lyapunov vectors. The Lyapunov exponents are the exponential growth rate of singular values and the backward Lyapunov vectors are the evolved singular vectors, which are given by operating $M(t_2, t_1)$ on forward singular vectors, in the limit of $(t_2 - t_1) \to \infty$ (Legras and Vautard 1996).

The Lyapunov exponents and backward Lyapunov vectors can be estimated numerically by using the following properties of small perturbations (Shimada and Nagashima 1979). Let $\lambda_i$, $i = 1, 2, \ldots, n$, be the Lyapunov exponents arranged in decreasing order and let

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln |M(t, 0)|.$$
\(\mathbf{g}_i(t), i = 1, 2, \ldots, n\) be the corresponding normalized backward Lyapunov vectors. When the time interval \(t_2 - t_1\) is sufficiently large, almost all perturbations at \(t_1\) grow with an exponent \(\lambda_i\) and turn toward the direction of the first backward Lyapunov vector \(\mathbf{g}_i(t_2)\) at \(t = t_1\); almost all surface elements spanned by two perturbations at \(t_1\) grow with an exponent \(\lambda_1 + \lambda_2\) and turn toward the surface spanned by the first two backward Lyapunov vectors at \(t_2\); generally, almost all \(k\)-dimensional volume elements spanned by \(k\) perturbations at \(t_1\) grow with an exponent \(\lambda_1 + \lambda_2 + \cdots + \lambda_k\) and turn toward the \(k\)-dimensional subspace spanned by the first \(k\) backward Lyapunov vectors at \(t_2\). Once we get the \(k\)-dimensional subspace spanned by the first \(k\) backward Lyapunov vectors at \(t_3 \geq t_2\) by operating the matrix \(\mathbf{M}(t_3, t_2)\mathbf{S}(t_2)\) where \(\mathbf{S}(t_2)\) represents the \(k\)-dimensional subspace spanned by the first \(k\) backward Lyapunov vectors at \(t_2\). Note that as for each backward Lyapunov vector, \(\mathbf{g}_i(t_2)\) is not necessarily parallel to \(\mathbf{M}(t_3, t_2)\mathbf{g}_i(t_2)\), otherwise \(i = 1\).

d. Kaplan–Yorke (Lyapunov) dimension

The Lyapunov exponents are the most important quantity to characterize the global property of attractors in nonlinear dynamical systems. We can classify attractors by the spectrum of the Lyapunov exponents into limit cycle, torus, chaos, and so on (see, e.g., Goldhirsch et al. 1987). Kaplan and Yorke (1979) defined a dimension \(\text{D}_{\text{KY}}\), which is called Kaplan–Yorke dimension or Lyapunov dimension, with the Lyapunov exponents \(\lambda_i\)

\[
\text{D}_{\text{KY}} = j + \frac{1}{|\lambda_{j+1}|} \sum_{i=1}^{j} \lambda_i,
\]

where \(j\) is the integer that satisfies \(\sum_{i=1}^{j} \lambda_i \geq 0\) and \(\sum_{i=j+1}^{j} \lambda_i < 0\). Since the sum of the first \(k\) Lyapunov exponents \(\sum_{i=1}^{k} \lambda_i\) is the exponential growth rate of \(k\)-dimensional volume element following the solution \(\mathbf{x}(t)\), \(\text{D}_{\text{KY}}\) represents the dimension of the volume with zero-growth rate. The Lyapunov dimension \(\text{D}_{\text{KY}}\) is considered to correspond to the attractor dimension.

e. Singular value analysis and Lorenz index on a \(k\)-dimensional subspace \(S_k(t)\)

A perturbation contained in the subspace \(S_k(t)\) at \(t = t_1\) is written as

\[
\mathbf{y}(t_1) = \sum_{i=1}^{k} \mathbf{p}_i \mathbf{g}_i(t_1) = \mathbf{G}(t_1) \mathbf{p},
\]

where \(\mathbf{G}(t)\) is an \(n \times k\) matrix, \(\mathbf{G}(t) = [\mathbf{g}_1(t), \ldots, \mathbf{g}_k(t)]\) and \(\mathbf{p}\) is a \(k\)-dimensional vector, \(\mathbf{p} = (p_1, \ldots, p_k)^T\). The amplification rate of this perturbation is given by

\[
\gamma(t_2, t_1, \mathbf{y}(t_1)) = \sqrt{\langle \mathbf{p}^\top \mathbf{G}^\top \mathbf{M}^\top \mathbf{M} \mathbf{G} \mathbf{p} \rangle / \langle \mathbf{p} \cdot \mathbf{p} \rangle}.
\]

The analysis of the singular values and singular vectors of the \(n \times k\) matrix \(\mathbf{G}\) gives information on the evolution of perturbations in \(S_k(t)\) in the same manner as in \(\mathbb{R}^n\). The first forward singular vector of the matrix \(\mathbf{G}\) has a maximum amplification rate for the time interval on \(S_k(t)\), which is equal to the first singular value of \(\mathbf{G}\).

