A Vector Derivation of the Semigeostrophic Potential Vorticity Equation

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ABSTRACT

The semigeostrophic potential vorticity equation is derived in a vector-based notation for the shallow-water and primitive equations, as models for atmospheric flow. The derivation proceeds from knowledge of the functional form of potential vorticity and starts directly from a vector form of the governing equations. The method makes use of highly involved vector identities and provides for a clearer picture of the nature of the final result, as compared to a component-based derivation. It is, however, limited by the need to know a priori the appropriate functional vector form of the dynamically relevant quantity, such as potential vorticity.

1. Introduction

In the semigeostrophic (SG) model, introduced by Hoskins (1975) and Hoskins and Bretherton (1972) and discussed earlier by Eliassen (1949), only the geostrophic part of momentum is advected by the full flow, which is referred to as the geostrophic momentum approximation. This approximation leads to the SG equations—where this terminology is used here in the same sense as used in section 9.2.2 of Holton (1992)—as a model of atmospheric flow. The SG equations are, on one hand, balanced in that they cannot propagate gravity waves but, on the other hand, describe properties of the full unapproximated equations much more realistically than the quasigeostrophic equations.

The mathematical structure of the SG model has been the subject of many studies (e.g., Cullen and Douglas 2003; McIntyre and Roulstone 2002; Ren 2000; Purser 1999; Craig 1993; Salmon 1988). It is well known that the SG equations contain a potential vorticity (PV) conservation principle that is similar to the PV equation that results from the unapproximated equations (see, e.g., section 9.4 in White 2002; section 3 in Hoskins 1975). However, the derivation of the SG PV equation appears impeded by mixed advection terms in the momentum equation [see, e.g., Eqs. (2.1) and (3.1)]. Thus, for example, Allen et al. (1990) have considered a procedure that facilitates this derivation in the context of the shallow-water model by solving, on the basis of the momentum equation, for the ageostrophic-wind components in terms of the geostrophic-wind components (see also, White 2002). It is the purpose of this paper to indicate a more direct vector-based derivation of the SG PV equation for both the shallow-water and the primitive-equation model.

2. The shallow-water model

a. The geostrophic momentum approximation

Introducing the geostrophic momentum approximation leads to the following SG form of the momentum and continuity equations of the shallow-water (SW) model (see, e.g., White 2002):

\[ \frac{\partial \mathbf{v}_g}{\partial t} + \mathbf{v} \cdot \nabla h \mathbf{v}_g + f_0 \mathbf{k} \times \mathbf{v}_a = 0, \]  

(2.1)

\[ \frac{\partial H}{\partial t} + \mathbf{v} \cdot \nabla H + H \nabla \cdot \mathbf{v}_g = 0, \]  

(2.2)

\[ f_0 \mathbf{k} \times \mathbf{v}_g = -g \mathbf{\nabla}_v \gamma, \]  

(2.3)

where symbols have their usual meaning, the Coriolis parameter is replaced by the constant value \( f_0 \), and the full two-dimensional wind \( \mathbf{v} = \mathbf{v}_g + \mathbf{v}_a \) is written as the sum of the (nondivergent) geostrophic component \( \mathbf{v}_g \), defined in (2.3), and the ageostrophic component \( \mathbf{v}_a \). Further, \( \nabla_v \) denotes the horizontal gradient operator, \( \mathbf{k} \) is a constant unit vector in the upward vertical direction, \( H \) is fluid depth, defined as \( H = \overline{h} + \gamma - h_b \), where \( \gamma \) is the deviation of the free surface from a reference level \( \overline{h} \), and \( h_b \) is bottom topography. In comparison to the full SW equations, the only approximation in the SG model (2.1) and (2.2) is made by neglecting the term \( (\partial \mathbf{v} / \partial t + \mathbf{v} \cdot \nabla) \mathbf{v}_a \) in the momentum equation (2.1), which ensures the absence of gravity waves from the flow.
b. The vector derivation of the SG PV equation

