Gravity Wave Breaking in Two-Layer Hydrostatic Flow

QINGFANG JIANG* AND RONALD B. SMITH

Department of Geology and Geophysics, Yale University, New Haven, Connecticut

(Manuscript received 23 April 2002, in final form 19 November 2002)

ABSTRACT

To better understand mountain-induced gravity wave breaking and potential vorticity generation in the troposphere, a two-layer hydrostatic flow over a three-dimensional Witch-of-Agnesi type of mountain is investigated. It is suggested that a two-layer model is the simplest model in which the partitioning of upper- and lower-level wave breaking and dissipation can be studied.

High-resolution shallow water model runs are carried out with unsheared upstream flow and a wide variety of mountain heights. A regime diagram is constructed, in which gravity wave breaking is classified based on shock number, location, and type. It is demonstrated that different types of shocks identified in the numerical simulations can be consistently described using a shock regime diagram, derived from viscous shock theory. Four curious shock properties are shown to influence orographic flow: the steepening requirement, the tendency for external jumps to amplify shear, the bifurcation in external jumps, and the “double shock.” Some results are compared with continuously stratified flow simulations by a nonhydrostatic mesoscale model.

It is demonstrated that vertical wind shear controls the vertical distribution of wave breaking and potential vorticity generation.

1. Introduction

a. Motivation

The two leading prototypes of stratified flow over mountains are the deep continuous stratification model and the single layer hydraulic model. The former has an extensive elegant literature dealing with linear theory and a number of fully nonlinear numerical studies (e.g., Queney 1948; Sawyer 1956; Clark and Peltier 1977; Smith 1980; Pierrehumbert 1984). The latter has a nonlinear two-dimensional analytic formulation (Long 1972; Baines 1984, 1995; Houghton and Kasahara 1968; hereafter HK68) and a quasi-analytic three-dimensional formulation with powerful conservation laws and jump conditions (Schär and Smith 1993a, hereafter SS93a).

To consolidate these points of view, it is natural to consider a multilayer formulation. Intuitively, a deep continuous stratification can be approximated by a large number of finely spaced homogeneous layers with different densities. In practice, the two-layer model is the first step toward this approximation, but it has two limitations. First, as the two-layer stratification does not continue upward to infinity, waves which would otherwise propagate vertically will be reflected. The other two models would also exhibit this limitation if they were overlaid with a homogeneous layer or rigid lid. Second, as the two-layer stratification is coarse in the vertical, wave motions with fine vertical structure will not be properly treated. For example, the vertical wavelength for steady hydrostatic gravity waves is \( \lambda_z = \frac{2\pi U}{N} \) (Queney 1948), where \( U \) is wind speed and \( N \) is buoyancy frequency. If \( U \) is small, \( \lambda_z \) becomes too small to be resolved by the two-layer system. The deep continuous stratification model would also exhibit this problem for very slow flow (e.g., \( U < NH \), where \( H \) is vertical length scale). While these limitations are rather severe, analysis of the two-layer system may illuminate certain aspects of mesoscale wave breaking, as it has already done for large-scale balanced flows (e.g., Phillips 1954; Pedlosky 1963). In certain situations, real atmospheric, oceanic, or other fluid density profiles may resemble two-layer flow and our analysis will be qualitatively applicable.

A particular goal of this paper is to deepen the understanding of mountain wave breaking, including its sensitivity to upstream shear. While linear theory captures some aspects of this sensitivity (Smith 1989a), steady-state wave breaking is an inherently nonlinear process (e.g., Peltier and Clark 1979; Smith 1985; Schär and Durran 1997; Grubišić and Smolarkiewicz 1997). Direct numerical simulation is a powerful tool for understanding wave breaking, but it does not always provide a connection with theory. To clarify this subject,
we examine the two-layer system in which upper- and lower-level wave breaking is represented by a classifiable set of internal and external hydraulic jumps.

The outline of this paper is as follows. In the remainder of this section, a brief review is given of previous research. The governing equations of the two-layer shallow water model are introduced in section 2. A regime diagram is created for unsheared steady-state flow over an isolated bump in section 3. In section 4, some flow fields are diagnosed using a two-layer shock regime diagram. In section 5, the effect of vertical shear is examined. A couple of ARPS simulations with continuously stratified flow are shown in section 6. A summary is given in section 7.

b. Wave breaking and potential vorticity generation

Mountain-induced gravity wave breaking is associated with the onset of stagnation, overturning, and turbulence. Huppert and Miles (1969) found that the critical hill height for gravity wave breaking was \( h_0 = h_m U^2 / N^2 = 0.85 \) (where \( h_m \) is the mountain peak height) for flow across a simple ridge. An isolated 3D hill must have a greater height to induce wave breaking, due to wave dispersion aloft. For an axisymmetric Gaussian hill in a 3D flow, this critical height was found to be \( h_0 = h_m N U = 1.1 \pm 0.1 \) (Smith and Grönås 1993). These critical heights are also functions of mountain aspect ratio and vertical wind shear (Smith 1989a). It was observed in both laboratory experiments and numerical simulations that whenever these critical heights were exceeded, a stagnation point formed aloft over the lee slope leading to the overturning of isentropes and the occurrence of severe downslope winds and turbulence (e.g., Long 1956; Clark and Peltier 1977; Durran and Klemp 1987). The influence of ambient shear on wave breaking has been examined by Clark and Peltier (1984), Grubišić and Smolarkiewicz (1997), and Dörnbrack and Nappo (1997).

