Nonlinear Energy Transfer through the Spectrum of Gravity Waves for the Finite Depth Case

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ABSTRACT

An algorithm for calculation of the nonlinear kinetic integral is described for the case of finite depth. The use of an effective approximation of the exact dispersion relationship for gravity waves in finite depth permits modification of the analytical part of the calculation technique derived earlier by the author for the deep water case.

A representative series of two-dimensional spectral shapes was investigated, and numerous peculiarities of nonlinear energy transfer through the spectrum for the finite depth case were revealed. The analytical parameterization on nonlinear transfer was constructed as a function of spectral shape parameters. The first approximation of such a parameterization is presented, and further investigation of the problem discussed.

1. Introduction

As it is well known, there have been several attempts at numerical calculation of the nonlinear kinetic integral (for references see Hasselmann and Hasselmann 1985) since the pioneering paper by Hasselmann (1962). Herewith, the first reliable results of such calculations have been obtained nearly 20 years later (Webb 1978; Masuda 1980; Hasselmann and Hasselmann 1981), but these studies were not detailed. They encompassed only the conceptual investigation of the peculiarities of nonlinear interactions between waves.

The most detailed investigation of the problem, for the case of deep water, was carried out by the author recently (Polnikov 1989, 1990). In terms of physics, four distinct properties of the nonlinear energy transfer as a function of two-dimensional spectrum shape parameters were found and formulated (Polnikov 1989). In Polnikov (1990), the peculiarities of a long-scale temporal evolution of the two-dimensional frequency-angular wave spectrum $S(\omega, \theta)$, provided by the nonlinearity of waves only, were studied numerically. Many new and interesting results regarding the role of nonlinearity of waves were obtained.

This paper is a natural continuation of the previous studies, focusing on the case of finite depth. The necessity of such an investigation is based on the following reasons: First, it is induced by the lack of a detailed description of nonlinear energy transfer for the case of finite depth at present. The only two papers dealing with the point (Helterich and Hasselmann 1980; Hasselmann and Hasselmann 1985) are insufficient for the full investigation of the problem. Second, the numerical technique of computation in this case is more complicated than for the deep water case, and it should be described in detail.

The initial point of investigation is the well-known nonlinear kinetic equation of the kind

$$\frac{dn_i}{dt} = 4\pi \int |M_{1,2,3}|^2[n_i n_2(n_3 + n_4) - n_i n_2(n_1 + n_2)] \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \delta(k_i + k_2 + k_3 + k_4) \times dk_1 \cdot dk_2 \cdot dk_3 \cdot dk_4 = I,$$

where $n_i = n(k_i)$ is the wave action spectrum in $k$ space related to the energy spectrum $S(k)$ by the formula

$$n(k) = \frac{\gamma\omega S(k)}{\omega(k)}, \quad (\gamma = 4\pi^2);$$

where $M_{1,2,3,4} = M_{k,k,k,k}$ are the matrix elements of the four-wave interactions and $\omega_i = \omega(k_i)$ and $k_i$ are the frequency and the wave vector related to each other by the dispersion relationship of the kind

$$\omega(k) = gk \tanh(kh).$$

The explicit expressions for the elements $M_{1,2,3,4}$ for the case of finite depth are presented, for example, in the paper by Kratiskii (1994).

There are several techniques of integration of the kinetic integral (1) (Webb 1978; Masuda 1980; Hasselmann and Hasselmann 1981). For these calculations a modified method of Masuda, derived by the author (Polnikov 1989), was used. In this method, for the deep water case, integration of the threefold $\delta$-function in (1)
is carried out analytically in an explicit form. Unexcludable singularities under integral (1), connected with the zero values of the Jacobian denominator arising due to the integration of the complicated \( \delta \) function, are integrated analytically in the vicinity of singular points as well. In this way, the mathematical difficulties of numerical integration are totally overcome. Details of the algorithm are given in Polnikov (1989).

However, for the case of finite depth, the direct utilization of relationship (3) in the algorithm stated above is impossible due to the irrational equation for the frequency resulting from the function \( \delta (\omega_1 + \omega_2 - \omega_0 - \omega_0) \). Thus, the problem requires adaptation of the previous algorithm for the case of finite depth. This problem is solved and described below.

The main aim of this paper is to find and describe the principle peculiarities of the nonlinear energy transfer (NET) through the spectrum of surface gravity waves for the finite-depth case. It will be shown that these peculiarities are not confined only by a trivial increase of the NET intensity with decreasing depth \( h \), but they are more numerous and complicated.

In section 2 the modification of the deep water algorithm of the kinetic integral calculation is described. Methods and results of the calculations are presented in sections 3 and 4, respectively. Analysis and analytical parameterization of the NET properties for the case of finite depth are given in section 5. Finally, a discussion of the results is carried out in section 6.

2. Modification of calculation algorithm

According to Polnikov (1989), the algorithm of kinetic integral calculation comprises the following items:

1) analytical integration of the \( \delta (\mathbf{k}) \) function with respect to \( \mathbf{k} \);

2) transformation to polar coordinates \((\omega_0, \theta_0)\) and analytical integration of \( \delta (\omega_1 + \omega_2 - \omega_0 - \omega_0) \) with respect to \( \theta_1 \);

3) determination of singular points of the Jacobian denominator (by the analysis of its explicit expression) and the limits of integration with respect to \( \omega_0 \);

4) analytical integration over the vicinity of singular points and further numerical integration over the domain outside these points.