The essential step in the vector-based derivation of the SG PV equation, as given here, is the recognition that the vertical component of the SG absolute vorticity \( \xi_{SG} \), defined, for example, in Eq. (9.41) of White (2002), may, due to the nondivergence of \( \mathbf{v} \), be written in vector form as

\[
f_0 \frac{\partial \xi_{SG}}{\partial t} = f_0^2 + f_0 \mathbf{k} \cdot \nabla \times \mathbf{v}^* - \frac{1}{2} (\nabla \cdot \mathbf{v}) (\nabla \cdot \mathbf{v}^*), \tag{2.4}
\]

where \( \xi^* = \mathbf{k} \cdot \nabla \times \mathbf{v} \) is the vertical component of the (relative) geostrophic vorticity. The double-dot notation in the last term on the right-hand side (rhs) of (2.4) denotes the double contraction (see, e.g., Aris 1962) of the outer product with itself of the second-order tensor \( \nabla \cdot \mathbf{v} \), describing the gradient of \( \mathbf{v} \) [see also, Eqs. (A.1) and (A.2) in appendix A]. Due to this last term, the SG absolute vorticity \( \xi_{SG} \) is only indirectly related to the curl of the velocity field; on the basis of this definition, however, a conservation principle for SG PV may be derived, that is not achievable if only the curl of the velocity field is considered. To derive the equation governing the time evolution of \( \xi_{SG} \), the time derivative of the defining relation (2.4) is taken:

\[
f_0 \frac{\partial \xi_{SG}}{\partial t} = f_0 \frac{\partial \xi^*}{\partial t} - (\nabla \cdot \mathbf{v}) (\nabla \cdot \mathbf{v}^*), \tag{2.5}
\]

where the chain rule has been used to differentiate the double contraction. As the next step, the SG equation of motion (2.1) is used to determine the time tendencies appearing on the rhs of (2.5). Operating with \( f_0 \mathbf{k} \cdot \nabla \times \) on (2.1) yields

\[
f_0 \frac{\partial \xi}{\partial t} + f_0 \mathbf{k} \cdot \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) + f_0 \nabla \cdot \mathbf{v} = 0, \tag{2.6}
\]

where identity (B.1) (see appendix B) was used. Further, taking the gradient of (2.1) and then applying the operator \( \left[ (\nabla \cdot \mathbf{v}) \right] \) on the result yields

\[
(\nabla \cdot \mathbf{v}) \cdot \frac{\partial}{\partial t} (\nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{v}) \cdot [\nabla (\mathbf{v} \cdot \nabla \mathbf{v})] + (\nabla \cdot \mathbf{v}) \cdot [\nabla (f_0 \mathbf{k} \times \mathbf{v})] = 0. \tag{2.7}
\]

Replacing the terms on the rhs of (2.5) through the results given in (2.6) and (2.7) yields after reordering

\[
f_0 \frac{\partial \xi_{SG}}{\partial t} = -f_0 \frac{\partial z}{\partial t} \mathbf{v}^*_a - f_0 \mathbf{k} \cdot \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) - (\mathbf{v} \cdot \mathbf{v}^*) \cdot [\nabla (\mathbf{v} \times \mathbf{v}^*)] + (\mathbf{v} \cdot \mathbf{v}^*) \cdot [\nabla (\mathbf{v} \cdot \nabla \mathbf{v})]. \tag{2.8}
\]

Using the vector identities (B.2) and (B.3) (see appendix B) to rewrite the rhs of (2.8) leads to

\[
f_0 \frac{\partial \xi_{SG}}{\partial t} = -\nabla \cdot \left( \mathbf{v} \cdot f_0^2 \right) + \nabla \cdot \left( f_0 \mathbf{k} \cdot \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) \right) - \nabla \cdot \left( \mathbf{v} \cdot \left[ \frac{1}{2} (\nabla \cdot \mathbf{v}) \cdot (\nabla \cdot \mathbf{v}^*) \right] \right). \tag{2.9}
\]