The root-mean-square amplification rate of the perturbations distributed equally in \(S_k(t)\) at \(t = t_1\) is also given by the root-mean-square of \(k\) singular values of the matrix \(\mathbf{G}\). We call this quantity subspace Lorenz index. The subspace Lorenz index \(\alpha_k(t_2, t_1)\) is given by

\[
\alpha_k(t_2, t_1) = \sqrt{\text{tr}(\mathbf{G}^\top \mathbf{M}^\top \mathbf{M} \mathbf{G}) / k}.
\]

If \(k = n\), then \(\alpha_k(t_2, t_1) = \alpha(t_2, t_1)\).

3. Results

We examine the properties of the quantities described in the previous section using a simple atmospheric model. We compare the subspace Lorenz index \(\alpha_k(t_2, t_1)\) with the Lorenz index \(\alpha(t_2, t_1)\), and then compare the first forward singular vector in the subspace with the first forward singular vector in the whole phase space and the first backward Lyapunov vector.

a. Model and a chaotic solution

We use the same atmospheric circulation model that was used in Yamane and Yoden (1998), which is a modification of the Legras and Ghil (1985) model. This model is governed by the barotropic vorticity equation on a rotating sphere with forcing, dissipation and surface-topography terms. Its nondimensional form is as follows:

\[
\frac{\partial}{\partial t} \Delta \psi + J[\psi, \rho \Delta \psi + \mu (1 + h)] = \kappa \Delta (\psi^* - \psi) - \nu \Delta^3 \psi,
\]

where \(\psi(\lambda, \mu, t)\) is the streamfunction, \(\lambda\) the longitude, \(\mu\) the sine of the latitude, \(t\) the time, \(\psi^*(\mu)\) the streamfunction for a zonally symmetric forcing, \(h(\lambda, \mu)\) the topographic height, \(\Delta\) the spherical Laplacian operator, \(J\) the spherical Jacobian operator, \(\kappa\) the relaxation time, \(\nu\) the artificial hyperdiffusion coefficient added for smooth numerical behavior, and \(\rho\) the nondimensional number that corresponds to the Rossby number. The potential vorticity is defined as \(\rho \Delta \psi + \mu (1 + h)\) in this model. The parameter is fixed as follows: \(\psi^*(\mu) = -75 \sqrt{3} (a \Omega)^{-1} \mu^3\) where \(a\) is the radius of the earth (=6.37 \times 10^6 m) and \(\Omega\) is the angular speed of rotation.
A time-dependent solution of the model, \( x_t \), is obtained by numerical time integrations with the fourth-order Runge–Kutta method, starting from a nearly rest initial condition (\( \psi = 0 \)). The first 1000-day integration is regarded as the period of initial transient behavior, and the subsequent 15 000-day integration is regarded as the period when the solution is on the attractor of the system and this period is labeled “day 1–15 000.” The solution is irregular and nonperiodic as described by Yamane and Yoden (1998) in detail.

We applied the calculation method of Lyapunov exponents developed by Shimada and Nagashima (1979) to the solution \( x_t \) for the period of day 1–15 000 and obtained the first 10 Lyapunov exponents \( \lambda_1, \ldots, \lambda_{10} \) and the corresponding first 10 backward Lyapunov vectors \( g_1(t), \ldots, g_{10}(t) \) for the period of day 5001–15 000. Here we considered that a set of 10 orthonormal vectors chosen at random initially converged to the set of the first 10 backward Lyapunov vectors after the first 5000-day calculation. In this study, we used the enstrophy norm to measure the magnitude of perturbation, that is, the inner product is defined as

\[
\langle a, b \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \Delta \psi_a(\lambda, \mu) \Delta \psi_b(\lambda, \mu) \, d\mu \, d\lambda,
\]

where \( \psi_a(\lambda, \mu) \) and \( \psi_b(\lambda, \mu) \) are the streamfunctions corresponding to the \( n \)-dimensional vectors \( a \) and \( b \), respectively.

