A Continuously Stratified Nonlinear Ventilated Thermocline

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ABSTRACT

Three exact, closed-form analytical solutions for the subtropical gyre are presented for the ideal fluid thermocline equations. Specifically, the flow is exactly geostrophic, hydrostatic, and mass and buoyancy conserving. Ekman pumping and density can be chosen as fairly arbitrary functions at the surface. No flow is permitted through the ocean’s eastern boundary, or through its bottom. The solutions are continuous extensions of existing layered models. The first solution, discovered simultaneously with Janowitz’s solution, uses a deep resting isopycnal layer; the surface density may only be a function of latitude for this solution. A second nonunique solution requires velocities to tend to zero at great depth, giving an additional degree of freedom which permits surface density to be specified almost arbitrarily. This second solution is unphysical in the sense that depth-integrated mass fluxes and energies are infinite. However, a small change in the solution (which returns surface density to a function of latitude only) permits solutions with finite fluxes once more. A third solution requires partial homogenization of the potential vorticity of fluid layers which, while overlying a deep resting isopycnal layer, are not directly ventilated from the surface. Again, fairly arbitrary surface density and Ekman pumping are permitted. All the problems reduce to linear homogeneous second-order differential equations when density replaces depth as the vertical coordinate. The importance of the bottom boundary for closing the problem is stressed.

1. Introduction

One of the main goals of physical oceanography has been to understand the way in which the ocean responds to wind and buoyancy forcing at its surface. A great deal of work has been published on a model set of equations, known as the “thermocline equations” (e.g., Welander, 1959; Robinson and Stommel, 1959; or see the review by Veronis, 1969), describing the advection of buoyancy by an ideal geostrophic fluid.

Similarity solutions for the thermocline equations have been thoroughly studied, cf. Welander (1971). These solutions cannot satisfy the boundary conditions believed to hold for them, so that only very special, and often unrealistic, forcing functions have been used. The “standard” set of surface boundary conditions is to specify the Ekman pumping $w_E$ and the density $\rho_S$ at the base of the mixed layer (both are frequently functions of latitude only), and to require no normal flow either through the floor of the ocean or through the eastern boundary. Because of many technical difficulties, this last condition is frequently relaxed.

Recently, oceanographic datasets have become sufficiently accurate that maps of potential vorticity on density surfaces can be used as indicators of geostrophic streamlines (e.g., McDowell et al., 1982; Kefler, 1985) in parallel with atmospheric studies. This has led to the production of several inverse methods (e.g., Wunsch, 1978; Schott and Stommel, 1978) which involve a “best fit” to the idealised dynamics of the thermocline model, and which suggest that such dynamics may describe much of the interior of ocean gyres.

Luyten et al. (1983, and later papers with co-workers), arguing that the similarity solutions were too constrictive, abandoned continuous fluids in favor of a multilayered model above a resting deep isopycnal layer. At a stroke, this ingenious simplification reduced the solution of the problem to straightforward, though tedious, algebra. Isopycnals (now layer interfaces) could be followed as they descended in the subtropical gyre, and deep flow was permitted. The technical problem concerned with the eastern boundary condition mentioned above caused some problems for these papers. It is easy to show for a layered fluid that the depths of all but one of the layers must vanish on the eastern boundary if there is no geostrophic flow normal to that boundary. Luyten et al. (1983) chose the depth of the remaining layer to be a nonzero constant. This was an explicit choice; the correct boundary condition remains unknown. Because this upper layer had a nonzero depth, an unventilated “shadow” region could extend westwards and southwards from the eastern boundary, comprising fluid of the same density as other, moving fluid, save that in the shadow region the fluid had never been in contact with the forcing. As the depth of the eastern layer shallowed, so the area occupied by the shadow region diminished.

Part of the problem is that the northward fluid ve-
Velocity in the lower layers near the eastern coast in the model of Luyten et al. (1983) becomes infinite as the coast is approached, although the model has finite energy. This problem is generic in ideal fluid theory, and will recur here.

Such technical problems recur when considering a continuously stratified model. Killworth (1983; cf. Young and Ierley, 1986) demonstrated that the only eastern boundary conditions for a fluid with an infinitely differentiable density field are either uniform density on the boundary or globally stagnant flow; neither condition is particularly physical. (A singularity in density of some kind must occur at the surface on the eastern boundary, since we apply at the surface a density varying northward, and yet require no northward gradient of density on the wall because of thermal wind.) One can avoid the problem either by introducing nondifferentiability at some point, or nonideal dynamics. No suitable eastern boundary dynamics are yet formulated for such problems.

As the number of layers in Luyten et al.'s model becomes very large, either the depth of the one nonzero layer must tend to zero, or the boundary condition must be relaxed (cf. Pedlosky, 1983, for the latter case). Within the confines of the geostrophic model, one would probably prefer the former option, particularly in the Luyten et al. model, since the depth of the layer which outcrops exactly on the gyre boundary should be zero there by definition, and hence zero down the eastern boundary by geostrophy.

It is thus natural to require all isopycnals to surface on the eastern boundary for a continuous model (cf. Young and Ierley, 1986), as this removes the problem of any flow through the eastern boundary, at the cost of an unrealistic circulation there. It also removes the possibility of trajectories leaving the eastern boundary, implying the lack of shadow zones for an ideal continuous fluid.

Assume for the moment that the surface conditions involve \( \rho_S \) being a function of latitude only. Luyten et al.'s (1983) solution would take a similarity form, with all quantities functions of \( x \) (measured east from the eastern boundary), \( y \) (north) and a similarity variable \( z(\bar{x})^{-1/2} \), where \( z \) is vertical. We shall use a more general form of this to permit a more general form for \( w_y \). Solutions of this form occur naturally if one examines solutions which allow vertical and horizontal stretching, together with rescaling (cf. Killworth, 1983, or solutions near the eastern boundary in appendix A). The resulting functional relationship between linear potential vorticity \( q \), density \( \rho \) and linear Bernoulli function \( B \) (introduced by Wandel, 1971) now becomes

\[
qB = \text{function}(\rho). \quad (1.1)
\]

The resulting equations are, however, still fully nonlinear despite only involving two independent variables rather than three.

A drastic analytical simplification to the problem occurs if density is used as an independent variable (cf. Killworth, 1986a for another use of this technique, which was initiated by Hodnett, 1978 for the thermocline equations). A paper by Janowitz (1986), which appeared while this paper was being written, uses the technique to produce similar solutions to those in section 2. Equation (1.1) reduces to a linear homogeneous second-order ordinary differential equation for the Bernoulli function [also discovered independently by Long (private communication) at about the same time as Janowitz and the present paper]. The deep boundary condition and the surface density condition are both homogeneous, so that the arbitrary scaling on the solution can be used to satisfy the surface Ekman pumping condition, which supplies the amplitude of the depth structure for the system. Only under simple circumstances (longitude independent forcing functions) is this scaling local; integration along characteristics of constant surface density is normally necessary.