Since \( \nabla \cdot \mathbf{v} = 0 \), it follows that \( \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{v}^* = \mathbf{v} \) and hence (2.9) can be rewritten, by using the definition of \( \xi_{SG} \) given in (2.4), as

\[
\frac{\partial \xi_{SG}}{\partial t} = -\nabla \cdot (\mathbf{v} \xi_{SG}), \tag{2.10}
\]

which is, of course, the desired result that governs the time evolution of \( \xi_{SG} \). Finally, the standard combination of the continuity equation (2.2) with (2.10) leads to the well-known SG PV equation (see, e.g., White 2002):

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left( \xi_{SG} \right) = 0. \tag{2.11}
\]

3. The primitive-equation model

a. The geostrophic momentum approximation

The SG form of the primitive equations (PEs) in their hydrostatic, Boussinesq form is given by the three-dimensional equation of motion, the continuity equation, and the thermodynamic equation, respectively:

\[
\frac{d\mathbf{v}}{dt} + f_0 \mathbf{k} \times \mathbf{u} = 0, \tag{3.1}
\]

\[
\nabla \cdot \mathbf{u} = 0, \tag{3.2}
\]

\[
\frac{d\theta}{dt} = 0, \tag{3.3}
\]

with the individual time derivative given as

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{3.4}
\]

with \( \nabla \) denoting the three-dimensional gradient operator, and the following three-dimensional relation between the full three-dimensional wind \( \mathbf{u} \) and the ageostrophic wind \( \mathbf{u}^*_a \):

\[
f_0 \mathbf{k} \times \mathbf{u} = -\nabla \phi + f_0 \mathbf{k} \times \mathbf{u}^*_a, \tag{3.5}
\]

where, as in section 2, \( f_0 \) is the constant value of the Coriolis parameter, the difference of full and ageostrophic wind is defined as the (two-dimensional non-divergent) geostrophic wind \( \mathbf{v}_a = \mathbf{u} - \mathbf{u}^*_a \), and \( \mathbf{k} \) is a constant unit vector in the upward vertical direction. Further, \( \phi \) is the geopotential, and \( \theta = g \theta \theta_0 \), where \( \theta \) is potential temperature with constant reference value \( \theta_0 \), and gravitational acceleration \( g \). Equations (3.1)–(3.5) represent a vector formulation of Eq. (10) given by Hoskins and Bretherton (1972) based on the Hoskins and Bretherton (1972) pressure-related vertical pseudoheight coordinate \( z \). As in the SW case, these equations differ from
the unapproximated PEs only by neglecting the term \((d/dt)u_n\) in the momentum equation (3.1) resulting in the removal of gravity waves.

b. The vector derivation of the SG PV equation

For the vector-based derivation of the SG PV equation relevant for the above geostrophic-momentum-approximated PEs it is essential to recognize that the quantity \(\xi_e\) [introduced by Hoskins (1975) in (ii) after his Eq. (10)] that appears in the definition of the SG PV denoted by \(q_e\), defined as

\[
q_e = \xi_e \cdot \nabla \theta, \tag{3.6}
\]

may be written, by using the nondivergence of \(v_e\), in vector form as

\[
\xi_e = \nabla \times v_e, \tag{3.7}
\]

with the following definitions:

\[
\xi_e = \nabla \times v_e, \tag{3.8}
\]

\[
2A = (\nabla v_e) \cdot (\nabla v_e), \tag{3.9}
\]

\[
R = k \times [(k \times \xi_e) \cdot \nabla (k \times v_e)]. \tag{3.10}
\]

The equation governing the time evolution of \(\xi_e\) representing a form of vorticity equation is obtained by time differentiating the defining relation (3.7):

\[
\frac{\partial \xi_e}{\partial t} = \frac{\partial \xi_e}{\partial t} - \frac{1}{f_0} \frac{\partial A}{\partial t} + \nabla \bullet \theta = 0, \tag{3.11}
\]