Figure 1 shows the values of the obtained first 10 Lyapunov exponents (crosses). As the first Lyapunov exponent \( \lambda_1 \) is positive, the solution \( x_t \) is found to be unstable and chaotic. The reciprocal of \( \lambda_1 \) is about 7.8, which is given by 

\[
h(\lambda_1) = 0.4\mu^2(1 - \mu^2) \cos 2\lambda_1 \text{ for } \mu = 0 \text{ and } h(\lambda, \mu) = 0 \text{ for } \mu < 0, \quad \kappa^{-1} = 10 \text{ days in dimension}, 
\]

\( \nu \) is set to give a damping time of 0.5 day at the largest total wavenumber, and \( \rho = 0.18 \). Equation (11) is discretized through an expansion in spherical harmonics with a triangular truncation of total wave-number 42 (T42). Then the model becomes a forced-dissipative dynamical system of 1848 real variables (\( n = 1848 \)).

\[ (= 7.292 \times 10^{-5} \text{ s}^{-1}) \]

The attractor dimension is also estimated by the power of the correlation integral \( C(r) \), of which computational method is introduced by Grassberger and Procaccia (1983). This method has been applied to various meteorological phenomena (e.g., Sharifi et al. 1990; Fraedrich et al. 1990; Zeng and Pielke 1993). The correlation integral \( C(r) \) is the probability that the distance between two points on the attractor is less than \( r \), and is computed by counting the number of pairs with the distance less than \( r \) in all pairs of points for a long-time series on the attractor. The correlation dimension is the slope of the linear region in a plot of \( \ln C(r) \) versus \( \ln r \). We also computed the correlation integral \( C(r) \) for the solution \( x_t \) of the present model, using one hundred thousand points in the 1848-dimensional phase space for the period of day 5001–105 000. Figure 2 shows the plot of \( \ln C(r) \) versus \( \ln r \). The correlation dimension of the solution \( x_t \) can be estimated to be 6.9 from the slope of the region \(-4 < \ln r < -3\), which is the linear region in Fig. 2. The correlation dimension 6.9 is roughly comparable to the Lyapunov dimension \( D_{KY} \approx 7.8 \), of which the slope is shown by a dashed line in Fig. 2. The difference in value between the correlation dimension and the Lyapunov dimension is not surprising, and either one is a reasonable candidate for the dimension of the attractor.

**b. Subspace Lorenz index versus Lorenz index**

We calculated the Lorenz index \( \alpha(t_i + \tau, t_i) \) for the time interval \( 1 \leq \tau \leq 20 \) days with the initial time \( t_i \) changed every 1 day for the period of day 8001–8100, and the subspace Lorenz index \( \alpha_s(t_i + \tau, t_i) \) for the time

\[ (\text{day}^{-1}) \]

\[ 0.1 \]

\[ -0.1 \]

\[ \lambda_1 \]

\[ 0 \]

\[ 5 \]

\[ 10 \]

\[ \text{Fig. 1. Lyapunov exponents } \lambda_i (\times) \text{ and accumulated Lyapunov exponents } \sum_{i=1}^{10} \lambda_i (\cdot) \text{ for } 1 \leq i \leq 10. \text{ The Lyapunov dimension } D_{KY} = 7.8. \text{ The unit of the ordinate is day}^{-1}. \]
Fig. 2. Correlation integral \( C(r) \). A solid line indicates the slope of the correlation dimension 6.9, estimated from \( C(r) \) for \(-4 \leq \ln r \leq -3 \). A dashed line indicates the slope corresponding to the Lyapunov dimension \( D_L = 7.8 \).

interval \( 1 \leq \tau \leq 20 \) days with the initial time \( t_i \) changed every 1 day for the period of day 5001–15 000. The four-dimensional subspace \( S_4(t) \) for \( \alpha_4(t_i + \tau, t_i) \) is spanned by the backward Lyapunov vectors associated with nonnegative Lyapunov exponents (Fig. 1). The subspace \( S_4(t) \) is considered to be a tangent space of the attractor of the solution \( x(t) \). Note that the analysis period for the Lorenz index \( \alpha \) is much shorter than that for the subspace Lorenz index \( \alpha_4 \). The reason is that the cost of calculation for \( \alpha_4 \) is lower than one percent of that for \( \alpha \) once we get the subspace \( S_4(t) \), since \( \alpha_4 \) is the mean amplification rate of the perturbations in the four-dimensional subspace while \( \alpha \) is that in the full 1848-dimensional phase space.

Figures 3a and 3b show the dependence of the Lorenz index \( \alpha_4(t_i + \tau, t_i) \) and the subspace Lorenz index \( \alpha_4(t_i + \tau, t_i) \) on the time interval \( \tau \), respectively, for the same period of day 8001 ≤ \( t_i \) ≤ 8100. The geometric average of \( \alpha \) denoted by a solid line in (a) shows that the averaged Lorenz index decreases with \( \tau \) initially for \( \tau < 4 \) days and increases with \( \tau \) afterward. The increase of \( \alpha \) with \( \tau \) is due to the nature of chaos of the solution \( x(t) \); \( \alpha \) grows with the first Lyapunov exponent \( \lambda_1 \) in the limit of \( \tau \to \infty \). The slope of the solid line in (a) is closer to the slope of a dot-dashed line, which is \( \lambda_1 \), as \( \tau \) becomes larger. The initial decrease of \( \alpha \), on the other hand, is due to the dissipative nature of the system; the Lorenz index \( \alpha \) grows with an exponent \( \text{tr}(J[x(t)])/n \) at \( \tau = 0 \), since the Lorenz index is approximated by

