Objective Calculations of Divergence, Vertical Velocity and Vorticity

JOHN C. BELLAMY *

Department of Meteorology, University of Chicago

ABSTRACT

Methods of calculating horizontal divergence, vertical velocity and vorticity directly from wind observations without analyzing the wind field are presented. The appropriate formulae for these quantities in terms of the finite areas or volumes determined by the observation points are derived. Convenient nomographs and tabular forms for their calculation are described and illustrated.

INTRODUCTION

SOME methods of calculating the horizontal divergence, vertical velocity and vorticity directly from wind observations without first analyzing the wind field are described here. For these calculations volumes or areas determined by the actual locations of the points of observation, rather than by arbitrary coordinate systems, are used. Convenient volumes of this type are of unit vertical extent and bounded horizontally by the sides of triangles formed by any three non-collinear observation points.

When three adjacent observation points, say A, B and C, are used to define the triangular volume, essentially nothing is known of the wind distribution between these points. A convenient assumption for this distribution is that the wind field is a linear function of space between the points. This assumption allows us to treat each wind observation separately. For example, first it can be considered that the wind at point A is as observed but that the winds at points B and C are calm. The “partial horizontal divergence,” $\Delta_A$, for these conditions can then be calculated. Similarly the partial horizontal divergence, $\Delta_B$, for the wind at B and calm at A and C, and the value of $\Delta_C$, for the wind at C and calm at A and B can then be calculated. The horizontal divergence, $\Delta$, from the triangular volume is then the sum of these partial divergences, or

$$\Delta = \Delta_A + \Delta_B + \Delta_C.$$  

HORIZONTAL DIVERGENCE

The horizontal projection of a typical triangular volume with observation points at A, B and C is represented in Figure 1. The wind observation at point A is represented by the vector $AA'$, plotted as the straight line distance on the map that the air could travel in the direction of the wind in any convenient arbitrary time $T$. If then the winds at points B and C were calm the rate of outflow of air, in $T$ hours, from the volume of unit thickness is represented by the area $A'AB + A'AC = A'BC - ABC$. Defining the partial horizontal divergence, $\Delta_A$, as the rate of outflow under these conditions through the unit vertical sides of the volume, per unit volume, we have that

$$\Delta_A = \frac{A'BC - ABC}{ABC}. \quad (2)$$

Defining $h_A, h_A'$ and $n_A$ as shown in Figure 1 we have that $A'BC = (BC)h_A'$ and $ABC = (BC)h_A$, or that

$$\Delta_A = \frac{h_A' - h_A}{h_A},$$

* Present address: Cook Research Laboratories, Inc., 1457 Diversey Parkway, Chicago, Ill.
Equation (3) can be solved conveniently with the aid of a nomograph as illustrated in Figure 2.

![Figure 2](image_url)

The solid lines represent the lines on a transparent overlay to be placed on a map containing a plot (dashed lines) similar to Figure 1. This overlay is constructed as follows:

1. The rectangular grid is entirely arbitrary, being used only to give directions.
2. The lines radiating from point P are drawn with an arbitrary uniform spacing along any (and all) vertical lines, such as line RS.
3. The horizontal radiating line, PQ, is labelled −1.0 and the other radiating lines are numbered consecutively (in tenths in the illustration) from line PQ.

These numbers are then the values of $\Delta_A$ if the following procedure is used:

1. The overlay is placed on the triangle considered with the −1.0 line (line PQ) along the base of the triangle (line BC) and with the observation point being considered (point A) on the O-line (line PO).
2. Point $A''$ is located at the intersection of the horizontal line through point $A'$, the end point of the wind vector, and the vertical line through point A.

Equation (3) can be solved conveniently with the aid of a nomograph as illustrated in Figure 2.

3. The value of the partial horizontal divergence, $\Delta_A = \frac{h_A'}{h_A} - 1$, is the value corresponding to the radiating line passing through point $A''$. In the example shown this is +0.5, corresponding to half the volume considered flowing out of the volume in the time $T$.

In similar fashion the values of $\Delta_B$ and $\Delta_C$ can be obtained with the nomograph, using the observed winds at points B and C plotted from their respective observation points. The horizontal divergence from this triangular volume is then the algebraic sum of these three values, as in Equation (1), with negative values corresponding to convergence, or flow into the volume.

The values obtained in this way are in units of the reciprocal of the time unit, $T$, for which the winds are plotted. For example, if the winds are plotted to represent straight-line displacements on the map in a three hour period, the numbers represent fractional outflows per three hours. They can be converted to, say, fractional outflows per second by dividing by $3 \times (60 \times 60) = 10,800$.