To express the time tendencies on the rhs of (3.11), the equation for the geostrophic vorticity is first formed by taking the curl of the SG equation of motion (3.1) as

\[
\frac{\partial \xi_e}{\partial t} + \nabla \times (u \cdot \nabla v_e + f_0 k \times u_k) = 0, \tag{3.12}
\]

which is rewritten, through use of \(u = v_e + u_n\) and use of Eqs. (A.3) and (A.4) (see appendix A) and by observing that both \(v_e\) and \(\xi_e\) are nondivergent, as

\[
\frac{\partial \xi_e}{\partial t} + v_e \cdot \nabla \xi_e - \xi_e \cdot \nabla v_e + \nabla \times (u_n \cdot \nabla v_e)
\]

\[
- f_0 k \cdot \nabla u + k \times \nabla \theta = 0, \tag{3.13}
\]

where (3.5) was used to express \(\nabla \times (f_0 k \times u_n)\), in addition to observing (3.2). Further, observing (3.9) and proceeding analogously to the derivation of (2.7), the time tendency of \(A\) follows from (3.1) as

\[
\frac{\partial A}{\partial t} + (\nabla v_e) \cdot [\nabla (u \cdot \nabla v_e)]
\]

\[
+ (\nabla v_e) \cdot [\nabla (f_0 k \times u_n)] = 0. \tag{3.14}
\]

In addition, taking the time derivative of (3.10) leads to

\[
\frac{\partial R}{\partial t} = k \times \left[ \left( k \times \frac{\partial \xi_e}{\partial t} \right) \cdot \nabla (k \times v_e) \right]
\]

\[
+ k \times \left[ \left( k \times \xi_e \right) \cdot \nabla \left( k \times \frac{\partial v}{\partial t} \right) \right], \tag{3.15}
\]

which by inserting (3.12) and (3.1), respectively, becomes

\[
\frac{\partial R}{\partial t} = -k \times \left[ \{ k \times [\nabla \times (u \cdot \nabla v_e)] \} \cdot \nabla (k \times v_e) \right]
\]

\[
- f_0 k \times \{ [k \times \{ k \times (k \times u_n) \}] \cdot \nabla (k \times v_e) \}
\]

\[
+ f_0 k \times \{ [k \times \xi_e] \cdot \nabla u_n \}
\]

\[
- k \times \{ [k \times \xi_e] \cdot \nabla [k \times (u \cdot \nabla v_e)] \}. \tag{3.16}
\]

The vorticity equation for \(\xi_e\) is obtained on the basis of (3.11) by inserting Eqs. (3.13), (3.14), and (3.16) into (3.11) and collecting terms as

\[
\frac{d\xi_e}{dt} = \xi_e \cdot \nabla v_e + f_0 k \cdot \nabla u - k \times \nabla \theta + M
\]

\[
- u \cdot \nabla \left( \frac{A}{f_0} \right) + u \cdot \nabla \left( \frac{R}{f_0} \right) + \frac{1}{f_0} N, \tag{3.17}
\]

with the following abbreviations for the vectors \(M\) and \(N\):

\[
M = u_n \cdot \nabla \xi_e - \nabla \times (u_n \cdot \nabla v_e)
\]

\[
+ k(\nabla v_e) \cdot [\nabla (k \times u_n)]
\]

\[
+ k \times \{ [k \times \xi_e] \cdot \nabla u_n \}
\]

\[
- k \times \{ [k \times \nabla (k \times u_n)] \cdot \nabla (k \times v_e) \}, \tag{3.18}
\]

\[
N = k(\nabla v_e) \cdot [\nabla (u \cdot \nabla v_e)]
\]

\[
- k \times \{ [k \times \nabla (u \cdot \nabla v_e)] \cdot \nabla (k \times v_e) \}
\]

\[
- k \times \{ [k \times \xi_e] \cdot \nabla [k \times (u \cdot \nabla v_e)] \}. \tag{3.19}
\]

On the basis of identity (A.4) (see appendix A) and observing (3.2), result (3.17) is reexpressed as

\[
\frac{d\xi_e}{dt} = \xi_e \cdot \nabla v_e + f_0 k \cdot \nabla u - k \times \nabla \theta + M + \frac{N}{f_0}
\]

\[
+ \nabla \times \left( \frac{R}{f_0} - \frac{A}{f_0} k \right) \times u + u \nabla \left( \frac{R}{f_0} - \frac{A}{f_0} k \right)
\]

\[
+ \left( \frac{R}{f_0} - \frac{A}{f_0} k \right) \cdot \nabla u \tag{3.20}
\]

and is further, on the basis of result (C.1) (see appendix C) and by referring back to (3.7), rewritten as
\[
\frac{d\zeta_s}{dt} = \zeta_s \cdot \nabla u - k \times \nabla \theta - \zeta_s \cdot \nabla u_n + M \\
+ \frac{1}{f_0}[(\nabla u) \cdot (\nabla v)](k \times \zeta_s).
\] (3.21)