\[
\alpha(t_i + \tau, t_i) = \sqrt{1 + \frac{2\tau}{n} \times \text{tr}(J[x(t)])}
\]

in the limit of \( \tau \to 0 \) (Mukougawa et al. 1991). In the present model, the trace of the Jacobian matrix is independent of \( x(t) \), and \( \text{tr}(J)/n = (-2.73 \text{ days})^{-1} \), which is determined by the coefficients, \( \kappa \) and \( \nu \), of the two dissipative terms, the relaxation term and the numerical diffusion term, respectively. [If the numerical diffusion term does not exist in the model, \( \text{tr}(J)/n = -\kappa = (-10 \text{ days})^{-1} \).] The initial decrease of \( \alpha \) can be explained well by the slope of a dot-dot-dashed line in (a), which is
growth rate of an $n$-dimensional volume element, which is represented by three dot-dot-dashed lines for the three dissipation rates.

The geometric average of the subspace Lorenz index $\alpha_i$, denoted by a solid line in Fig. 3b shows a monotonous increase with $\tau$ from the beginning. This is because the average of the exponential growth rate of a four-dimensional volume element in $S_i(t)$, which is $\Sigma_{i=1}^{\infty} \lambda_i$, is positive. In fact, the average of $\alpha_i$ grows with an exponent $\Sigma_{i=1}^{\infty} \lambda_i = (54.1 \text{~day}^{-1})$ at $\tau = 0$, which is denoted by a dot-dot-dashed line in Fig. 3b. The solid line in Fig. 3b for $\tau > 10$ days is almost parallel to the dot-dashed line, which indicates the growth with the largest Lyapunov exponent $\lambda_i$.

The variability of $\alpha$, which is represented by two dashed lines of $\pm 1$ standard deviation in Fig. 3a is smaller than that of $\alpha_i$ in Fig. 3b, especially for $\tau < 4$ days. The reason for small variability of $\alpha$ for $\tau < 4$ days is that the dissipative structure of the present system is independent of the position $x$ in $n$-dimensional phase space, that is, the instantaneous exponential growth rate of an $n$-dimensional volume element is independent of the position $x$. On the other hand, the instantaneous exponential growth rate of a four-dimensional volume element in $S_i(t)$ is dependent on the position $x$, so the variability of $\alpha_i$ for $\tau < 4$ days is not so small.

Variations of $\alpha(t_1 + \tau, t_1)$ and $\alpha_i(t_1 + \tau, t_1)$ for the period of day $8001 \leq t_1 \leq 8100$ at $\tau = 3, 7, 15$ days are shown in Figs. 5a and 5b, respectively. Dominant timescales in the variations of $\alpha$ and $\alpha_i$ are about 10 days or longer. The variations of $\alpha$ and $\alpha_i$ for the same time interval are correlative. However, there exist more correlative pairs of $\alpha$ and $\alpha_i$, if we permit time lag. A dashed line in the bottom panel of (a) shows the variation of the subspace Lorenz index $\alpha_i(t_1 + 15 \text{~days}, t_1 + 8 \text{~days})$, which is identical to the solid line in the middle panel of (b) shifted backward for 8 days. The dashed line and the solid line in the bottom panel of (a) are very correlative; the coefficient of correlation between these two lines is 0.92. The relation that the time interval for $\alpha_i$, 7 days, plus the shifted period, 8 days, is the time interval for $\alpha_i$, 15 days, holds in this case. Similar relations are found for other pairs with a high correlation. We calculated the coefficient of correlation between $\alpha(t_1 + \tau, t_1)$ and $\alpha_i(t_1 + \tau^\prime, t_1)$ with different time interval under the condition that $\alpha_i(t_1 + \tau^\prime, t_1)$ is shifted backward for $\tau^\prime$, that is, the coefficient of correlation between $\alpha(t_1 + \tau, t_1)$ and $\alpha_i(t_1 + \tau, t_1 + \tau^\prime - \tau^\prime)$. Figure 6 shows the results for $1 \leq \tau \leq 20$ days and $1 \leq \tau^\prime \leq 20$ days. The coefficient of correlation in the case stated above is indicated at $\tau = 15$ days and $\tau^\prime = 7$ days in Fig. 6. Highest correlation between $\alpha$ and $\alpha_i$, of which the coefficient is more than 0.9 (dark shading), is seen in the region for $\tau - \tau^\prime = 6-10$ days and $\tau \geq 12$ days, not in the region for $\tau = \tau^\prime$. This means that the variations of the Lorenz index for $\tau \geq 12$ days can be well estimated by the variations

![Fig. 4. Geometric average of Lorenz index $\alpha(t_1 + \tau, t_1)$ for the control experiment (solid line, which is the same as the solid line in Fig. 3a), a half-dissipation experiment (long-dashed line), and a double-dissipation experiment (short-dashed line). Three dot-dot-dashed lines indicate the slope of $\text{tr}(J)/n$ for the three experiments.](image)
of the subspace Lorenz index for 6−10 days shorter time interval.