The possibility of potential vorticity (PV) generation related to turbulent dissipation due to mountain-induced gravity wave breaking was proposed about a decade ago (Smith 1989a,b; Haynes and McIntyre 1990), which has been verified by some recent work (e.g., SS93a, Schär and Smith 1993b, hereafter SS93b; Schär 1993; Schär and Durran 1997; Smith et al. 1997). Mesoscale eddies created by wave breaking have been observed and numerically simulated downstream of some major mountains (e.g., Thorpe et al. 1993; Smith and Grubišić 1993; Smith et al. 1997). Identifying such PV sources in the atmosphere is important to intermediate range weather forecasting.

c. Wave breaking in layered flow

SS93a used a single-layer shallow water flow with a free surface to simulate the subcritical stratified flow over a two-dimensional bump. Steady-state solutions were classified using a regime diagram, which was extended to supercritical flow by Jiang and Smith (2000, hereafter JS00). The bottom friction effect on the single-layer solutions was investigated by Grubišić et al. (1995). It was found that the primary contribution of bottom friction is to stabilize the lee vortices. The vorticity generation due to bottom friction was smaller than the jump-related dissipation effect. Vorticity generation across a jump and pseudoinviscid vorticity generation with a drifting vortex were examined by Smith and Smith (1995, hereafter SS95).

However, a single-layer model is an imperfect representation of the real atmosphere, especially in the sense that it cannot simulate vertical wind shear or baroclinicity, both of which play important roles in atmospheric dynamics. To include these effects, it is natural to extend the single-layer system to two layers.

A critical element in the current study is the behavior of two-layer shocks. Two-layer shock theories have been proposed by a number of authors (Yih and Guha 1955; Chu and Baddour 1977; Wood and Simpson 1984; Armi 1986). Unfortunately, these theories are not appropriate for our application because they either do not predict Bernoulli loss correctly from first principles or they empirically take into account between layer mixing processes, which are not included in our simple shallow water model (SWM). Also, they do not use a steepening condition for shock existence. Therefore, as a prerequisite for this study, the authors have recently completed a new theory of ideal shocks (Jiang and Smith 2001a,b, hereafter JS01a and JS01b, respectively), which specifically excludes mixing. The new theory includes a shock classification system based on the end states of the shock. A particular question in the current analysis is whether four curious shock behaviors found in JS01b will appear in orographic flows: the requirement for wave steepening, the tendency for external jumps to amplify shear, the bifurcation of external shocks, and the sequential “double shock.”

2. Governing equations and numerical model

a. Governing equations and conservation laws

For two-layer density stratified Boussinesq fluids with a free surface under a third passive layer, the nondimensional shallow water equations can be written as

\[
\frac{D\mathbf{V}_1}{Dt} + \nabla (h_1 + rh_2 + h) = 0, \quad (1)
\]

\[
\frac{D\mathbf{V}_2}{Dt} + r\nabla (h_1 + h_2 + h) = 0, \quad \text{and} \quad (2)
\]

\[
\frac{Dh_i}{Dt} + h_i \nabla \cdot \mathbf{V}_i = 0, \quad (3)
\]

where \( \mathbf{V}_1 = (u_1, v_1) \) and \( \mathbf{V}_2 = (u_2, v_2) \) are velocity vectors in the lower and upper layers, respectively; \( h \), \( h_1, h_2 \) are the mountain height, the lower-layer depth,
and the upper-layer depth, respectively; and \( r = (\rho_2 - \rho_1)/(\rho_1 - \rho_3) \) is the density difference ratio. Equations (1)–(3) are scaled by the following scales: the radius of the obstacle \( L \) for horizontal length scale, the upstream depth of the lower layer \( H_1 \), the vertical scale, \( \sqrt{g' H_1} \) for the horizontal velocity scale, and \( L/\sqrt{g' H_1} \) for the timescale, where 
\[
g' = g(\rho_1 - \rho_3)/\rho_1
\]
is the reduced gravity between the lower layer (density \( \rho_1 \)) and the passive layer (density \( \rho_3 \)).

Inviscid equations (1)–(3) require that each layer materially conserve potential vorticity; that is,
\[
\frac{D(\text{PV})}{Dt} = 0,
\]
where \( i = 1, 2 \) corresponds to the lower and upper layers. The potential vorticity \( (\text{PV})_i = k \cdot (\nabla \times V_i)/h_i \), where \( k \) is the vertically oriented unit vector. Equation (5) implies that if flow in a layer is initially irrotational, it remains so forever, if only the flow is fully inviscid. Therefore, in the absence of explicit friction, the only factor that can generate PV in a two-layer shallow water system is the internal dissipation within a shock. However, the PV generated by shocked internal dissipation has two distinctive features. First, this type of PV generation is controlled by macroparameters such as Froude numbers and depth ratio and is independent of the magnitude of a dissipation parameter like viscosity (SS93a). Second, PV is generated without violation of the circulation law (Smith 1989b; SS93a). The PV generation in a layer can be measured by potential enstrophy (PE), which (in the absence of ambient rotation) is defined as
\[
(\text{PE})_i = \int \int \frac{\xi_i^2}{2h_i} \, dA,
\]
where \( \xi_i = k \cdot (\nabla \times V_i) \) is vertically oriented vorticity, and \( \int \, dA \) represents an integration over the domain area under consideration. Using Eqs. (1)–(3), one can show that, in the absence of friction, PE is conserved in each individual layer. As will be shown, PE can be produced by shock-induced internal dissipation and destroyed by explicit lateral diffusion and numerical viscosity.