Use of the divergence equation (C.3) together with result (C.2) (see appendix C) implies that the sum of the last three terms in Eq. (3.21) vanishes, resulting in
\[
\frac{d\zeta_s}{dt} = \zeta_s \cdot \nabla u - k \times \nabla \theta,
\] (3.22)
as the desired equation that governs the time evolution of \(\zeta_s\), which is precisely the result stated by Hoskins (1975) [his result (ii) after his Eq. (10)]. Finally, the SG PV equation relevant for the PE model, namely,
\[
\frac{dQ_s}{dt} = 0,
\] (3.23)
is immediately obtained for the SG PV \(Q_s\) defined in (3.6) through the standard combination of Eqs. (3.22) and (3.3), while observing results (A.5) and (A.6) (see appendix A).

4. Remarks

A direct vector-based derivation of the SG PV equation has been illustrated in the context of the SW and the PE model. In both models, this technique proceeds directly from a vector formulation of \(\xi_{SG}\) or \(\zeta_s\), respectively.

This technique makes use of vector formulations of the governing SG equations and, in the SW context, avoids the explicit componentwise expression of \(v_s\) in terms of \(v_s\) on the basis of the momentum equation (see, Allen et al. 1990). As such, this vector-based technique appears applicable for the derivation of relationships that involve quantities whose functional vector form is known a priori, such as \(\xi_{SG}\) or \(\zeta_s\). Clearly, the extent to which dynamically important quantities are known in functional form a priori poses a limitation on the discussed methodology, as does the necessity to verify highly involved vector identities such as (B.2) and (B.3) for the SW model or (C.1) and (C.2) for the PE model. Nevertheless, proceeding directly and in vector-based notation from the governing equations, as done here, seems to provide for a clearer picture of the nature of the final result.

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APPENDIX A

Standard Vector Identities

1) In Cartesian tensor notation (see, e.g., Simmonds 1994), the double contraction of the outer product of two second-order tensors \(P\) and \(Q\) is written as
\[
P \cdot Q = P_{ij}Q_{ij},
\] (A.1)
where summation over both \(i\) and \(j\) (from one to three) is implied [see, e.g., section 7.31 of Aris (1962) or section 2.3 of Pichler (1997)]. In this notation, the double contraction expression \((\nabla v_s) \cdot (\nabla v_s)\) introduced in Eq. (2.4) becomes
\[
(\nabla v_s) \cdot (\nabla v_s) = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial v_{si}}{\partial x_j} \frac{\partial v_{sj}}{\partial x_i},
\] (A.2)
where \(v_{si}\) denotes the \(i\)th component of \(v_s\), and coordinates are denoted by \(x_i\). Since \(v_s\) is two-dimensional in the PE model context, too, expression (A.2) applies also for the definition of the quantity \(2A\) in Eq. (3.9).

2) The following identities for vectors \(A\) and \(B\) may be found in section 3.24 of Aris (1962) or chapter 4 of Spiegel (1959):
\[
A \cdot \nabla A = \frac{1}{2} \nabla (A \cdot A) - A \times (\nabla \times A)
\] (A.3)
\[
B \cdot \nabla A - A \cdot \nabla B = \nabla \times (A \times B) + B \nabla \cdot A - A \nabla \cdot B.
\] (A.4)

3) The interchanging relationship
\[
\nabla \frac{d}{dt} s - \frac{d}{dt} \nabla s = (\nabla u) \cdot \nabla s
\] (A.5)
holds in conjunction with definition (3.4) for an arbitrary scalar \(s\) [see, e.g., Eq. (20) in Kahlig 1974].