Figures 7a and 7b show the evolution of enstrophy spectrum of the Lorenz index \( \alpha(t_1 + \tau, t_1) \) and the subspace Lorenz index \( \alpha_4(t_1 + \tau, t_1) \), respectively, averaged over the same one hundred cases as those for Figs. 3 and 5. We use the term “enstrophy spectrum of \( \alpha(t_1 + \tau, t_1) \)” to refer to the average enstrophy spectrum at \( t = t_1 + \tau \) of the perturbations distributed equally on \( \mathbb{R}^n \) at \( t = t_1 \), and the term “enstrophy spectrum of \( \alpha_4(t_1 + \tau, t_1) \)” to refer to the average enstrophy spectrum at \( t = t_1 + \tau \) of the perturbations distributed equally on \( S_4(t_1) \) at \( t = t_1 \). The enstrophy spectrum of \( \alpha \) for \( \tau = 0 \), which is the average enstrophy spectrum of the perturbations distributed equally on \( \mathbb{R}^n \), shows a gradual increase with wavenumber (solid line in Fig. 7a), since the number of spherical harmonics with which the governing Eq. (11) is expanded increases with wavenumber. On the other hand, the enstrophy spectrum of \( \alpha_4 \) for \( \tau = 0 \) denoted by a solid line in Fig. 7b has a peak at wavenumber 7, which is a characteristic of the subspace \( S_4(t_1) \). As the time interval \( \tau \) increases, the enstrophy spectrum of \( \alpha \) becomes similar in the shape of distribution to the enstrophy spectrum of \( \alpha_4 \) by losing the enstrophy at highest wavenumbers and at lowest wavenumbers (\( \tau \leq 6 \) days). For \( \tau > 6 \) days the enstrophy spectrum of \( \alpha \) increases its amplitude with little change in its shape, while that of \( \alpha_4 \) increases its amplitude from the beginning without the change in its shape.

c. Forward singular vector in subspace

We calculated the first forward singular vector (hereafter referred to as the first FSV) for 3-day time interval in the whole phase space \( \mathbb{R}^n \) and the first FSV for 3-day time interval in the subspace \( S_4(t_1) \) with the initial time \( t_1 \) changed every 1 day for the period of day 8001−8100. The first FSV in \( \mathbb{R}^n \) represents the perturbation with the largest amplification rate for the time interval in \( \mathbb{R}^n \), and the first FSV in \( S_4(t_1) \) represents that in \( S_4(t_1) \). The properties of the first FSV in \( S_4(t_1) \) are examined by comparing them with those of the first FSV in \( \mathbb{R}^n \) and
FIG. 7. Evolution of enstrophy spectrum of (a) Lorenz index \( a(t_1 + \tau, t_1) \) and (b) subspace Lorenz index \( a_4(t_1 + \tau, t_1) \), averaged over 100 cases with different initial time \( t_1 \) for the period of day 8001±8100.

Figure 8 shows the variations of the amplification rate \( \gamma \) for 3 days of (a) the first forward singular vector for 3-day time interval in the whole phase space \( \mathbb{R}^n \), (b) the first forward singular vector for 3-day time interval in the subspace \( S_4(t) \), and (c) the first backward Lyapunov vector. Dashed lines show the amplification rates based on the energy norm for the same vectors.

The time variations of the amplification rate of the first FSV in \( \mathbb{R}^n \) are not so correlative with those of the Lorenz index \( a \) for 3 days in the top panel of Fig. 5a (the coefficient of correlation between the two is 0.62), while the variations of the amplification rate of the first FSV in \( S_4(t) \) are highly correlative with those of the subspace Lorenz index \( a_4 \) for 3 days in the top panel of Fig. 5b (the coefficient of correlation is 0.96). The first FSV in \( S_4(t) \) is representative of perturbations in \( S_4(t) \) at the amplification rate in the present model.