Three families of solutions are presented. Each can be solved only for subtropical, or nearly subtropical, gyres (i.e., ones for which the surface Ekman pumping is predominantly downward). For the case of a deep resting layer (section 2, which reproduces Janowitz's solution, with some extensions), any given function of density in (1.1) implies a specific structure for \( \rho_S(y) \); the inverse problem, to determine the function, given \( \rho_S(y) \), seems difficult. It is straightforward to find typical oceanic solutions where \( \rho_S(y) \), relative to the deep density value, varies inversely as a power of Coriolis vector \( f \) (ignoring the unrealistic surface values required in the vicinity of the equator), and one such solution, possessing continuous derivatives of density except at gyre boundaries, is given. However, appendix B shows that there are \( \rho_S(x, y) \) arbitrarily close to \( \rho_S(y) \) which do not have solutions close to those for \( \rho_S(y) \), so that the flow field may be a sensitive function of the surface conditions.

For an infinitely deep ocean, a traditional bottom boundary condition is the requirement that horizontal velocities tend to zero at depth (although numerically rather slowly). This permits an entire family of explicit solutions for any surface density (subject to a few physical limitations), in section 3. The solutions are not uniquely determined by this boundary condition. The vertically integrated mass transport is not finite in this latter model, so that the upwelling velocity tends to infinity at great depth. Nonetheless, the resulting density field is not unrealistic. A small change to the solution (which removes the extra freedom to choose surface density varying with \( x \)) removes the unphysical behavior at depth and yields finite transports.

The solutions of sections 2 and 3 include fluid layers which are in motion but are never directly ventilated at the surface. Instead of requiring these layers to satisfy (1.1), we can seek solutions in which their potential vorticity has been homogenized. The problem now in-
volves an active, ventilated layer above an active, but homogenized layer, which itself overlies a deep resting isopycnal layer. Nonunique solutions, with q discontinuous between the two layers, may readily be found, and an example is given in section 4. The region of homogenization grows westwards, and does not quite achieve the desired distribution of potential vorticity. It is shown that the westward extent of such a solution is limited by the increase in effective depth of the homogenized region, until it dominates the flow field.

Section 5 concludes with a discussion of the results and a comparison and connection with previous, more restrictive, solutions. The importance of the bottom boundary for closing the problem—and hence the importance of deep water sources—is stressed.

2. The problem for a deep resting layer

The thermocline equations as normally posed are fundamentally nonlinear (e.g., Welander, 1971). Some simplification is obtained by using density as an independent vertical coordinate in place of the depth z, as used by Hodnett (1978), Killworth (1986a), et c., and more recently by Janowitz (1986) whose work this section parallels. The Bernoulli function

\[ B = (p + \rho g z) / \rho_0 \]  

(2.1)

replaces pressure p as the basic dependent variable, where g is the acceleration due to gravity and \( \rho_0 \) the mean density. The actual density \( \rho \) is measured relative to some deep resting isopycnal value. The hydrostatic relationship becomes

\[ B_\rho = g z / \rho_0, \]  

(2.2)

goestrophy implies

\[ -f \nu = -B_x, \]  

(2.3)

\[ f \mu = -B_y, \]  

(2.4)

(\( f \) varies with \( y \); a beta-plane is not assumed, and the entire analysis applies to the spherical earth case). Buoyancy conservation gives, for an ideal fluid, the vertical velocity

\[ w = u z_x + v z_y, \]  

(2.5)

and mass conservation yields the linking equation

\[ (u z_\rho)_x + (v z_\rho)_y = 0, \]  

(2.6)

where \( z_\rho \) takes the role of depth \( h \) for a layered fluid. This can also be written in more fundamental form (Welander, 1971) as

\[ q = Q(\rho, B) \quad \text{for some } Q, \]  

(2.7)

where the linear potential vorticity \( q \) is given by

\[ q = f g / (\rho_0^3 B_\rho) = f / \rho_0^2 g_B = (f / \rho_0) \partial \rho / \partial z. \]  

(2.8)

Rewriting (2.7) gives

\[ B_\rho = -f F(\rho, B) \quad \text{for some } F, \]  

(2.9)

where the sign is natural, since \( B_\rho \) is negative.

The boundary conditions are taken as follows: on the surface \( z = 0 \), the density and vertical velocity take prescribed values

\[ w = w_F(x, y), \quad z = 0 \quad \text{on } \rho = \rho_S(x, y), \]  

(2.10)

while on the floor \( z = -H \), there is no upward flow

\[ w = 0 \quad \text{on } z = -H, \]  

(2.11)

although as we shall show, this can be replaced with a little care by

\[ B = u = v = w = 0, \quad \rho = 0, \]  

(2.12)

where \( \rho = 0 \) represents some arbitrarily chosen abyssal value of density. On the eastern boundary, we require no flow, so that

\[ u = 0, \quad x = 0. \]  

(2.13)

The boundary conditions require some comment. The eastern boundary condition for an ideal fluid is discussed by Killworth (1983), who concluded that if no flow passed through the eastern boundary and the solution was infinitely differentiable, then either the eastern boundary was an isopycnal or the fluid was at rest (a similar argument holds for a quasi-geostrophic regime below the mixed layer). This somewhat unphysical conclusion is extended by Young and Ierley (1986) who examined the effect of small diffusion on a simplified model equation. As diffusion becomes smaller, they show that some singularity—in their model a density jump, but in most cases a jump in density gradient only—must occur somewhere in the fluid. Since the surface density condition has \( \rho \) varying along the top of the eastern boundary, the remainder of which must be isopycnal, it is clear that a (benign) singularity must occur precisely along the top of the eastern boundary, and maybe elsewhere.

In this model, all isopycnals rise to intersect the line \( x = z = 0 \). Thus, both north–south flow and potential vorticity will become infinite at this corner, as in the Luyten et al. (1983) model, and there are no shadow zones. This will prevent the propagation of Rossby wave information from the eastern boundary to the interior. (Appendix A shows, inter alia, that it is possible to permit \( \int udz = 0 \) on the eastern boundary, as in Pedlosky (1983), without gross modifications to the problem.)

The bottom boundary condition is also worthy of note. (Janowitz uses it without comment.) Layered models (e.g., Luyten et al., 1983) usually examine flow above some deep, resting layer. If the surface of this lowest layer is at a height \( h \) above some horizontal surface, then the hydrostatic relationship requires the Bernoulli function \( B \) to have the value \( g'h \) in the layer immediately above the one at rest, where \( g' \) is reduced gravity. If the number of layers becomes infinite as the fluid becomes continuously stratified, \( g' \) tends to zero and so \( B \) must take the value zero on the interface \( \rho = 0 \). In this case \( u, v \) and \( w \) all vanish on the interface,
avoiding velocity jumps, and provided \( z_\rho \) also vanishes there, the gradient of density will be continuous also (a discontinuity only appearing in second derivatives of density).

Finally, there is assumed to be a western boundary layer whose properties are conveniently such that the layer can accept in the south and return further north water with essentially any required density and potential vorticity. In other words, the western boundary layer is assumed passive. This is obviously incorrect; the layer must help to set the functional in (2.9) by mixing both density and potential vorticity. In the case of an outcropping density strata (section 4) the work of Rhines and Young (1982) suggests that potential vorticity will be homogenized on density contours.