Especially, for a steady state, we can define Bernoulli functions along the bottom and along the interface as (in nondimensional form)
\[
B_i = \frac{1}{2} V_i \cdot V_i + h_i + h_1 + r h_1,
\]
\[
B_2 = \frac{1}{2} V_2 \cdot V_2 + r h_i + r h_1 + r h_2,
\]
where \( B_1 \) and \( B_2 \) are scaled with \( g' H \). Using Eqs. (1)–(3), it can be derived that, in the absence of both bottom and lateral friction, the Bernoulli functions along both the bottom and the interface must be constant along each streamline; that is,
\[
\frac{DB_i}{Dt} = V_i \cdot \nabla B_i = 0.
\]

Following the argument of SS93a for a single-layer shallow water flow, (9) can be violated only in regions with internal dissipation such as hydraulic jumps. Across a wave breaking or hydraulic jump region, the turbulence can dissipate energy, reduce Bernoulli constant, and generate potential vorticity. Vorticity in steady flow can be directly related to the gradient of Bernoulli function (Schär 1993; SS93a): that is,
\[
\xi_i = -\frac{1}{V_i} \frac{\partial B_i}{\partial n},
\]
where \( i = 1, 2 \) and \( n \) is the distance normal to the flow direction. Hence, when nonuniform energy dissipation occurs in one layer, cross-stream Bernoulli variations and potential vorticity generation will occur in that layer.

b. Wave modes and shock types

The linearized version of Eqs. (1)–(3) admits two wave modes; the internal (slower) mode and the external (faster) mode. The critical condition for a stationary mode is (Armi 1986; JS01b)
\[
(F_1^2 - 1)(F_2^2 - r) - r^2 = 0.
\]
On an \( F_1 \) by \( F_2 \) diagram, condition (11) has two branches separating flow into three distinct regimes: both internally and externally subcritical (BB), internally supercritical but externally subcritical (BP), and both internally and externally supercritical (PP) regimes. Five types of normal shocks were identified by Jiang and Smith (JS01b): internal jump (type \( \hat{a} \)), internal drop (type \( \hat{b} \)), external lower-layer jump (type \( \hat{c} \)), external two-layer jump (type \( \hat{d} \)), and external upper-layer jump (type \( \hat{e} \)). Internal shocks begin in BP and end in BB. External jumps begin in PP and end in BP. The three types of external jump differ in the partitioning of dissipation between the two layers. Certain high-speed flow states in PP allow three different end states corresponding to jump types \( \hat{c}, \hat{d}, \) or \( \hat{e} \). A double shock is composed of two closely spaced shocks going from PP to BP to BB. The first external shock creates a layer asymmetry (i.e., thinner layer depth and faster flow in one layer and thicker layer depth and slower flow in the other), which allows an internal shock to follow.

c. Numerical model

The shallow water model used here has been used to study hydraulic jumps in two-layer flow by JS01a and JS01b. The single-layer version of this model has been used by SS93a,b, Grubišić et al. (1995), SS95, Smith et al. (1997), and JS00.

The nondimensional time-dependent shallow water equations (1)–(3) in flux form are integrated on a rectangular grid using the “upstream scheme.” Iterations
are performed to satisfy the momentum conservation requirements. This so-called conservative and synchronous flux-corrected transport technique has been described by Schär and Smolarkiewicz (1996) in detail. The advantage of this scheme for the present study is that it conserves mass and momentum in shocks, allowing the proper generation of potential vorticity and the destruction of kinetic energy by internal dissipation. This dissipation occurs in the absence of surface friction and interfacial shear stress. This formulation allows momentum exchange with the underlying topography and momentum exchange between the two layers due to hydrostatic pressure. The method guarantees that the numerical formulation is consistent with the ideal-shock hydrostatic pressure. The method guarantees that the momentum exchange between the two layers due to momentum exchange with the underlying topography and interfacial shear stress. This formulation allows momentum exchange between the two layers due to hydrostatic pressure. The method guarantees that the numerical formulation is consistent with the ideal-shock assumptions (JS01a,b). Test runs show that the model is numerically stable if the Courant–Friedrich–Lewy criterion is satisfied.

The idealized obstacle used in this study is a circular topography described by two parameters, the maximum mountain height \( h_m \) and radius \( a \):

\[
h(x, y) = \frac{h_m}{[1 + (x/a)^2 + (y/a)^2]^{1/2}}.
\]

A high hill is allowed to pierce through the lower interface, giving the lower-layer zero mass and velocity on the dry peak. At \( t = 0 \), the interface and free surface are assumed to be level, and the fluids are impulsively started from rest. A forward integration of the time-dependent shallow water equations in flux form is carried out to nearly steady state. Typically, the integration time is chosen to allow the slowest wave mode to propagate out of the domain.