Dashed lines in Fig. 8 show the amplification rates based on the energy norm for the same vectors. The amplification rate based on the energy norm is much smaller than that based on the enstrophy norm for the first FSV in \( \mathbb{R}^n \) (Fig. 8a). This means that the first FSV in \( \mathbb{R}^n \), which is obtained based on the enstrophy norm, does not increase its energy as much as its enstrophy. The geometric average of the dashed line for the first FSV in \( \mathbb{R}^n \) is 1.72, which is a little larger than that for the first FSV in \( S_4(t) \), 1.54, and that for the first BLV, 1.17. The amplification rate of the first FSV in \( \mathbb{R}^n \) is very sensitive to the norm measuring its magnitude. The other two amplification rates for the first FSV in \( S_4(t) \) and the first BLV, on the other hand, show similar variations and those are not so sensitive to the choice of norm.

The investigation of the evolution of enstrophy spectrum helps us to understand the dependence of the amplification rate on the norm more clearly. Figure 9 shows...
The evolution of enstrophy spectrum of the first FSV in $\mathbb{R}^n$ (a), the first FSV in $S_\gamma(t)$ (b), and the first BLV (c), averaged over one hundred cases with different initial time $t_i$ for the period of day 8001–8100. The enstrophy spectrum of the first FSV in $\mathbb{R}^n$ for $\tau = 0$ has a maximum at the lowest wavenumber 1 (Fig. 9a). As the time interval $\tau$ increases, perturbation enstrophy at wavenumbers 1 and 2 decreases, while that increases at higher wavenumbers to have similar spectrum in shape to the first BLV (Fig. 9c). There exists a transfer of perturbation energy from low wavenumbers to high wavenumbers. This downward transfer is more favorable for the increase of perturbation enstrophy because the contribution of the components with high wavenumbers to the enstrophy is relatively large compared to that to the energy. It turns out that the large difference between the two lines in Fig. 8a is caused by the downward transfer of perturbation energy, which is a property of the first FSV based on the enstrophy norm. On the other hand, the enstrophy spectrum of the first FSV in $S_\gamma(t)$ for $\tau = 0$ has a peak at wavenumber 7 and does not change in shape very much as $\tau$ increases (Fig. 9b). There is little downward transfer of perturbation energy, and thus the two lines in Fig. 8b show similar variations. This property of the first FSV in $S_\gamma(t)$ is similar to that of the first BLV. The first BLV does not change the shape or transfer the energy because of its nature (Fig. 9c).

Figure 10 shows an example of the evolution of the perturbation vorticity field initialized with the first FSV in $\mathbb{R}^n$ (a), the first FSV in $S_\gamma(t)$ (b), and the first BLV (c) together with the potential vorticity field. The initial time $t_1$ is set to day 8041, when the amplification rate of the first FSV in $S_\gamma(t)$ has a maximum (see Fig. 8b). The potential vorticity field is characterized by two belts of large gradient in midlatitudes corresponding to strong jet stream. The first FSV in $\mathbb{R}^n$ has large amplitude on the equator side of the strong jet [left panel of (a)]. The vorticity perturbation propagates downstream along the belt of large gradient of potential vorticity and amplifies rapidly. The first FSV in $S_\gamma(t)$, shown in the left panel of (b), does not have very similar spatial pattern as the first FSV in $\mathbb{R}^n$, although it has large amplitude around the area where the first FSV in $\mathbb{R}^n$ does have. The first FSV in $S_\gamma(t)$ evolves in the same way as the first FSV in $\mathbb{R}^n$ and becomes very similar in spatial distribution at $\tau = 3$ days. The amplification rate of the first FSV in $S_\gamma(t)$ is less than one-third of that of the first FSV in $\mathbb{R}^n$. Such a close resemblance in the evolution between the first FSV in $S_\gamma(t)$ and the first FSV in $\mathbb{R}^n$ as in Figs. 10a and 10b is often seen for the other initial time $t_i$. The spatial distribution of the first BLV [left panel of (c)], on the other hand, is quite different from those of

![Fig. 9. Evolution of enstrophy spectrum of (a) the first forward singular vector for 3-day time interval in the whole phase space $\mathbb{R}^n$, (b) the first forward singular vector for 3-day time interval in the subspace $S_\gamma(t)$, and (c) the first backward Lyapunov vector, averaged over one hundred cases with different initial time $t_i$ for the period of day 8001–8100.](http://journals.ametsoc.org/jas/article-pdf/58/9/1074-1520-0469(2001)058_1074_fteosp_2_0_co_2.pdf)
the first FSV in $\mathbb{R}^n$ and in $S_4(t)$, because the first BLV is the perturbation that has been evolved since time immemorial and has no specific meaning for the evolution in future. Thus the evolution of the first BLV is also essentially different from those of the first FSVs.