Since the construction and use of this overlay is independent of the scale of the map or the scale for which the winds are plotted it is convenient for use in:

1. Investigations of small scale phenomena where the relative positions of the observation points (balloon positions) cannot be considered as constant so that different triangles are used for each set of observations at each level.
2. Investigations using observations from varying positions such as from weather reconnaissance aircraft.
3. Investigations using synoptic reports of surface or upper air winds in which the calculations are made too infrequently to warrant the construction of special overlays or calculation tables for given fixed triangles.

A convenient graphical method of making many calculations of the horizontal divergence from a triangle of essentially fixed points of observation is illustrated in Figure 3. The solid lines represent lines drawn on a transparent overlay. Each set of lines is drawn parallel to one side of the triangle with the −1 line along that side. The 0 line of each set is drawn through the opposite observation point. The other lines of each set are then equally spaced, the spacing being determined by the spacing of the −1 and 0 lines. If then the
winds are plotted on the map as straight line T-hour displacements (dashed lines) the partial horizontal divergences can be read directly at their end points \(\Delta_A = +0.5, \Delta_B = -0.6, \Delta_C = -0.7\), in the illustration) and their sum \(\Delta = -0.8\) is the value of the horizontal divergence. This overlay can also be placed over separate hodographs of the three wind observations to determine rapidly the horizontal divergence at each level of the pilot balloon reports.

If numerical tables are preferred for routine computations, a double entry (wind direction \(\alpha\), and wind speed \(v\)) table can be constructed for each station of each triangle. It is seen from Equation (4) that the partial horizontal divergence is equal to the component of the wind normal to the opposite side of the triangle divided by the corresponding altitude of the triangle. Thus the values in the tables can be obtained from the formula

\[
\Delta_A = \frac{v}{h_A} \sin (\beta_{BC} - \alpha),
\]

where \(\beta_{BC}\) is the azimuth of the opposite side of the triangle. If \(v\) is in miles per hour and \(h_A\) is in miles, \(\Delta_A\) is in \((\text{hrs})^{-1}\). Using three such tables the horizontal divergence in any given triangle can be rapidly obtained directly from the wind reports without the necessity of plotting the reports graphically.

**Vertical Velocities**

For simplicity in calculating vertical velocities from horizontal convergence it is assumed that horizontal density gradients and local density changes can be neglected. Many times it is probably sufficiently accurate to neglect also the effect of the vertical density gradient. In this case the continuity equation reduces to the statement that the horizontal divergence through the unit-thick sides of the volume considered is equal to the difference of the vertical velocity, \(W_2\), down into the top of the volume and the vertical velocity, \(W_1\), out of the bottom of the volume, or

\[
\Delta = W_2 - W_1.
\]

The sign of the vertical velocity has been chosen as positive downward for convenience of calculation.

For use with wind reports for each 1000-foot level it is convenient to choose the unit height of each volume to be 1000 feet, extending between the reported levels. The value of \(\Delta\) in Equation
(6) should then be used as the average value of $\Delta$ at the two levels considered, or

$$W_2 = W_1 + \frac{\Delta_1 + \Delta_2}{2}$$

or

$$2W_2 = 2W_1 + (\Delta_1 + \Delta_2). \quad (7)$$

$W_2$ and $W_1$ are then the vertical velocities, positive downward, at the top and bottom levels, respectively, in units of 1000 feet per $T$ hours when $\Delta$ is given in units of fractional flow per $T$ hours.

The boundary condition that the vertical velocity at the surface of the earth is zero then permits the calculation, with Equation (7), of the vertical velocity at each of the reported levels. The following procedure has been found to be convenient.

1. Determine the values of $\Delta_A$, $\Delta_B$, and $\Delta_C$ for each reported level, including the surface.
2. Add these values at each level to obtain the values of $\Delta$.
3. Calculate the sum of each adjacent pair of values of $\Delta$. That is, calculate $\Delta_{sfc} + \Delta_{1000}$ ft, $\Delta_{1000}$ ft $+ \Delta_{2000}$ ft, etc.
4. Multiply the first of these $(\Delta_{sfc} + \Delta_{1000}$ ft) by $1000 - Z_0/1000$ where $Z_0$ is the average elevation, in feet, of the surface at the three observation points. (If $Z_0$ is between 1000 ft and 2000 ft the multiplicative factor is $2000 - Z_0/1000$, etc.) This value is then the value of $2W_{1000}$ ft since $W_{sfc}$ is considered to be zero.
5. Add $\Delta_{1000}$ ft $+ \Delta_{2000}$ ft to $2W_{1000}$ ft, thus obtaining the value of $2W_{2000}$ ft. The cumulative totals obtained by successively adding the values of step 3 to this sum are then the values of twice the vertical velocities at each 1000-foot level. These values can be interpreted directly as the downward vertical velocities at the various levels expressed in units of 500 feet per $T$ hours.