The problem, then, is to find a functional in (2.9), and a solution to the resulting equation, which can satisfy the conditions (2.10), (2.12) and (2.13). The similarity solutions to date (Wandel, 1971) require that \( \rho_S(x, y) \) and \( w_E(x, y) \) be related in some manner, and one would like to remove this requirement, since such solutions do not satisfy (2.13). There are two separate paths that lead to the solutions in this section. The first is to examine modifications to existing solutions (cf. Killworth, 1983). Suppose for the case of \( \rho_S \) and \( w_E \) independent of \( x \), we have some solution \( B(x, y, z) \) (reverting to Cartesian coordinates for the moment). Then another solution is \( B' = LB(x/L, y, z/L) \) for some constant \( L \). This particular transformation is designed to retain the same values of density and Ekman velocity at the surface \( z = 0 \). Under what conditions can (2.9) retain its form? Since \( q' = q/L \), it follows that (2.9) must reduce to

\[
B_{pp} = -fF(p)B, \quad (2.14)
\]

which is a linear second-order o.d.e. in \( B \) as a function of \( p \), with \( x \) and \( y \) entering only parametrically. Equation (2.14) in finite-difference form is exactly the relationship satisfied by layered solutions (Luyten et al., 1983) with vanishing surface layer depth at the eastern coast.

A second path to (2.14) is outlined in appendix A. For small \( x \), i.e., near the eastern boundary, the vanishing of \( B \) is shown to require a leading-order solution of the form (2.14). Furthermore, this solution can by itself satisfy all the boundary conditions so that no further expansion is required, provided that

\[ \rho_S(x, y) = \rho_S(y) \]

a restriction adopted for this section.

The solution of (2.14) is straightforward to obtain for any given function \( F(p) \), the \( B \) takes the form

\[ B = A(x, y)G(y, p); \quad G_{pp} = -fF(p)G, \quad (2.15) \]

where \( G \) satisfies the boundary conditions

\[ G = 0, \quad \rho = 0 \]

\[ G_{p} = 0, \quad \rho = \rho_S(y) \]

so that the system for \( G \) is homogeneous. Notice that the depth of the resting interface, \( z_0(x, y) \), for example, is not yet determined; nor does the condition on Ekman pumping need to be applied until the scaling \( A \) is required. Then (2.10) implies

\[ w_E = \rho_0 B_x B_p/fg, \quad \rho = \rho_S(y), \quad (2.18) \]

and we have

\[ 0 = \partial/\partial y \{ B_x(y, \rho_S) \} = B_{yy} + \rho_S B_{p\rho}, \quad \rho = \rho_S \quad (2.19) \]

so that (2.18), using (2.14) and (2.15), becomes

\[ w_E = AA_xG^2(y, \rho_S)F(p_S)\rho_0\rho_S/g. \quad (2.20) \]

Since \( G(y, \rho) \) can be chosen with an arbitrary scaling, (2.20) determines \( A(x, y) \) as

\[ \frac{1}{2} A^2(x, y) = g \int_0^{\infty} w_E dx/[G^2(y, \rho_S)F(p_S)\rho_0\rho_S/g], \quad (2.21) \]

where we can recognize the "Sverdrup function" (Luyten et al., 1983). There is a clear requirement that the rhs of (2.21) be positive. Apparently, in the subtropical gyre \( w_E < 0 \) this requires the natural \( \rho_S > 0 \), while in the subpolar gyre \( w_E > 0 \) the unnatural \( \rho_S < 0 \).

Appendix C shows that only the former case can occur, so solutions can only be found for the subtropical gyre. \( q \) cannot be conserved in an ideal fluid in the subpolar gyre, at least for our boundary conditions.) The boundaries of this gyre are thus where \( A \) vanishes and the depth scale becomes zero, and not where \( w_E \) vanishes; \( w_E \) may be positive over part of the solution domain. Appendix A shows the changes which occur if the Pedlosky (1983) eastern boundary condition \( \int_0^L u dz = 0, \quad x = 0 \) is used; the changes merely redefine \( A \).

The shape of the other physical variables may now be deduced:

\[ v = A_xG/f \sim (-x)^{-1/2}, \quad x \to 0; \quad (2.22) \]

\[ u = -(AG)_x/f \sim (-x)^{1/2}, \quad x \to 0 \]

\[ z = \rho_0 AG_{x\rho}/g \sim (-x)^{-1/2}, \quad x \to 0 \]

\[ q = f\rho_0\rho_S AG_{pp} \]

\[ = g/\rho_0^2 AF(p)G^2(-x)^2, \quad x \to 0 \quad (2.24) \]

\[ w = w_E (G_{pp} - G_{x}G_{p})/fG^2(y, \rho_S)F(p_S)\rho_S \]

so that \( v \) and \( q \) become infinite at the wall \( x = 0 \), and all isopycnal surfaces tilt up to the corner \( x = z = 0 \) as indicated schematically in Fig. 1a. (Although \( v \) becomes infinite, the southward flux and kinetic energy remain finite in the corner.) Note that the ratio \( w/w_E \) is independent of \( x \) on a density surface. If \( w_E \) is independent of \( x \), (2.21) shows that the solution, and hence the depth, scales naturally with \((-w_E)^{1/2}\), as in Luyten et al. (1983). The full form of (2.23) shows that the thermocline depth scale is the familiar
It is easy to see that there can be no closed contours in this solution (except for that of the boundary layer being assumed to absorb or erect precisely as much material as the solution requires). Hence Rhines and Young's (1982) homogenization—unlike Cox's (1985), as noted above—may not necessarily occur here (but see section 4). Similarly, because the eastern boundary is stagnant, there can be no shadow zone: all fluid of a density that outcrops somewhere in the gyre has been in contact with the surface. The depth of the deep isopycnal, which marks the maximum depth influenced by the wind, is given by (2.23) as

$$z_{0}(x, y) = \frac{\rho_{0}}{g} A(x, y) G_{p}(y, 0).$$  

Not all functions $F(\rho)$ give physically realistic solutions, as Janowitz notes. To avoid a discontinuity in $\rho_{g}$, which is zero in the deep layer, we need $F$ becoming infinite as density tends to zero (a singularity in density space frequently corresponds to a well-behaved solution in real space). However, there will be a jump in all density gradients at the deep isopycnal unless $\rho_{g}$ also vanishes, so that a physically well-behaved solution needs

$$F(\rho) \sim (-\rho)^{-\alpha}, \quad 1 < \alpha \leq 2, \quad \rho \to 0.$$  

Finally, the mode of solution is to provide an $F(\rho)$, and determine the $\rho_{g}(y)$ which matches it. However, one would probably wish to determine the form of $\rho_{g}(y)$, and deduce the form of $F(\rho)$ from it. I know of no way this can be performed, or even whether all surface densities have a matching $F$ function.

As an example, consider a problem similar to that of Luyten et al. (1983), who produced an approximate layered solution for the North Atlantic, using natural winter outcrops of density to provide surface values. They took $\sigma = 26.20$ at $25^\circ$N, $27.00$ at $37.5^\circ$N, and $27.40$ at $51.5^\circ$N. If the deep resting isopycnal is $\sigma = 27.75$, (one might prefer a deeper value, say $28.25$, perhaps), relative surface densities become $-1.55$, $-0.75$ and $-0.35$, respectively. A rough fit to this variation with latitude is given by

$$\rho_{S} \propto f^{-2},$$  

although the surface density becomes badly behaved near the equator. We shall limit attention to the subtropical gyre, thus avoiding the bad behavior. Now it is easily shown that

$$F(\rho) = \sigma(-\rho)^{-r} \Leftrightarrow G = \sigma^{1/2} J_{1/(2-r)} \{2\sigma^{(2-r)/2}/(2 - r)\},$$  

$$\sigma = -\rho(\mu f)^{1/(2-r)}$$  

for some constant $\mu$. The surface condition (2.17) then implies the variation of $\rho_{S}$ with $y$. For the case $r = 3/2$, we have

$$G = \sigma^{1/2} f_{s}(4\sigma^{1/8}), \quad \sigma = -\mu^{2} f^{2} \rho,$$  

whose shape is shown in Fig. 2. Here $G$ reaches a finite

Fig. 1. Schematics of the density structure (a) near the eastern boundary and (b) near the northern boundary of the subtropical gyre.
of different density must be subducted underneath each successive lighter layer as the fluid moves southward and downward. If the layers are brought closer together at the surface (large density gradient) then each layer must dive deeper than before in order to maintain static stability.