A specified upstream boundary condition is used to represent the large-scale balanced flow field. At the lateral boundaries, the relaxation scheme of Davies (1983) is used. The spatial resolution for these simulations is \( \Delta x = \Delta y = 0.1 \). The temporal resolution is \( \Delta t = 0.025 \). Such temporal and spatial resolutions guarantee that the Courant–Friedrichs–Lewy criterion is satisfied throughout all our simulations. The domain size is chosen as \( 80 \times 40 \) (i.e., \( 800 \times 400 \) grid points), which is large enough to leave sufficient room upstream for propagating wave to disperse and downstream to display the features of wakes.

3. Classification of wave breaking in two-layer flow

a. Regime diagram

The time-dependent solutions of the nondimensional equations (1)–(3) can be schematically expressed as

\[
\psi(x, y, t) = \psi_i(x, y, t; F_i, F_2, r, K, M),
\]

where \( \psi_i \) is one of the field variables \( (h, u, v) \). In (13), \( K = H_i/H_2 \) is the layer depth ratio \( (H_i \) is the upper-layer depth), \( M = h_m/H_i \) is the nondimensional mountain height based on the lower-layer ambient depth and

\[
F_i = U_i \sqrt{g H_i}, \quad (i = 1, 2)
\]

are the two ambient Froude numbers based on the reduced gravity \( (4) \). For unsheared incoming flow, with equal depth and a fixed \( r \), the steady-state solutions can be classified using a regime diagram of \( F \) by \( M \), where \( F = F_1 = F_2 \) (Fig. 1). We created Fig. 1 using some theoretical considerations and more than 40 high-resolution model runs. Using (11), we divide the regime diagram into three parts: BB, BP, and PP. Ten different types of flow are identified and labeled by their ambient and disturbance flow characteristics (Table 1). Nine of them are displayed in Fig. 2. Although these solutions are essentially two-dimensional, most of the distinctive features in the different regimes can be concisely described by the behavior along the centerline.

It is important to keep in mind the differences between the one-dimensional and two-dimensional hydraulic solutions. First, PV generation can only be analyzed in two-dimensional solutions as its definition includes lateral derivatives of the velocity field. Second, horizontal convergence and divergence in two dimensional solutions allow the existence of reversed flow, stagnation points, and horizontal shear instability. Finally, one-dimensional solutions are more dependent on the boundary conditions. Upstream blocking and wave reflection from the boundaries have a greater influence.
on the interior flow field. For instance, Gutman et al. (1996) demonstrated that the emergence of the stationary jumps in the 1D problem of Houghton and Kasahara (1968) is stipulated by the downstream boundary condition. Our 2D solutions are more robust than this.

b. Ambient flow in BB

The features of solutions in BB are similar to those of the single-layer subcritical solutions (SS93a,b). This ambient flow regime can be divided into three disturbance regimes, namely, internally subcritical, internal normal shock, and lower-layer flow splitting with internal flank shocks. These are discussed below.

1) INTERNALLY SUBCRITICAL REGIME (BB1)

With a low hill, flow in this regime remains internally subcritical through the entire domain. In the absence of shocks and lee waves, flows are inviscid in both layers, and the steady-state solutions are fore–aft symmetric. Right over the peak, there is a dip at the lower interface where wind speed reaches maximum in the lower layer. Corresponding to the dip at the lower interface, there is a rise at the upper free surface with a much reduced amplitude (not shown). These solutions can be well predicted by linear theory.

2) INTERNAL NORMAL JUMP REGIME (BB2, BB3)

With a higher hill, the subcritical ambient flow in this regime experiences a smooth-state transition and locally becomes internally supercritical downstream of the peak. An internal jump appears on the lee slope of the hill with dissipation and vorticity generation in the lower layer. Flow in the upper layer is almost inviscid.

There are two types of flow patterns in this regime depending on the hill height. For a lower hill, the lee jump is weak (BB2). Fluid parcels with potential vorticity or reduced Bernoulli constant are advected downstream by the ambient flow, which results in a long wake trailing downstream with decelerated flow (Fig. 2a). As the hill becomes higher, the lee jump can be so strong that vorticity self-advection generates reversed flow near the centerline (BB3; Fig. 2b). A pair of stationary eddies appears downstream of the jump with closed streamlines in the lower layer. With strong reversed flow, left–right asymmetric perturbations can grow rapidly leading to shedding vortices similar to eddies in single-layer flow (SS93b).

3) FLOW SPLITTING WITH INTERNAL FLANK SHOCKS REGIME (BB4, BB5)

Flow in this regime is characterized by the inability of the lower-layer flow to climb over the peak, leading to lower-layer flow splitting. There are three stagnation points in the lower-layer flow. The first stagnation point appears on the windward slope where the central streamline is split. As demonstrated by SS93a, there is a high Froude number region along the intersecting curve around the dry peak associated with the shallow fluid depth. Accordingly, there exist a pair of internal jumps on the flanks of the hill with strong vorticity generation and reversed flow along the centerline in the lower layer. The second stagnation point is located on the lee slope and the third stagnation point is at the end of the eddies. The lack of a potential flow solution (Drazin 1961) has been attributed to the edge waves around the cut curve (SS93a).