To include the effect of the variation of the density in the vertical the volume transport equality, Equation (7), should be changed to a mass transport equality, as

$$2\rho_2W_2 = 2\rho_1W_1 + (\rho_1\Delta_1 + \rho_2\Delta_2),$$

or

$$2W_2 = 2W_1 \frac{\rho_1}{\rho_2} + \left(\frac{\rho_1}{\rho_2} + \Delta_2\right). \quad (8)$$

The calculation procedure could then be:

1. Calculate the values of $\Delta_A$, $\Delta_B$, and $\Delta_C$ for each station at each level.
2. Calculate the values of $\Delta$ at each level.
3. Calculate the values of $(\rho_1/\rho_2)\Delta$, at each level. It is usually sufficient to use the constant value of $\rho_1/\rho_2 = 1.03$ for all levels since in the U. S. Standard Atmosphere $\rho_9$ ft/$\rho_{1000}$ ft $= 1.030$ while $\rho_{19,000}$ ft/$\rho_{20,000}$ ft $= 1.035$.
4. Calculate the values of $\left(\Delta_1 + \frac{\rho_1}{\rho_2}\Delta_2\right)$ for each successive pair of levels.
5. Multiply the first of these

$$\left(\Delta_{sfc} - \frac{\rho_{sfc}}{\rho_{1000}} + \Delta_{1000}\right)$$

by $\frac{1000 - Z_0}{1000}$ to obtain $2W_{1000}$ ft.
6. Multiply $2W_{1000}$ ft by $\frac{\rho_{1000}}{\rho_{2000}}$ and add to

$$\left(\Delta_{1000} - \frac{\rho_{1000}}{\rho_{2000}} + \Delta_{2000}\right)$$

to obtain $2W_{2000}$ ft.
7. Continue this process for each reported level.

VORTICITY

The vertical component of the vorticity, $\xi$, is defined as the line integral of the tangential component of the wind around a horizontal area divided by that area. Since a rotation of a wind by ninety degrees changes tangential components to normal components the graphical methods of computing divergence can be used for computing vorticity merely by plotting the winds normal to their reported directions. Positive divergence values obtained in this way correspond to counterclockwise vorticity if each wind is rotated ninety degrees to the right when facing downstream. The vorticity is then obtained in units of radians per $T$ hours.

For tabular vorticity computations, Equation (5) can be altered to

$$\xi_A = -\frac{v}{h_A} \cos (\beta_{BC} - \alpha), \quad (9)$$

where $\xi_A$ is the partial vertical component of the vorticity in radians per hour and positive values are counterclockwise. The vertical component of vorticity is then obtained from the sum,

$$\xi = \xi_A + \xi_B + \xi_C. \quad (10)$$

It should be noted that tables for the horizontal divergence can also be used for the vorticity by entering with wind direction values ninety degrees greater than reported.
VALIDITY OF THE CALCULATIONS

Since these computations involve the calculation of line integrals around horizontal areas, the assumption of a linear wind distribution between points is essentially only an assumption that the average value of the wind between two points is approximately the average of the values at the end points. Thus their validity depends primarily upon the amplitude of the wind variations of a spatial scale smaller than the triangle used. Almost certainly the values obtained in this way occur somewhere in the chosen area. In the lack of more closely-spaced observations the most likely place for its occurrence is at the centroid (the intersection of the medians) of the triangle. Thus these values can be placed at the centroids of their respective triangles for analysis of their spatial distributions. These methods should give more representative values, and are certainly easier to use, than methods which depend upon interpolations between points of observations.

Calculations of horizontal divergence and vertical velocities by the author, using pilot balloon and aircraft reports, have indicated that representative patterns, as qualitatively checked with cloud and precipitation reports, are obtained. Extensive calculations of divergence, using small triangles in thunderstorm studies have been made by Byers and Hull.\(^1\) Also, quite satisfactory results, using pilot-balloon reports from widely-scattered points in Florida, have been obtained by Rodebush.\(^2\)


### Daily Sunspot Numbers for October and November 1948

The American relative ($R_A$) and provisional relative Zurich ($R_Z$) sunspot numbers are reduced by the National Bureau of Standards from observations of the members of the American Association of Variable Star Observers and others; furnished by courtesy of the Solar Division of the AAVSO, N. J. Heines, Director. (See Sept. 1948, *Bull.*, p. 381.)

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