The “bowl” of the wind-driven flow does not maintain a symmetrical position within the gyre. Figure 4 shows pressure on surfaces of constant depth; note how the bowl shrinks, and the center of the bowl moves northward as the depth increases, in good agreement with IGY data. This is also seen in Fig. 5, which shows a meridional slice of the solution. The maximum westward return flow occurs well below the surface, whereas the eastward flow maximum occurs at the surface. At 10°N, this flow is about 10 cm s⁻¹, which also agrees with Huang’s (1986) results. Typical southern surface flows are of order 1 cm s⁻¹ away from the eastern boundary.

3. The problem for an infinitely deep ocean

In this section we modify the bottom condition to permit an infinitely deep ocean, such that the density tends to its abyssal value and horizontal velocities tend to zero as the depth becomes infinite. Such boundary conditions are fairly traditional (e.g., Welander, 1959) and are still used at present (e.g., Young and Ierley, 1986). This section will show that such conditions (i) do not permit a unique solution; (ii) allow (fairly) arbitrary \( \rho_S(x, y) \) rather than \( \rho_S(y) \) as previously; (iii) permit solutions with an infinite energy and downwelling at great depth; but (iv) yield density structures which look fairly realistic.

We pose Eq. (2.15)

\[
B = A(x, y)G(y, \rho),
\]

(3.1)

and boundary conditions

\[
G \rightarrow 0, \quad G_\rho \rightarrow -\infty, \quad \rho \rightarrow 0
\]

(3.2)

\[
G_\rho = 0, \quad \rho = \rho_S(x, y)
\]

(3.3)

\[
-\nabla x B_{\rho x} + B_x B_{\rho y} = \frac{w_E}{\rho_0}, \quad \rho = \rho_S(x, y).
\]

(3.4)

We choose a form for \( F(\rho) \) as

\[
F(\rho) = K\rho^{-2},
\]

(3.5)

(where \( K \) is an arbitrary constant) so that \( G \) becomes \( G(y, \sigma) \) where \( \sigma = \rho/\rho_S > 0 \), and \( G \) satisfies

\[
G_{\sigma\sigma} = -K\sigma^{-2}G.
\]

(3.6)

Now both solutions of (3.6) satisfy (3.2); previously only one such solution was permitted. Thus a linear combination can satisfy (3.3) without constraining the solution, namely

\[
G = r_+ \sigma^- - r_- \sigma^+,
\]

(3.7)

where

\[
r_\pm(y) = \frac{1}{2} \pm \left(1 - 4(\pm K)^2/4\right)^{1/2}, \quad 0 < K < \frac{1}{4}.
\]
FIG. 3. Isopycnal depths for various density strata for the solution in section 2. Dashed line shows outcrops. Contour interval (in m) is shown by each diagram. Axes are $x$ (km) and latitude (degrees). Also shown, except for the deep isopycnal, are contours of Bernoulli function (units for contour interval are m$^2$ s$^{-2}$). The contours for potential vorticity are parallel to those for Bernoulli function.
Equation (3.4) gives after a little algebra
\[ g w_E/K \rho_0 = \partial \partial y [(1 - 4/K) A^2] \partial \partial x (\rho_s^{-1}) - \partial \partial x [(1 - 4/K) A^2] \partial \partial y (\rho_s^{-1}), \]  
whose characteristics are \( \rho_s = \text{constant} \). So, providing that \( w_E \) is negative at least at the eastern boundary, (3.9) can be integrated to give
\[ (1 - 4/K) A^2(x, y) = \left( g \rho_s^2 / K \rho_0 \right) \int_0^x w_E dx / \rho_s, \]  
integrated along \( \rho_s = \text{constant} \). This solution has several features. It is clear that the surface density must increase monotonically poleward to avoid singularities in (3.10). As a corollary, there can be no closed contours of \( \rho_s \) (shown more generally by Killworth, 1979). The gyre boundary is located where \( A \) vanishes for this solution, and bears little resemblance to maps of either \( w_E \) or \( \rho_s \). The solution is nonunique because the constant \( K \) can be chosen arbitrarily within the limits given by (3.8).

Because \( B \) tends to zero at depth, so do \( u \) and \( v \) (algebraically with \( z \), as \( z \to -\infty \)). However, although this is the usual boundary condition for such problems, it does not preclude unphysical behavior. Algebra shows that
\[ w \propto \rho^{2z-1} \to \infty, \quad \rho \to 0, \]  
so that the downwelling becomes infinite (although mass-conserving). Hence the northward transport at a point
\[ \int udz \propto \lim \rho^{2z-1} \to \infty \]  
is also infinite. It should be stressed that the inadequacy lies in the boundary conditions traditionally imposed, not in the solution. (The addition of diffusion can be shown to be a small perturbation for small \( \rho \), so that the inadequacy is also not due to the physics assumed.)

The vertical structure of the solution is shown in...
Fig. 5. Contours of density (in sigma-theta units) and eastward velocity \( u \) in cm s\(^{-1} \) in the meridional plane. Axes are latitude in degrees and depth in m. Dashed \( u \) contours denote westward flow; the dash-dotted contour in both sets of diagrams shows the resting isopycnal (the jerkiness in the \( u \) diagrams is a contouring artifact).

Fig. 6. Here \( G \) decays slowly with \( \sigma \), giving rise to the infinite downwelling noted above. Nonetheless, solutions for density are not unrealistic. Figure 7 shows contours of isopycnal depth and Bernoulli function for a more realistic surface density field than used in section 2 (the actual function is a polynomial in \( y \)), for the case \( K = 0.24/f(45^\circ \text{N}) \). Note that the isopycnals now reach a greater depth than in section 2 (partly because \( \rho_0 \) has been modified). The poleward shift of the dome of the wind-driven layer with depth is now much less marked, varying only by a few degrees. The westward return flow is now very strongly concentrated in the southern part of the gyre, with most of the flow oriented eastwards. Figure 8 shows zonal sections of density and southward flow. The depths of a given isopycnal vary only weakly with position away from the eastern boundary so that the \( v \) field decays very slowly with depth as noted.

Not all infinitely deep solutions share the unphysical aspects of this solution. For example, if we modify (3.5) slightly, so that

\[
F(\rho) = \frac{(K - L \rho^2)}{\rho^2},
\]

then only one solution vanishes at \( \rho = 0 \), namely

\[
G = J_K \rho [\rho(Lf)^{1/2}] .
\]

(3.13)
As a result, $\rho_S$ only depends on $y$ once again. Provided $Kf_{\text{max}} < 1$, where $f_{\text{max}}$ is the largest value of $f$ in the gyre, $z$ becomes infinite as $\rho$ tends to zero, as required. The vertical velocity now varies as $\rho^{2(Kf)^{1/2} - 1}$ for small $\rho$, so that $w$ tends to zero at depth provided

$$\frac{1}{4} < Kf_{\text{min}} < Kf_{\text{max}} < 1,$$

(3.15)

which is satisfied if $f$ varies by less than a factor of 4 in the gyre (e.g., 12° to 45°N). Hence there exist solutions for infinite depth which are physically acceptable, although I have been unable to find any which allow both Ekman pumping and surface density to vary arbitrarily with $x$ and $y$.