4. Discussion

The reason for the high correlation between the Lorenz index $\alpha$ and the subspace Lorenz index $\alpha_4$, as is seen in the bottom panel of Fig. 5a and Fig. 6, can be considered as follows. Most of the perturbations distributed equally on $\mathbb{R}^n$ at $t = t_1$, of which the root-mean-square amplification rate is the Lorenz index $\alpha(t_1 + \tau, t_1)$, decrease their magnitude with time initially due to the dissipation of the system, as shown for $\tau < 4$ days in Fig. 3a. During this period, the perturbations quickly lose the components in the directions associated with negative Lyapunov exponents and only the components in the subspace $S_4(t)$ are left, since most of the negative Lyapunov exponents, of which the number is much larger than that of nonnegative Lyapunov exponents, have larger absolute values than the largest Lyapunov exponent $\lambda_1$ (Fig. 1). After this period, the distribution of the perturbations is quite restricted on the subspace $S_4(t)$. This is consistent with the fact that the enstrophy spectrum of $\alpha$ at $\tau = 6$ or 8 days is similar to the enstrophy spectrum of $\alpha_4$ (Fig. 7). As the variance of $\alpha$ is not so large at $\tau = 6$ or 8 days (Fig. 3a), the manner of the deformation of the perturbations for 6 or 8 days does not change very much with the initial position $x(t_1)$. Therefore, the variations of the Lorenz index $\alpha$ for $\tau > 8$ days can be characterized by the variations of the subspace Lorenz index $\alpha_4$ without the period for
the initial deformation, since the subsequent evolutions of the perturbations for \( \tau > 8 \) days can be characterized by the evolutions of the perturbations only on \( S_\alpha(t) \). Thus a high correlation between \( \alpha \) and \( \alpha \) can be observed for the shift of about 8 days. This interpretation is equivalent to saying that most of the solutions with errors around the true solution \( x(t) \) are rapidly absorbed onto the attractor for initial 6 or 8 days and depart from \( x(t) \) on the attractor afterward.

As for the results on \( k \)-dimensional subspace, only those for four-dimensional subspace were described in this paper, but we also calculated the Lorenz index in the subspace with different dimension and examined the relation between the Lorenz index \( \alpha(t_1 + \tau, t_1) \) and the subspace Lorenz index \( \alpha (t_1 + \tau, t_1) \) for \( k = 1 \sim 10 \) in the same way as Fig. 6. It is found that highest correlations between \( \alpha \) and \( \alpha \) for \( k = 1 \sim 10 \) exist for \( \tau - \tau' = 3 \sim 10 \) days and \( \tau \geq 10 \) days, not for \( \tau = \tau' \), as Fig. 6 shows. Figure 11 shows the maximum values of the coefficient of correlation between \( \alpha(t_1 + \tau, t_1) \) and \( \alpha (t_1 + \tau, t_1) \) for \( k = 1 \sim 10 \). For example, the dotted line labeled “4” shows the maximum value for each \( \tau \) in \( 1 \leq \tau' \leq 20 \) days in Fig. 6. The maximum coefficient of correlation has a tendency to increase with the dimension of the subspace \( S_\alpha \). It is not surprising because the maximum coefficient of correlation becomes 1 at \( \tau = \tau' \) as \( k \to \infty \). It appears in Fig. 11 that the differences between the lines for \( k = 1 \sim 4 \) are larger than the differences between the lines for \( k = 4 \sim 10 \). There seems to be a certain kind of convergence at \( k = 4 \). This may support the significance of the subspace related to the nonnegative Lyapunov exponents in the analysis of the dynamics of perturbations.

It is well known that the leading FSVs in \( \mathbb{R}^n \) are dependent on the norm that measures the perturbation. Palmer et al. (1998) has investigated the dependence using the ECMWF T42L19 tangent model and pointed out that the leading FSVs based on the enstrophy norm have a tendency to transfer their perturbation energy downward while those based on the energy norm have a tendency to transfer upward. In this study, we examined the first FSV based on the enstrophy norm using the idealized barotropic model and confirmed the existence of the downward transfer of perturbation energy (Fig. 9a). The downward transfer of perturbation energy plays an important role in the rapid growth of the first FSV based on the enstrophy norm, because the perturbation can increase the total perturbation energy by transferring the perturbation energy downward while conserving the total perturbation energy. On the other hand, it is considered that the upward transfer of perturbation energy by transferring the perturbation enstrophy upward while conserving the total perturbation enstrophy. Anyhow, the first FSV in \( \mathbb{R}^n \) is characterized by the transfer of perturbation energy or enstrophy, and then the amplification rate of it is sensitive to the measuring norm as is seen in Fig. 8a. The first FSV in \( S_\alpha(t) \), however, transfers little perturbation energy or enstrophy (Fig. 9b) and the amplification rate is not so sensitive to the norm (Fig. 8b).