Over the “dry” (i.e., no lower-layer fluid) peak, the upper layer responds directly to the terrain. This regime can be divided into two subregimes based on the different behaviors of the upper layer over the peak (the boundary is approximately indicated by a dashed line). For a lower hill, the upper layer is still subcritical over the peak (governed by $F_2 < 1$), and therefore remains inviscid (BB4; Fig. 2c). For a higher hill, the upper layer becomes supercritical over the peak and a normal jump forms over the lee slope with a long wake, or a pair of eddies (BB5; Fig. 2d). We do not consider the case of a hill high enough to penetrate into the passive layer.
c. Ambient flow in BP

In regime BP, the ambient flow is faster than the internal wave mode, but slower than the external mode. Regime BP can also be divided into three disturbance regimes: the V-wave regime, the external normal shock regime, and a regime with lower-layer flow splitting and external flank shocks.

1) INTERNAL V-WAVE REGIME (BP1)

The flow is internally supercritical everywhere, as the hill is not high enough to force flow into an externally supercritical state. There are a pair of internal V waves trailing in the lee side (JS00). No shocks form and no state transition occurs through the domain (Fig. 2e).

2) EXTERNAL NORMAL SHOCK REGIME (BP2, BP3, BP4)

With a higher hill, the internally supercritical ambient flow becomes externally supercritical near the hill. Flow in this regime is characterized by both an external jump and internal V waves. With increasing hill height, three subregimes with different jump structures are identified. For a moderate hill, a weak external jump appears with relatively weak Bernoulli losses and long wakes in both layers (BP2; Fig. 2f). As mountain height increases, dissipation in the upper layer becomes dominant. Vertical asymmetry develops in the postshock flow, namely, a thicker and slower upper layer, and a thinner and faster lower layer. Approximately across the dashed line in the BP section (Fig. 1), this asymmetric structure promotes a second jump, an internal jump, which brings the flow back toward symmetry to match the downstream flow state (BP3). There is a short distance (four grid points or 0.4 nondimensional length units) between the two shocks for the case shown in Fig. 2g. This allows the flow after the first jump to adjust itself toward a jump-allowed state. This jump–jump pair produces eddies or long wakes in both layers.

The double shock structure in Fig. 2g (case BP3) bears some resemblance to the structure of the well-known Boulder windstorm of January 1972 (Lilly and Zipser 1972). Their observations show an upper-level jump several kilometers ahead of the lower-level jump. Lilly and Zipser pointed out, however, that the upper and lower regions were observed by aircraft at different times and that a single jump with changing position fits the data equally well.

Some two-dimensional fields from case BP3 are displayed in Fig. 3. Figures 3a,b show strong wind on the lee slope in both the lower and the upper layers, a well-defined external jump on the lee slope, and a wake region after the jump with a velocity deficit. Reversed flow exists in the lower layer. The internal V wave is evident in the lower-layer height field (Fig. 3c). The vorticity in the lower and upper layers are contoured in Figs. 3e,f. Only those parcels that have passed trough a shock have vorticity.

As the hill becomes even higher (across the bold dashed curve in Fig. 1), the energy dissipation in the external jump switches suddenly to the lower layer and we enter BP4 (Fig. 2h). This lower-layer dissipation creates an opposite asymmetry, with the lower layer thick and slow and the upper layer shallower and faster. The second shock, an internal drop, occurs eight grid points (i.e., 0.8 nondimensional length units) downstream of the first jump, with energy dissipation in the upper layer. Unlike the other regime boundaries in Fig.
1, the boundary between BP3 and BP4 represents a bifurcation, probably with hysteresis (JS01b).

3) **Flow splitting with external flank shocks regime (BP5)**

As the hill becomes still higher, the lower layer is pierced through by the peak and we enter BP5 (Fig. 2i). On the flanks of the peak, the Froude number of the lower layer can be very large, and a strong response of the upper layer brings flow into the external supercritical regime. A pair of external jumps forms on the flanks of the hill with a strong wake and reversed flow in the lower layer. Also, over the peak, there is an upper-layer normal shock, which is strong enough to induce reversed flow in the upper layer. Because the ambient flow is internally supercritical, there is an internal V wave coexisting with wave breaking.

It is interesting to note that none of the regimes in Fig. 1 include upstream bow shocks. This is so because bow shocks require either strong layer asymmetry or externally supercritical flow, neither of which is present in the upstream conditions for this set of runs.

4. **Diagnosis of solutions using the shock diagram**

An interesting question is whether shocks in the numerically generated flow can be compared with ideal viscous shock solutions. Specifically, are shocks in a two-dimensional time-dependent numerical model classifiable, and do their jump conditions agree with shock theory?

As the ability of the SWM to reproduce the correct jump conditions for bow shocks has already been considered in JS01a and JS01b, we limit our demonstration to a few cases of lee-side normal shocks. The most convenient way to compare shock theory and the model is to follow streamlines along the centerline where, due to left–right symmetry, any shock encountered is a normal shock. This allows us to diagnose the flow field by plotting the Froude numbers along the center streamline on a shock regime diagram. We call these curves “Froude number trajectories” (Fig. 4). The reference critical curves on the shock regime diagram are invariant, as the only other parameter in (11) is the fixed density ratio.