4. Inclusion of a region of homogenized potential vorticity

A feature of the solution in section 2 is the existence of density strata which overlie the deep resting isopycnal but do not outcrop at the surface anywhere in the subtropical gyre. Such regions are termed unventilated by Luyten et al. (1983). We now assume that most unventilated water particles will slowly recirculate via the western boundary current, and by the arguments of Rhines and Young (1982) their potential vorticity will eventually become homogenized on a density surface. Baroclinic instability at the southern end of the gyre (Cox, 1985) would have a similar effect. To include this effect one merely requires that the functional form (1.1) change between the actively driven surface flow and the unventilated homogenized flow. In the lower layer $q$ is a function of density only, as in the original solutions of Welander (1971). Huang (1986) has patched together regions of differing functions (1.1) in a numerical study; also cf. Pedlosky and Young (1983). As we shall see, this modification allows again a fairly arbitrary surface density field.

We pose the situation sketched in Fig. 9, with an actively driven layer overlying a homogeneous layer, which in turn overlies a deep resting isopycnal layer. The interface between active and homogeneous fluids occurs at some (so far unknown) density $\rho_S(x, y)$, to be determined. It can be shown that the apparently natural choice, to require the interface to occur at the largest value of outcropping surface density, (or indeed any constant value of density) cannot easily be solved if the problem possesses a resting abyssal layer, irrespective of the $\rho, q, B$ relation assumed. (This remains true even if one adopts the idea of intermediate matching regions of $\rho, q, B$ relations as used by Huang, 1986.)

The interface between homogeneous and isopycnal (zero density) fluid has an unknown depth where $B_p = \phi(x, y)$, a function which has also to be found. Each layer is solved in turn, beginning with the active layer. Here

$$B_{pp} = -jF(\rho)B$$

(4.1)

once more, giving

$$B = \sum_{i} A_i(x, y)G_i(y, \rho),$$

(4.2)

where $G_i$ are the two linearly independent solutions of (4.1). The surface boundary conditions yield

$$B_p = 0, \quad B = \alpha(x, y), \quad \rho = \rho_S(x, y),$$

(4.3)

where

$$\alpha = \left[2g/(F(\rho_S)\rho_0) \int_0^z (\rho_S/\rho) dx \right]^{1/2}$$

(4.4)

integrated along $\rho_S = \text{constant}$ as in section 3. Hence the $A_i(x, y)$ are given by

$$A_i = \left(-1\right)^{i} \alpha G_{i-1}(y, \rho_S)/W,$$

(4.5)

where

$$W(y) = G_2 G_{1p} - G_1 G_{2p}$$

(4.6)

is the Wronskian of the solutions. The solution is fully specified by the surface conditions and the choice of $F$.

In the homogeneous layer, the potential vorticity structure is given by

$$B_{pp} = -jP(\rho)$$

(4.7)

where $P(\rho)$ is some arbitrary function of density which could be constant if required. For smooth matching with the isopycnal layer, one would probably prefer $P \to \infty, \rho \to 0$, with the integral of $P$ finite. Varying $P$ will yield families of solutions with the same surface
forcing; solutions are not unique. Integrating (4.7) from the isopycnal layer,

\[ B_x = \phi - f \int_0^\rho P(\rho')d\rho'; \quad B = \phi - f \int_0^\rho \int_0^\rho P(\rho')d\rho'' \]

(4.8)

We match the solutions at the unknown density level \( \rho_b(x, y) \), requiring both \( B \) and \( B_x \) to be continuous, ensuring continuity of pressure, density and velocities. A choice ensues: if \( q \) is also continuous, a pair of transcendental equations for \( \rho_b \) and \( \rho_S \) are formed. I have not been able to prove whether the freedom in the choice of \( P \) and \( F \) is sufficient to allow fully general \( \rho_S \).

The alternative, pursued here, allows a jump in density gradient at the interface. This implies a discontinuity in \( q \) (removed, in a real fluid, by a narrow diffusive layer), but permits a (fairly) free choice for \( \rho_S(x, y) \). Thus,

\[ \phi \rho_b - f \int_0^{\rho_b} P(\rho')d\rho' = B(x, y, \rho_b) \]  

(4.9)

\[ \phi - f \int_0^{\rho_b} P(\rho)d\rho = B(x, y, \rho_b), \]  

(4.10)

so that eliminating \( \phi(x, y) \) yields

\[ f \left[ \rho_b \int_0^{\rho_b} P(\rho)d\rho - \int_0^{\rho_b} \int_0^\rho P(\rho')d\rho'' \right] = B(\rho_b) - \rho_b B(\rho_b), \]  

(4.11)

giving \( \rho_b(x, y) \). A sensible solution must have \( 0 > \rho_b \) \( > \rho_S \), so that the interface occurs between the isopycnal layer and the surface. (We also require \( \phi(x, y) < 0 \), which is trivially so.) Appendix D gives a geometrical argument to show that the solution is physical provided

\[ \rho_S(x, y) > \tilde{\rho}_S(y) \]  

(4.12)
be a finite x-domain in which the solution can occur. By choosing $P$ sufficiently large (and the homogenized potential vorticity sufficiently small), the western edge of this domain can be moved beyond the western boundary of the ocean. If this is not the case, the depth of the homogenized region will increase relative to the ventilated region as one moves westwards, with the depth of the ventilated region eventually becoming zero when (4.13) ceases to be satisfied. Beyond this point, there is no solution.

How wide the ocean can be for such solutions to exist is sensitive to quantities like surface density. Estimating the lhs of (4.13) gives $\alpha \sim g f (\rho_S/\rho_0)^2/2|g|$, where $q$ is an estimate of the homogeneous potential vorticity, typically of order $10^{-12}$ cm$^{-1}$ s$^{-1}$ (Keffer, 1985). Since $\alpha$ varies as $|x|^{1/2}$ for $x$-independent forcing, (4.15) implies that the width depends on the fourth power of surface density.

If we take the figures of section 2, for example, the maximum extent of the gyre with a homogenized layer would vary from an infinite width at the gyre boundary, down to 3000 km at 37°N and back up to 24 000 km at 23°N. While these estimates are rather dependent on the parameters chosen, the predicted widths are not physically unrealistic.

Thus, one effect of (partial) homogenization of potential vorticity in ventilated layers may be to limit the east–west extent of the ventilated gyre. Given that the (wide) Pacific and (narrower) Atlantic have similar values of homogenized vorticity, the model would predict that the ventilated density relative to abyssal values would have to be larger for the Pacific (which is the case; Keffer, 1985). The way in which the homogenization occurs, and the interaction between the gyre and the western boundary current, plus the limits on the width of the gyre, must all be intricately involved in setting the density field in the subtropical gyre.