We examined the fundamental properties of the evolution of perturbations using an idealized barotropic model. Since the model is very simple and has only barotropic instability, the solution of the model is not as unstable as the real atmosphere. Zeng and Pielke (1992) analyzed the time series of daily surface temperature and pressure and pointed out that the error e-folding time is several days and the attractor dimension is greater than 8 in the atmosphere, while, in this model, the error e-folding time is more than 20 days and the attractor dimension is only 7.8. The aim of this paper is, however, to clarify the validity and usefulness of analyzing the perturbations in the subspace spanned by the leading BLVs. The properties of the perturbations in the subspace were examined in detail by comparing them with those in the whole phase space, and some affirmative results were obtained by using this simple model; for example, a high correlation exists between the Lorenz index and the subspace Lorenz index, and the first FSV in the subspace does not depend on the norm very much. It is a further study to investigate whether the present analysis in the subspace is useful and efficient in more complicated atmospheric models including baroclinic instability and convective instability.

There are two methods of the ensemble forecasts that
are in operation at the centers for medium-range weather forecasts in the world. One is that the initial ensemble members are produced by adding the breeding vectors to the initial value obtained in the regular analysis, which method has been developed at NCEP (Toth and Kalnay 1993, 1997). The breeding vectors are the vectors that have been bred for a long time in the analysis cycle, and correspond to the leading BLVs in the linear limit. The other is that they are produced by adding the leading FSVs to the initial value, which method has been developed at ECMWF (Mureau et al. 1993; Molteni et al. 1996). The leading FSVs in operational forecast models are computed by an iterative Lanczos procedure using the adjoint model of a simplified version of the forecast model, as the dimension of the models is too large to deal with the error matrix practically. The advantage of the breeding method is that the initial ensemble members tend to be restricted on the attractor of the real atmosphere, while the advantage of the FSV method is that the ensemble members diverge rapidly with time, which seems to be important when the number of ensemble members is limited. A detailed comparison of both methods has been made with a low-resolution GCM by Szunyogh et al. (1997).

As another method to produce initial ensemble members, one can propose a mixture of both of the methods that we introduced in this paper; initial ensemble members are produced by adding the leading FSVs in the subspace spanned by the leading BLVs to the initial value. Toth et al. (1996) have speculated about the possibility of this method. This method has the both advantages described above: the initial ensemble members are restricted on the attractor and diverge rapidly with time. The computational cost for this method is almost equal to that for the breeding method, unless the dimension of the subspace is extremely large. The adjoint model is not necessary for the calculation since the error matrix in the subspace is directly given by the breeding vectors. We think that the application of this method to operational ensemble forecasts is worth investigating.

5. Conclusions

We examined the fundamental properties of the finite-time evolution of perturbations using an idealized barotropic model on a rotating sphere, which is a forced-dissipative system of \( n = 1848 \) real variables. A time-dependent solution of the model is a chaotic solution, of which the first Lyapunov exponent is \((20.8 \text{ day})^{-1}\). The number of nonnegative Lyapunov exponents is 4 and the Lyapunov dimension of the solution is 7.8, which are much smaller than the total dimension of the system (Fig. 1). We focused our attention on the subspace spanned by the first four backward Lyapunov vectors (BLVs), \( S_4(t) \), and defined the subspace Lorenz index and the forward singular vectors (FSVs) in the subspace \( S_4(t) \). The subspace Lorenz index \( \alpha_4 \) is the root-mean-square amplification rate of the perturbations distributed equally in the subspace \( S_4(t) \), which is an extension of the Lorenz index \( \alpha \) defined by Lorenz (1965). The four-dimensional subspace \( S_4(t) \) is considered to be a tangent space of the attractor of the chaotic solution.

The subspace Lorenz index \( \alpha_4 \) has a tendency to increase with time from the beginning while the Lorenz index \( \alpha \) decreases with time initially due to the dissipation of the system before increasing due to the nature of chaos (Fig. 3). The variations of the subspace Lorenz index \( \alpha_4 \) are highly correlated with those of the Lorenz index \( \alpha \) when the time interval of the Lorenz index is several days longer than that of the subspace Lorenz index in this model (Figs. 5 and 6). The reason for the high correlations is considered to be that most of the initial perturbations in the whole phase space rapidly decrease their magnitude initially with only the components in the subspace \( S_4 \) left, and they evolve on the subspace \( S_4 \) afterward.

The first FSV in \( S_4(t) \) has a property of transferring little perturbation energy during the period of evolution, and its amplification rate does not depend on the norm very much, which is similar to the property of the first BLV (Figs. 8 and 9). The first FSV in \( S_4(t) \) does not have very similar pattern to the first FSV in \( \mathbb{R}^4 \). However, the evolved pattern of the perturbation vorticity from the first FSV tends to become similar between the cases in \( \mathbb{R}^4 \) and \( S_4(t) \) (Fig. 10). Thus, the subspace related to the nonnegative Lyapunov exponents is very useful in the analysis of the evolution of perturbations.

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