4) Observing the order of scalar products as indicated by parentheses, the following identity holds (see also, section M2.4.3 in Zdunkowski and Bott 2003):
\[
(\nabla s) \cdot (A \cdot \nabla B) = A \cdot [(\nabla B) \cdot (\nabla s)].
\] (A.6)

APPENDIX B

Vector Identities for the SW Model

1) For a two-dimensional vector, such as \(v_{s}\) (with third component identical to zero), constant \(k\) allows for viewing the result
\[
k \cdot \nabla v_{s} = (k \times v_{s}) \cdot v_{s}
\] (B.1)
as a consequence of Eq. (A.4).

2) Using \(\nabla v_{s} = \nabla_{s} v_{s} = 0\) in addition to explicit use of \(v = v_{s} + v_{a}\), the following identity can be shown to be true:
\[
k \cdot \nabla_{s} (v \cdot \nabla_{s} v_{s}) - (\nabla_{s} v_{s} \cdot \nabla_{s} (k \times v_{s}))
\] (B.2)
3) Using the nondivergence of \(v_{s}\), the following identity can be shown to be true:
\begin{equation}
\n\left( \nabla \cdot (\mathbf{u} \times \nabla \cdot \nabla) \right) = \frac{1}{2} \nabla \cdot \left\{ \nu \left[ (\nabla \cdot \nabla) \right] \right\}. \quad (B.3)
\end{equation}

4) Proof of identity (B.2): Differentiating the term on the rhs of (B.2) and including one of the resulting terms, namely, \( \mathbf{v} \cdot \nabla \mathbf{z} \), on the left-hand side (lhs), yields the following expression in Cartesian coordinates, where \( x \) and \( y \) denote horizontal coordinates and \( u, u_x, u_y \) and \( v, v_x, v_y \), denote zonal and meridional full, geostrophic, and ageostrophic wind components, respectively:

\begin{align}
\mathbf{k} \cdot \nabla x (\mathbf{v} \cdot \nabla \mathbf{z}) - (\nabla \cdot \nabla) & \cdot [\nabla \cdot (\mathbf{k} \times \mathbf{v}_s)] - \mathbf{v} \cdot \nabla \mathbf{z} \\
& = \frac{\partial}{\partial x} \left( u \frac{\partial u_x}{\partial x} + v \frac{\partial u_y}{\partial y} \right) - \frac{\partial}{\partial y} \left( u \frac{\partial u_x}{\partial x} + v \frac{\partial u_y}{\partial y} \right) \\
& \quad - \frac{\partial u_x}{\partial x} \frac{\partial u_y}{\partial x} - \frac{\partial u_y}{\partial y} \frac{\partial u_x}{\partial y} - \frac{\partial v_x}{\partial x} \frac{\partial v_y}{\partial x} - \frac{\partial v_y}{\partial y} \frac{\partial v_x}{\partial y} \\
& \quad - \frac{\partial u_x}{\partial x} \frac{\partial v_x}{\partial x} - \frac{\partial u_y}{\partial y} \frac{\partial v_y}{\partial y}, \quad (B.4)
\end{align}

where in the second equality cancellations were observed and \( \mathbf{v} \) was replaced by \( \mathbf{v} - \mathbf{v}_x \). Observing further the pairwise cancellations on the basis of the nondivergence of \( \mathbf{v}_s \), as indicated by the underbraced numbers, as well as observing that the sum of the remaining terms is just \( \xi_x \cdot \nabla \cdot \mathbf{v} \), as indicated, establishes the correctness of identity (B.2).