The precise end point of the trajectory gives information about the velocity and Bernoulli deficit in the
wake. This is so because, in the irrotational limit \((f = 0)\), flow depth after the jump will recover its upstream value \((h_i = h_u)\) to reach a pressure equilibrium. With the depths restored to ambient values, a reduced Froude number implies a reduced speed and lower Bernoulli value. For instance, if \(F_i > F_u\), where \(s\) and \(e\) denote the start point and end point, respectively, the distance between \(F_i\) and \(F_u\) is a measure of the strength of the wake (or Bernoulli drop) in the lower layer.

As examples, single shock cases BB2 and BP2, and double shock cases BP3 and BP4 are diagnosed using a shock regime diagram in Fig. 4. In case BB2, the flow starts from the BB regime, crosses the critical curve near the topography and enters the BP regime. The asymmetry in two layers increases until an internal jump occurs, which appears as a sharp turn of the trajectory in Fig. 4a. The postjump flow state lies in the BB regime. The ideal shock jump conditions based on the prejump flow parameters derived from numerical simulations shows good agreement with the numerical solution.

Note that the flow does not return precisely its upstream state. The \(F_1\) at the final state is smaller than the upstream \(F_{1u}\), indicating a long wake extending to the downstream boundary of the domain.

Case BP2 is similar to case BB2, except that flow starts from the BP regime, crosses the external critical curve near the topography and enters the PP regime. While the flow over the lee slope is still symmetric (i.e., \(F_1 = F_2\)), a type "d" (see section 2b for definition) external jump occurs, which brings the flow back into the BP regime. The postjump state maintains approximate symmetry in two layers, and the distance between the equilibrium state and the upstream state suggests that there is a long wake in both layers with comparable Bernoulli losses. Again, there is good agreement between SWM and ideal shock jump conditions.

The double shocks in cases BP3 and BP4 can also be described using ideal shock theory (Fig. 4b). As the topography becomes higher and we enter regime BP3, a type \(\hat{e}\) external jump occurs slowing down the upper-layer flow. The postjump flow is asymmetric in the two layers, that is, faster and thinner flow in the lower layer, and slower and thicker flow in the upper layer. This asymmetric structure is necessary for an internal jump. After some adjustment, an internal jump (type \(d\)) occurs, which slows down the lower layer, and the brings flow into the BB regime. After the latter jump, flow recovers its layer symmetry.

As the hill becomes even higher, a shock bifurcation occurs (case BP4). The type \(\hat{e}\) jump is suddenly replaced by a type \(\hat{e}\) jump. The change in the Froude number trajectory is dramatic. The postjump flow has a thinner and faster upper layer and a thicker and slower lower layer, which is the necessary structure to form an internal drop. The subsequent internal drop (type "\(\hat{d}\)"") slows down the upper layer. In both cases, the Froude number trajectories intercept the \(F_1\) or \(F_2\) axis, indicating that flow stagnation occurs in the lower or upper layer associated with lee eddies. Because no unique streamline can be drawn through a stagnation point, the remaining part of the trajectories after flow stagnation is difficult to interpret.

Numerical simulations indicate that the transition from the jump–jump solution and jump–drop solution is very sharp. For example, with \(r = 0.5, K = 1, F = 0.7, M = 1.0\), there is a jump–jump pair, while for \(M = 1.30\), there is a jump–drop pair. The sharp transition indicates that a shock bifurcation occurs corresponding to the cusp structure identified by JS01b.

Double shocks, or more exactly, drop–jump pairs were found in laboratory experiments in two-layer flow with a rigid lid by Baines (1984). Baines noticed that this phenomena can occur when the depth ratio is small (thinner lower layer) and the obstacle is high. Our numerical experiments are different from Baines in several ways. We have an upper free surface under a third passive layer instead of rigid lid. We have two horizontal dimensions instead of one. Because of these model differences, we are not able to establish...
a physical connection between Baines’ double shocks and ours.

5. The influence of ambient shear

a. Sheared flow over a hill with moderate height

To examine the shear effect, an $F_1$ by $F_2$ regime diagram has been created for $K = 1$, $r = 0.5$, and nondimensional mountain height $M = 0.6$ based on physical constraints and several numerical runs (Fig. 5). The two branches of the critical curve (11) divide the diagram into three regimes: BB, BP, and PP. If the vertical shear is large enough, the environmental flow will experience Kelvin–Helmholtz shear instability even in the absence of any topography. The boundary lines in Fig. 5 are derived from linear theory (Baines 1995). The analysis of hill-induced disturbances is not possible in this unstable regime. Another interesting reference line in Fig. 5 is the dotted–dashed curve dividing BP into three subregimes: jump, drop, and no shock. This curve comes from shock properties alone (JS01a,b) and is relevant here because it controls bow shocks. With negative ambient shear falling in the “jump” regime, a bow shock in the form of an internal jump will occur upstream of the hill. Bow shocks of the “internal jump” type do not form with positive hills, even with forward shear. The dotted diagonal line $F_1 = F_2$, corresponds to the dotted line of $M = 0.6$ in Fig. 1, which cuts through the same regimes in both diagrams.