A numerical example considers a small modification to the solution in section 2. Although the surface density can now vary with both $x$ and $y$ in the solution,
we choose to modify its previous value only by adding a small constant density of 0.1σ, units at each point. For the homogenized fluid, we take the form

\[ P(\rho) = K(-\rho)^{1/2}, \]  

(4.15)

where \( K \) has the value \( 3.10 \times 10^{13} \) c.g.s., which is near the minimum value for the solution to exist in a basin of 6000 km width. It gives a maximum homogenized potential vorticity of \( 0.7 \times 10^{-12} \) cm\(^{-1}\) s\(^{-1}\), close to Kefari's (1985) estimates. Because the changes to the surface forcing are small, we can expect the changes to the solution to be so also (the comments of appendix B do not apply now). Figure 10 shows the depth of the interface \( \rho_0(x, y) \) which reaches a maximum of about 450 m in the west at 20°N, and approaches the surface in the west at around 37°N (in the northern half of the subtropical gyre, the interface is deepest at about 1000 km west of the eastern boundary). The depth of the isopycnal layer has a maximum of 1414 m, to be compared with 1913 m for the solution of section 2; this is not a small change. Figure 11 shows the depth of selected isopycnal surfaces, and the Bernoulli function on those surfaces. Contours of potential vorticity are parallel with those for Bernoulli function outside the homogenized region, and vanish within it. There is a bias in homogenization towards the northwestern edge of the subtropical gyre (easy to show for solutions "near" the power laws in section 2). The homogenized area shrinks as one rises through the water column. Figure 12 shows meridional sections of the density profile, with areas below the dashed line homogenized. The interface at which homogenization occurs is relatively flat up to 3000 km west, and then rises in the northwest to intersect the surface as shown.

The predicted density and flow fields are quite similar to those in section 2 (remembering the minor shift in surface density) so that it would be difficult to distinguish between the solutions by comparison with data unless the potential vorticity field was mapped.

5. Discussion

The solutions presented in this paper are all exact nonlinear solutions to the ideal fluid thermocline equations, with density and Ekman pumping prescribed at the surface and no flow permitted through the eastern boundary. They are exact parallels of the layered solutions of Luyten et al. (1983). Three different lower boundary conditions have been used; one involving a smooth match to a deep resting isopycnal water mass, a second which has horizontal density gradients and velocities decaying to zero for a basin of infinite depth, and a third which matches an actively ventilated layer of fluid to a region of homogenized potential vorticity which itself overlies a resting isopycnal layer. This last solution does possess discontinuities in density gradient, however, although solutions can be found with continuous \( q \).

These differing bottom conditions yield differing solutions. Other bottom conditions yield yet more solutions. We briefly consider two such changes. The first is a smooth match to some horizontally uniform resting stratification (rather than the isopycnal layer used previously). We specify some unknown \( B_0(\rho) \) for this stratification. If the fluid in motion intersects the resting fluid at density \( \rho_f(x, y) \) Bernoulli function and depth must match at this interface as before, giving a pair

\[ B(x, y, \rho_f) = B_0(\rho_f); \quad B_f(x, y, \rho_f) = B_0(\rho_f), \]  

(5.1)

which automatically ensures that all velocities vanish on the interface.

The second is to require that \( w \) vanish when \( z = -H \) (here assumed constant). This will occur provided

\[ -B_y \rho_{yf} + B_x \rho_{xf} = 0, \quad B_y = -\rho_0 H/g, \quad \rho = \rho_f \]  

(5.2)

where \( \rho_f(x, y) \) is now the unknown density on the flat bottom. (Since the eastern boundary in these solutions is an isopycnal, flow would have to pass in and out of the eastern boundary, possibly by relaxing the boundary condition and merely requiring no vertically integrated flow through this boundary.)

All such conditions share the feature that two conditions on \( B \) are provided at a density which is an unknown function of horizontal position. Some are more flexible than others but the number of conditions remains fixed. The surface boundary conditions totally specify the solution (as in section 4), so that matching the derivative of \( B \) at some location will always give a transcendental equation for the matching density. Requiring \( B \) itself to be continuous, however, usually implies a relationship between \( \rho_5 \) and \( w_x \), rather as in Welander (1971). So most of the possible lower bound-
Fig. 11. Isopycnal depths for various density strata for the active and homogenized solution. Dashed line shows outcrops. Contour interval (in m) is shown by each diagram. Axes are $x$ (km) and latitude (degrees). Also shown are contours of Bernoulli function (units for contour interval are $m^2 s^{-2}$). The region marked by small circles contains water with homogenized potential vorticity. Thus the contours for potential vorticity are parallel to those for Bernoulli function, and disappear within the area of circles. The close contours of Bernoulli function near outcrops are an artifact of the contouring routine. Axes as in Fig. 3. Note that the potential vorticity must be discontinuous at the boundary between active and homogenized regions.
ary conditions still require special surface forcings, unlike those considered in this paper.

The freedom to choose less restricted surface conditions here is won (just as in layered models) by permitting weak discontinuities in high-order derivatives; no physically realistic infinitely differentiable solutions to the thermocline equations with arbitrary surface conditions are known. These discontinuities typically occur near the floor. Thus a unique solution of the thermocline equations, given surface and sidewall conditions, is only possible if the lower boundary condition is extremely carefully defined, and explicit decisions are made concerning the degree of differentiability of the solution required. Since the details of the unventilated flow can and do modify the solution in the upper layers, it is necessary to know how the lower layers achieved the density structure they possess. Thus, details either of deep outflow from the western boundary current, or from deep-water sources poleward of the gyre, must in general be necessary to close the problem.

The approaches used here have connections with much recent work; in particular, the “patching” of $\rho$, $qB$ relationships in Huang (1986) is reminiscent of the solution in section 4. One of the main difficulties in judging how well a specific model performs is that when one analysis term balances in numerical models, one frequently finds that all terms, including diffusion, are important somewhere in the solution (so that even Gill’s, 1983, linear diffusive thermocline model can reproduce quite reasonable density structures). Inverse studies which include the effects of diffusion terms (e.g., Bigg, 1985) are clearly going to be relevant in our attempts to understand the subtle dynamics of model oceans, let alone real ones.

One last point needs to be made. All solutions of the ideal fluid equations, from Welander (1959) to the present, make an explicit assumption as to the shape of the $\rho$, $q$, $B$ relation. Hitherto it has not been possible to infer the shape of this relation from data, because although density and $q$ are—with reservations about accuracy—straightforward to measure with standard hydrographic data, information about the $B$ field has been lacking. Inverse methods do give us ways of estimating $B$, however. Do oceanic (or general circulation model) data confirm or deny any of the relations used hitherto?

Two sets of data were analyzed. The first are the averaged data of Cox’s (1985) eddy-resolving general circulation model. The Bernoulli inverse method (Killworth, 1986b) and other methods have been applied to various subsets of these data (Killworth and Bigg, 1987), thus providing $\rho$, $q$ and $B$ at each point. The inversions were very accurate at 32°N, 30°E (the datum occurs on the western boundary in Cox’s model) which lies in the Gulf Stream extension. (By “accurate” we mean simply that the predicted mean flows from the inversion were in very good agreement with Cox’s mean flows; thus the estimate of Bernoulli function can be taken as correct for current purposes.) To test if the functional form is (1.1), we plot a scatter diagram of $qB$ against density for a square of 5° × 5° “stations” centered at the above point, with a 1° spacing. Clearly $B$ needs an origin (a uniform shift changes none of the dynamical variables). A value fairly close to the largest $B$ in magnitude (which occurs at great depth) was chosen, to simulate the conditions of section 2 that $B$ should vanish at depth.