5) Proof of identity (B.3): Using the same notation for the wind components as in (B.4), the lhs and rhs of (B.3) may separately be written as

\begin{align}
(\nabla \cdot (\mathbf{u} \times \nabla) \cdot (\nabla \cdot \nabla)) & = \frac{\partial}{\partial x} \frac{\partial u_x}{\partial x} + \frac{\partial}{\partial y} \frac{\partial u_y}{\partial y} + \frac{\partial}{\partial x} \frac{\partial v_x}{\partial x} + \frac{\partial}{\partial y} \frac{\partial v_y}{\partial y} \\
& \quad + \frac{\partial}{\partial x} \frac{\partial u_y}{\partial y} + \frac{\partial}{\partial y} \frac{\partial u_x}{\partial x} + \frac{\partial}{\partial x} \frac{\partial v_y}{\partial y} + \frac{\partial}{\partial y} \frac{\partial v_x}{\partial x}, \quad (B.5)
\end{align}

and

\begin{align}
\frac{1}{2} \nabla \cdot \left\{ \nu [ (\nabla \cdot \nabla) \right\} & = \frac{1}{2} \frac{\partial}{\partial x} \left( \left( \frac{\partial u_x}{\partial x} \right)^2 + 2 \frac{\partial u_x}{\partial y} \frac{\partial v_x}{\partial y} + \left( \frac{\partial v_x}{\partial y} \right)^2 \right) \\
& \quad + \frac{1}{2} \frac{\partial}{\partial y} \left( \left( \frac{\partial u_y}{\partial x} \right)^2 + 2 \frac{\partial u_y}{\partial x} \frac{\partial v_y}{\partial x} + \left( \frac{\partial v_y}{\partial x} \right)^2 \right). \quad (B.6)
\end{align}

Detailed inspection of expressions (B.5) and (B.6) reveals immediately that terms containing \( u \) and \( v \) as undifferentiated factors are the same in both expressions, thus canceling in the difference of (B.5) and (B.6) and leaving for the difference, when observing the nondivergence of \( \mathbf{v}_s \),

\begin{align}
(\nabla \cdot (\mathbf{u} \times \nabla) \cdot (\nabla \cdot \nabla)) & = \frac{1}{2} \frac{\partial}{\partial x} \left( \left( \frac{\partial u_x}{\partial x} \right)^2 + 2 \frac{\partial u_x}{\partial y} \frac{\partial v_x}{\partial y} + \left( \frac{\partial v_x}{\partial y} \right)^2 \right) \\
& \quad + \frac{1}{2} \frac{\partial}{\partial y} \left( \left( \frac{\partial u_y}{\partial y} \right)^2 + 2 \frac{\partial u_y}{\partial x} \frac{\partial v_y}{\partial x} + \left( \frac{\partial v_y}{\partial x} \right)^2 \right) \\
& \quad - \frac{1}{2} \left( \frac{\partial u_x}{\partial x} \right)^2 - \left( \frac{\partial u_y}{\partial y} \right)^2 = 0. \quad (B.7)
\end{align}

Nondivergence of \( \mathbf{v}_s \) establishes that the difference is equal to zero, which is the case since the term in brackets in the second equality vanishes. Finally, given result (B.7), the correctness of identity (B.3) is established.

**APPENDIX C**

**Vector Identities for the PE Model**

1) Through use of the definitions in section 3, in particular Eqs. (3.19), (3.10), (3.9), and (3.8), the identity

\[ \mathbf{N} + \nabla \times [(\mathbf{R} - \mathbf{A}) \times \mathbf{u}] + \mathbf{u} \cdot (\mathbf{R} - \mathbf{A}) \]

\[ = [\nabla (\mathbf{u}) \cdot (\nabla \cdot \mathbf{z})] (\mathbf{k} \times \xi_s) \]

(C.1)

may be verified using the nondivergence of \( \mathbf{v}_s \). Referring, in addition, to (3.18) the result

\[ \mathbf{M} + [\mathbf{k} \cdot (\nabla \times \mathbf{u})] (\mathbf{k} \times \xi_s) = \xi_s \cdot \nabla \mathbf{u} \]

(C.2)

may be obtained. Results (C.1) and (C.2) may be verified for all three components by considerable manipulation starting from Cartesian tensor notation in a manner analogous to the verification of results (B.2) and (B.3) in appendix B.
2) Further, it is important to note that $\nabla \cdot v_g = 0$ implies the following divergence equation resulting from the PE momentum equation (3.1):
\[
\langle \nabla u \rangle \cdot (\nabla v_g) - f_0[k \cdot (\nabla \times u)] = 0. \quad (C.3)
\]

REFERENCES