There are two disturbance regimes in the BB regime in Fig. 5, namely, the subcritical and internal jump regimes. The no-shock regime can be further divided into two disturbance regimes, namely, the $V$ wave and the external shock. For the “subcritical,” “internal jump,” “$V$-wave,” and “external shock” subregimes, their steady-state solutions are qualitatively similar to the corresponding unsheared cases described in section 3. The bow shock has no unsheared analog. The vertical partitioning of dissipation varies strongly with shear, however, even within one subregime. This variation is discussed below.

b. The effect of shear on PV generation

The influence of shear on wave breaking and related PV generation is complicated even in a two-layer system. We start this section with one question in mind:
To what extent can vertical wind shear modify mountain wave breaking and PV generation?

The rate of PV generation in one layer is related to the potential enstrophy [defined in (6)] of the entire computational domain. In steady state, PE reflects a balance of PV creation and advection out of the domain.

To have a quantitative evaluation of the shear effect, we fix the nondimensional mean flow, that is, \( F_m = (F_1 + F_2)/2 = \text{const} \), in the two-layer system, and define shear as \( \delta F = F_2 - F_1 \). Therefore, for forward shear \( \delta F > 0 \), and for backward shear \( \delta F < 0 \). For simplicity, we choose two mean Froude numbers that fall inside large regimes in Fig. 5, so that the shear does not put the flow into another regime. For \( F_m = 0.35 \), all but one case lies in the internal jump regime. For \( F_m = 0.75 \), the flow is always in the external jump regime.

The response of nondimensional PE to changing vertical shear \( \delta F \) is plotted in Fig. 6. For a slow mean flow \( (F_m = 0.35) \), PE is generated in the lower layer due to the dissipation of an internal jump over the lee slope. Potential enstrophy in the upper layer is zero. In the lower layer, PE decreases rapidly as the shear changes from backward shear to forward shear. For example, with shear \( \delta F = -0.1 \), PE = 12.0, and with shear \( \delta F = 0.1 \), it drops to 2.0.

For a fast mean flow \( (F_m = 0.75) \), PE is generated in both layers by an external jump over the lee slope. Figure 6 shows that while the sum of PE in two layers is almost unchanged, the distribution of PE in the two layers varies rapidly with shear. For unsheared flow, PE in the upper layer is about twice that in the lower layer. As \( \delta F \) becomes negative (backward shear), PV generated in the upper layer becomes stronger. As \( \delta F \) becomes
positive (forward shear), PE of the lower layer becomes larger than the PE in the upper layer. Two examples of these sheared flows are shown in Fig. 7.

The case of strong PE generation in the upper layer is of particular interest as an analogy can be drawn to upper-level wave breaking in the continuously stratified case. Strong low-level flow prevents a large-amplitude wave from breaking there. However, when this wave reaches the slower upper layer, it causes breaking. The drag deposition and PV generation occur in a layer not in contact with the obstacle (Fig. 7b).

The underlying reason for the above behavior is the tendency of external jumps to magnify layer asymmetry. External jumps, if they are not too strong, concentrate dissipation in the slower moving layer, pushing the flow state away from the diagonal in Figs. 4 or 5 (JS01b).

6. Continuously stratified flow simulations

A mesoscale model, the Advanced Regional Prediction System (ARPS), is used in this study. ARPS is a three-dimensional nonhydrostatic model developed at the Center for Analysis and Prediction of Storms (CAPS) in the University of Oklahoma. The full nonlinear (momentum, continuity, and thermodynamic) equations are solved on discrete grid points. The terrain-following vertical coordinate is further stretched to allow higher vertical resolution in the lower troposphere (Xue et al. 1995). For the purpose of model validation, ARPS has been used to simulate mountain gravity waves induced by an idealized small ridge. Satisfactory agreement between linear theory and model predictions was found in wave phase, wave amplitude, and momentum flux. Recently ARPS participated in the Mesoscale Alpine Program (MAP) intermodel comparison for the 1972 gravity wave breaking (GWB) study, and reasonable agreement was observed with other mesoscale model predictions (Doyle et al. 2000). The 1.5-order turbulent-energy-based closure scheme is used after Deardorff (1972a,b).

The model covers a horizontal domain of 100 × 100 grid points and a depth of 18 km covered by 33 vertical layers. Rayleigh damping is applied to the uppermost 4 km to absorb gravity waves. A circular mountain described by Eq. (12) sits in the center of the domain with a mountain width of a = 40 km and a height of h_m = 2500 m. The model is initialized using a single sounding as plotted in Fig. 8. There is a thin stable layer between 4 and 5 km, and a tropopause at 9.5 km. Between the stable layers, there are two nearly neutral layers. Wind U_0 is uniform in the troposphere and decreases to zero in the stratosphere. There is a critical layer above 12 km (for U_0 = 15 m s^{-1}) and 14 km (for U_0 = 20 m s^{-1}).

We hope that the lower stable layer will serve as a density interface to separate the lower neutral layer from the layer above it. The thickness of the stable layer (∼1 km) prevents strong shears that might produce shear instability. The critical level above the tropopause in-
Fig. 8. The ambient flow profiles for ARPS runs. (a) Potential temperature. (b) Wind speed.

duces wave breaking and decouples the flow below the critical level from the flow above it.