Figure 13 shows the resulting fit, which possesses what seems to be small scatter; recall that this relates to a 5° square only, however. Nonetheless, $qB$ appears to collapse closely onto a function of density, approximately varying slightly slower than the square of ($\sigma$...
for small $x$. Assuming that differentiating w.r.t. $y$ or $\rho$ does not induce any more powers of $x$ (which would cause mathematical problems), the surface Ekman pumping condition yields

$$w_E(0, y) \approx -\rho_0 r(-x)^{\nu-1}/B_0B_\nu \bar{f}/g, \quad x \to 0, \quad \rho = \rho_S,$$

(A2)

so that $r = \frac{1}{2}$. We now expand $F(\rho, B)$ in powers of $B$, note that the vanishing of $B_\nu$ at $x = 0$ implies that $F$ contains no term independent of $B$, to find

$$F(\rho, B) \approx B F_1(\rho) + B^2 F_2(\rho) + \cdots.$$  

(A3)

We substitute (A1), and expand $B$ in powers of $(-x)^{1/2}$, assuming that we can reorder and collect terms. Thus

$$B \approx (-x)^\nu B_\nu(y, \rho) + (-x)^{\nu+1} B_1(y, \rho) + \cdots$$

(A1)

We then note that the truncation to a single term (with the modifications in section 2 to allow for $w_E$ varying with $x$ and $y$) can already satisfy all three boundary conditions. Hence the solution in section 2 can be readily derived from an expansion near the eastern boundary, and the truncation is consistent. I have been unable, however, to prove that the truncation is unique. Any further expansion of either $B$ or $F$ yields an infinite set of equations which are no longer of similarity form, and I know of no method for their solution. The comments in appendix B are relevant to this matter.

It is possible to repeat the Pedlosky (1983) extension to the eastern boundary condition (which assumes

![Figure 13](http://journals.ametsoc.org/jpo/article-pdf/17/11/1925/4420412/1520-0485(1987)017_1925_acsnvt_2_0_co_2.pdf)

**Figure 13.** Scatter diagram of $qB$ (with $B$ measured relative to a value corresponding to a fairly deep uniform value) against density, for $5 \times 5$ "stations" separated by $1^\circ$, at $32^\circ N$, $30^\circ E$ in the Cox (1985) eddy-resolving numerical solution. A quadratic in density fits all but the near-surface values well.

- 28.2) for the deeper data. The functional form in (2.14) would be the inverse of this. Oddly, there is more scatter near surface than deeper; one might have expected the reverse.

A similar calculation was performed on the Levitus (1982) data, inverted using the Bernoulli method (Killworth, 1986), but this time over a $40^\circ$ square in the North Atlantic. The inversion appears quite accurate by our classification (Killworth and Bigg, 1987), and again $B$ was measured relative to a deep value (one would, indeed, get fairly similar figures if one defined a level of no motion near the bottom). Figure 14 shows the resulting scatter which, although wider than the Cox scatter for several reasons, does seem to indicate that a rough power law dependence of $qB$ on density may hold in much of the North Atlantic. (Note that parts of the subpolar gyre were included in this diagram.) Obviously there is no "true yardstick" against which to judge the degree of scatter in these diagrams; but their similarity and degree of collapse do suggest that the solutions presented here may be of sufficiently wide application that they can be used to increase our understanding of the subtropical gyre circulation.

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**APPENDIX A**

**The Solution near the Eastern Boundary**

We assume, as in the text, that $B$ vanishes on $x = 0$ at the eastern boundary. Thus we set

$$B_\nu = -\xi F_1(\rho) B_\xi;$$

$$B_{1\rho} = -\xi F_1(\rho) B_1 - F_2(\rho) B_\xi^2;$$

(A4)

and we have the form of (2.14).
some nonspecified boundary layer whose function is to permit east–west flow but no net vertical integral of the flow. There is now no proof that the functional form is (2.14), although this is a consistent choice. The requirement of no net eastward flow on \( x = 0 \) forces \( \int BB_x(\rho)\,d\rho \) to vanish there, which merely acts to redefine \( A \) in (2.21).

**APPENDIX B**

The existence of Neighboring Solutions to Those in Section 2

Suppose we have found a solution of the type in section 2. Such solutions, as noted, have \( \rho_S(y) \) only. The natural extension of these solutions is to seek solutions for which

\[
\rho_S(x, y) = \rho_{S0}(y) + \epsilon \rho_{S1}(x, y),
\]

where \( \epsilon \) is a small nondimensional parameter. To this end we require the normal expansions

\[
B = B_0 + \epsilon B_1;
\]

\[
F = BF_S(\rho) + \epsilon F_1(\rho, B), \quad \omega_{ps} = \omega_{ps0} + \epsilon \omega_{ps1}.
\]  

(B2)

We now expand the equations and boundary conditions to first order in \( \epsilon \), giving

\[
B_{00} = -fF_S(\rho)B_0; \quad B_{10} = -fF_S(\rho)B_1 - fF_1(\rho, B_0)
\]

(B3)

\[
B_0 = B_1 = 0, \quad \rho = 0
\]

(B4)

\[
B_0 = 0; \quad B_1 = -fF_S(\rho)B_0 \rho_{S1}, \quad \rho = \rho_{S0}(y),
\]

and, after a little algebra similar to that in section 2,

\[
gw_{ps0}/\rho_0 = F_0 B_0 \rho_{S0}; \quad gw_{ps1}/\rho_0 = (AB_1)_{ps0}, G + F_0 AG(\rho_{S1}(y), \rho_{S0})
\]

\[
\rho = \rho_{S0} + \epsilon \rho_{S1},
\]

where the leading solution is as before:

\[
B_0(x, y, \rho) = A(x, y)G(y, \rho).
\]  

(B7)

Posing \( B_1(x, y, \rho) = B_1Q(x, y, \rho) \) for the perturbation, substitution into the second of (B3) and some algebra gives

\[
B_1 = \delta(x, y)B_0 - fB_0 \int_0^\rho \left( \frac{dp}{B_0} \right)^2 \int_0^\rho B_0 F_1(\rho, B_0)\,d\rho.
\]  

(B8)

Applying the second of (B5) yields

\[
\rho_{S1}(x, y) = \int_0^{\rho_{S0}} G(\rho, AG)\,d\rho / \{A F_0(\rho_{S0})G^2(y, \rho_{S0})\}
\]

(B9)

while the second of (B6) simply defines \( \delta(x, y) \). Turning attention to (B10), is it possible to choose the (hitherto arbitrary) function \( F \) to satisfy (B10) for any \( \rho_{S1}(x, y) \)? We show that, in general, this is not the case by demonstrating it for a specific example: a few moments thought will convince the reader that matching the x-structure of \( \rho_{S1}(x, y) \) will almost certainly force a specific y-structure.

Suppose—subject to convergence radii restrictions—we require that \( \rho_{S1}(x, y) \) have the form \( x' t(y) \), for some \( t(y) \) (the extension to arbitrary functions of \( x \) is easily made through an expansion). For the case of Ekman pumping being a function of \( y \) only to leading order, the solution takes the form

\[
B = (x')^{1/2}S(y)G
\]

(B10)

for some function \( S(y) \). To match the x-coefficients in (B10) we must have

\[
F_1(\rho, \phi) = F_0(\rho)\phi^{x'/2},
\]

so that (B9) reduces simplistically, for some \( T(y) \), to the form

\[
T(y)t(y) = \int_{\rho_{S0}}^\rho \tilde{F}(\rho)G^{x'/2}(y, \rho)\,d\rho.
\]  

(B12)

We further assume \( G \) to be one of the Janowitz and section 2 solutions in which \( G = G(\sigma = \rho_{S0}^{-1}) \), so that (B13) can be written

\[
T(y)t(y) = \int_1^\rho \tilde{F}(\rho)G^{x'/2}(\sigma)\,d\sigma.
\]  

(B13)

Now \( \rho_{S0}(y) \) is monotonic, so the lhs can be written as a function of \( \rho_{S0} \). If this is expanded in powers of \( \rho_{S0} \) (maybe even a finite expansion), the unknown function \( \tilde{F}(\rho) \) must take the form

\[
F(R) = \sum a_s R^s \quad \text{for some} \quad a_s, \quad r = 0, 1, \ldots.
\]  

(B14)

Provided that all these powers are positive, this presents no problem. But if negative powers enter, as they may easily, and one power is less than the \(-r + 5/2\)th, the integral in (B13) becomes undefined because of the singularity near zero \( (G \text{ varies as } \sigma \text{ near } \sigma = 0) \). This function cannot be matched, therefore.

Thus, there exist surface densities \( \rho_\phi(x, y) = \rho_{S0}(y) + \epsilon \rho_{S1}(x, y) \) which either have no solution or are not close to the section 2 solution for \( \rho_{S0}(y) \) only.

**APPENDIX C**

The North–South Surface Density Gradient

Consider \( D = \partial B/\partial \phi \), where \( \phi \) replaces \( x \) as northward independent variable. Then \( D \) satisfies

\[
D_{\phi\phi} = -fF(\rho)D - F(\rho)B.
\]  

(C1)

By multiplying (2.14) by \( D \) (C1) by \( B \) (subtracting, and integrating from \( \rho_{S0} \) to 0, and using

\[
D_\phi = \tilde{f}(\delta \rho_{S}/\delta \phi)F(\rho_{S})B, \quad \rho = \rho_{S}(\phi),
\]

we find

\[
\int_{\rho_{S0}}^\rho F(\rho)B^2(\rho)\,d\rho > 0,
\]

(C3)
since the sign of $F$ is required to maintain static equilibrium in the density field. Hence $\rho_S$ increases polewards:

$$f\partial\rho_S/\partial\rho > 0.$$  \hfill (C4)

One could have derived this physically. If a patch of surface water reached a local maximum in density in a region where the Ekman pumping did not vanish, where could this water be advected without causing static instability? Water both north and south would have a smaller density. This is a variation of a theorem in Killworth (1979).

**APPENDIX D**

**A Geometrical Argument for the Homogenized Vorticity Case**

Consider the lhs of (4.11). Its derivative w.r.t. $\rho_h$ is $f\rho_B P(\rho_h) < 0$ (Northern Hemisphere assumed throughout). Since the rhs vanishes at $\rho_h = 0$, the rhs is positive throughout. Now consider the rhs of (4.11). At the extreme case $\rho_h = \rho_S$, the rhs takes the value $\alpha(x, y) > 0$. Further, its derivative w.r.t. $\rho_h$ is $fF(\rho_h) B(x, y, \rho_h)$ whose sign is opposite to $B$, since $\rho_h$ is negative by supposition. Realistic solutions in the active layer will require $B$ to be positive.

To proceed, we need two observations. First, we examine the solution $\hat{B}(x, y, \rho)$, i.e., the solution given in section 2, where the surface density is $\bar{\rho}_S(y)$ determined purely by $F(\rho)$. We require that $\hat{B}$ takes the value $\alpha(x, y)$ at the surface also. Thus by definition $\hat{B}$ vanishes at $\rho = 0$. Now suppose that $\rho_S(x, y) < \bar{\rho}_S(y)$ at some location. (In other words, there is some horizontal point at which the surface density is less than the density that solves the problem without a homogeneous layer.) Since $B_{\rho_h} < 0$, and both $B$ and $\hat{B}$ satisfy the same equation, their curves as functions of $\rho$ (or $\rho_h$) are sketched in Fig. 15a. As a result, $B$ must vanish for some value of $\rho_h < 0$. Now suppose that the reverse occurs: $\rho_S(x, y) > \bar{\rho}_S(y)$ at some location. Then Fig. 15b shows that $B(x, y, 0)$ is positive.

![Fig. 15: The two geometric possibilities: (a) $\rho_S < \bar{\rho}_S$; (b) $\rho_S > \bar{\rho}_S$.](image1)

The second observation concerns the sign of $B - \rho_B^*$ or $B - \rho_B^*$ as in Fig. 16, we can determine the sign of $B - \rho_B^*$ by drawing a line from $B$ to the origin. If the gradient of $B$ lies above the line, $B - \rho_B^*$ is positive; if below, $B - \rho_B^*$ is negative.

We can now find under what conditions $\rho_S(x, y)$ is physically reasonable. Since the rhs of (5.11) is positive, so must the rhs be. Now in the case $\rho_S < \bar{\rho}_S$ (Fig. 17a), a tangent to the $B$ curve shows that only the area labeled “a” in Fig. 17a has the rhs positive. The curves for lhs, rhs thus resemble the sketch in Fig. 17b. For equality of the lhs, rhs to occur, the lhs must be less than $\alpha$ at $\rho = \rho_S$ (the lower of the two dashed curves in Fig. 17b). However, this cannot succeed throughout the domain. As $x$ becomes small (near the eastern boundary), $\alpha$ becomes small also, making the rhs small. However, the magnitude of the lhs is fixed, assuming $\rho_h$ changes only weakly with $x$ near the eastern boundary, so that for any magnitude of the lhs, there can be no solution sufficiently near the eastern boundary.

![Fig. 16: The geometric evaluation of the sign of $B - \rho_B^*$.](image2)

![Fig. 17: Schematics of (a) $B$, and (b) lhs, rhs of (4.11) when $\rho_S < \bar{\rho}_S$. Two possible sizes for the lhs are shown in (b). The region marked "a" is the only possible one for a solution, and the existence of one is disproved in the text.](image3)
The case \( \rho_S > \tilde{\rho}_S \) (Fig. 15b) does permit physical solutions, however. The curve for \( B \) (Fig. 18a) lies everywhere above the line connecting each point to the origin, so that the rhs is positive throughout, as in Fig. 18b. The lhs must have a point in common with the rhs if it exceeds the rhs at \( \rho_S \) (the upper of the dashed curves). (It is possible for the lhs to be below the rhs at \( \rho_S \) but curve upwards to cross it. However, this case fails by the arguments of the preceding paragraph.)

We have shown, therefore, that physically reasonable solutions exist provided

\[
\rho_S(x, y) > \tilde{\rho}_S(y)
\]  \hspace{1cm} (D1)

\[
f\left(\rho_S \int_0^{\rho_S} P(\rho) d\rho - \int_0^{\rho_S} d\rho \int_0^{\rho'} P(\rho') d\rho'\right) > \alpha(x, y).
\]  \hspace{1cm} (D2)

REFERENCES