Two examples with different $U_0$ values are shown in Fig. 9. With $U_0 = 15$ m s$^{-1}$, the steady-state solution (Fig. 9a) shows interesting similarity to the two-layer solution in the internal normal shock regime (BB2). There is a hydraulic jump riding over the lee slope with turbulent kinetic energy (TKE) generation in the lower neutral layer. The response of the second layer is opposite: upstream of the jump, the upper layer expands and downstream of the jump it contracts. TKE is generated in the second layer as well. The TKE maximum in the second layer is only about 20% of the TKE maximum in the lower layer. Therefore, we may conclude that this run corresponds to the internal shock solutions.

With a slightly stronger wind speed ($U_0 = 20$ m s$^{-1}$, Fig. 9b), the flow is similar to the two-layer shock solution in the external double shock regime (case BP3). First, we can see a strong low-level hydraulic jump over the lee slope with expansion and TKE generation. The behavior of the second layer is quite similar except for slight phase tilting.

Is it possible to mark these two runs on our regime diagram (Fig. 1)? The estimation of Froude number and nondimensional mountain height are crude due to the continuous stability profile, the uncertain domain top and non-Boussinesq effects. A rough estimation from Fig. 9 gives us $2H = 10$ km, and $\Delta \theta = 15$ K, where $2H$ is the total depth of two layers and $\Delta \theta$ is potential temperature increase from ground to the top surface. The nondimensional mountain height $M = 0.5$ and $g' = \Delta \theta / \theta_0 g = 0.5$. The corresponding Froude numbers for the two runs are $F = 0.3$ and $F = 0.4$ (Fig. 1). They fall slightly too low to explain their apparent flow regime.

7. Conclusions

In this final section, we assess the two-layer representation of mountain-induced gravity wave breaking. Most importantly, we showed that wave breaking is not one unclassifiable, incomprehensible state of fluid motion. Rather, it is composed of several discrete shock types with different vertical profiles of dissipation, finely controlled by the nondimensional control parameters of the problem. This dissipation, in turn, generates PV generation and propagation. The two-layer theory suggests ways to understand the vertical distribution of dissipation, the sensitivity to ambient shear, the hysteresis between different wave breaking states and the occurrence of double shocks within a wave breaking region.

The two-layer formulation confirms some previous results from models with continuous stratification, concerning the effects of mountain height, ambient flow speed, and shear on wave breaking. We mention here only the effects of ambient shear. Linear theory and nonlinear numerical simulation have already provided some insight into the effect of shear on wave breaking. The linear theory of wave action (e.g., Bretherton 1966; Smith 1979) indicates that standing mountain waves can alter their amplitude as they propagate up through layers with different wind speeds and static stability. This result, together with a local nonlinearity measure ($Nh/U$), indicates that waves increase their “horizontal velocity” amplitude and become more nonlinear within slow-moving layers. The onset of wave breaking becomes more likely. Slow moving layers near the ground will experience nonlinear flow (i.e., $Nh/U > 1$) leading to low-level wave breaking. Our results (Fig. 6) confirm this result and verify that under these conditions, little wave breaking occurs aloft. The nonlinearity measure $Nh/U$ is sometimes referred to as inverse Froude number, which should not be confused with the definition of Froude number in hydraulic flow. The former assumes a uniform stability, and the latter requires well-defined interface(s).

When the low-level flow is fast, waves are able to propagate into the upper levels. Slow flow aloft will increase the nonlinearity there and promote breaking of these waves. Faster flow aloft will further retard wave breaking and send the waves further aloft (to levels not captured in a two-layer model). These trends are also seen in Fig. 6. The current model verifies that these simple tendencies are robust and not limited to linear formulations.

Other effects seen in Fig. 6 are not predicted by linear wave action theory, particularly the effect of upper-level flow speed on low-level breaking. For example, two low-level flows with similar moderate flow speeds can
experience either severe wave breaking (i.e., small $F_m$, negative shear) or weak wave breaking (i.e., large $F_m$, positive shear) depending on the upper-level wind speed. As mentioned in section 1, other examples of poor correspondence between layered and continuous stratification can be identified, particularly with slow winds and no wave breaking.

Our analysis provides further examples of PV creation aloft, adding to the previous numerical simulations by Schär and Durran (1997) and Smith et al. (1997). A knowledge of the vertical distribution of PV generation provides more useful information than just knowing the drag or the vertical distribution of wave momentum flux deposition. The latter description does not suggest how the flow will respond to the momentum deposition or how the effects will be carried downstream.

Finally, the qualitative agreement between two-layer model and ARPS model simulations is encouraging. It suggests that some conclusions derived from layered models may be applicable to the real atmosphere. It is especially interesting that dissipation (i.e., TKE) occurs in neutral layers instead of within stable interfaces, which is one of the fundamental assumptions of layered models and our shock theory.

Acknowledgments. Christoph Schär kindly allowed us to use his shallow water code. The authors thank Dale Durran and four anonymous reviewers for their constructive comments and suggestions. This research was supported by the National Science Foundation, Division of Atmospheric Sciences (ATM-0112354).