The procedure stated above determines the needed modification of the algorithm provided by the dependence of the dispersion relationship (3) on depth.

For item 2 of the algorithm described above an analytical representation of wavenumber \( \mathbf{k} \) as a function of frequency \( \omega \) is needed; hence it is necessary to use an approximation of relationship (3). By means of a Taylor expansion of the hyperbolic tangent with respect to smallness parameter \( k_p h \) [\( k_p \) is the coordinate of the peak value of spectrum \( S(k, \theta) \) in the polar \((k, \theta)\)-space], it is easy to find an approximation of the kind

\[
k^2 = \left( \omega^2 + \omega_0^2 \omega^2 \right) / g^2, \tag{4}
\]

where \( \omega_0^2 = g / h \) is the theoretical parameter. Relationship (4) conserves all features of the exact one (3). Functions (3) and (4) coincide at the limits of very small and very large values of depth \( h \) or parameter \( k_p h \). The discrepancy is less than 10% in the domain of intermediate depth \((k_p h \approx 1)\). For this reason, one can expect that the use of approximation (4) is justified for the description of all principal peculiarities of the NET in the case of finite depth.

By introducing the definitions

\[
\mathbf{k}_n = \mathbf{k}_1 + \mathbf{k}_2, \tag{5a}
\]

\[
\omega_n = \omega_1 + \omega_2, \tag{5b}
\]

where vector \( \mathbf{k}_n \) in polar coordinates is defined by the couple \((k_n, \theta_n)\) (see appendix A), one can derive the following formula for the \( \delta \) function with respect to frequencies (taking into account the prior integration with respect to \( \mathbf{k}_2 \)):

\[
\delta (\omega) = \delta (\omega) + \left( \frac{g}{2h} \left( \left| 1 + 4h^2 (k_1^2 + k_2^2 - 2k_1 k_2 \cos (\theta_1 - \theta_2)) \right|^{1/2} - 1 \right) \right) - \omega_n. \tag{6}
\]

From (6) it is easy to find an analytical solution for zero points of argument of function \( \delta (\omega) \) with respect to \( \theta_1 \),

\[
\theta_1 = \theta_0 \pm \arccos \left( \frac{k_1^2 + k_2^2 - k_0^2}{2k_1 k_2} \right). \tag{7}
\]

Here \( k_1 \) and \( k_2 \) are expressed as functions of \( \omega_1 \) and \( \omega_2 = \omega_0 - \omega_0 \) by Eq. (4), respectively. The Jacobian of the \( \delta \)-function integration (at points \( \theta_1 = \theta_1^1 \)) is as follows:

Hereafter the tilde over subindex 2 is omitted every-
where. Consider now the denominator of the Jacobian more closely.

In the procedure for the kinetic integral calculation using permutation symmetry \((k_i \leftrightarrow k_j)\), one can restrict the integration interval in accordance with the condition \(k_1 = k_2\). Then the denominator of Jacobian (8) has only two multiplicands that may be equal to zero. One of them, defining the condition for the lower limit of integration with respect to \(\omega_1\), is of the kind

\[
d_1 = k_1 + k_a - k_2 = 0, \quad (9a)
\]

and the second, defining the upper limit of integration (taking into account the additional restriction \(\omega_1 \leq \omega_i/2\), is

\[
d_2 = k_1 + k_2 - k_a = 0. \quad (9b)
\]

In contrast to the case of deep water, the exact expressions for lower and upper limits of \(\omega_1\), denoted further by \(\omega_{\text{lim}}\) and \(\omega_{\text{lim}}\), cannot be calculated analytically in the case of finite depth. Because of this, for each set \(\{\omega_1, \theta_1, \omega_a, \theta_a\}\), determining a respective set \(\{k_1, \theta_1, \omega_a\}\), the solutions of Eqs. (9a,b) must be obtained numerically. For this purpose, Eqs. (9a,b) are rewritten in an explicit form using an approximation (4) for variables \(k_i(\omega_i)\) and \(k_i(\omega_i - \omega_j)\) (see appendix A).

As a result, after transformation to polar coordinates \((\omega, \theta)\) and nondimensional variables, with normalization of frequency by the peak value \(\omega_0\) and the spectrum \(S(\omega, \theta)\) by \(S_p = S(\omega_r, \theta_r)\), one can obtain the final expression for \(\frac{dS}{d\theta} = \hat{S}\) of the kind

\[
\hat{S}_\omega(\omega_i, \theta_i) = C \sum \int_{\omega_0}^{\omega_{\text{lim}}} \int_{\theta_0}^{\theta_{\text{lim}}} \int_{\theta_0}^{\theta_{\text{lim}}} [\mathcal{M}_{1234}^2 P_{1234}^2] \times \frac{d\omega}{(k_1 + k_a - k_2)^{\theta_r}(k_1 + k_2 - k_a)^{\theta_r}} \quad (10)
\]

\[
S_1 = \hat{S}_{\text{pm}}(\omega) * \cos^2(\Theta)
\]

Fig. 1. One-dimensional nonlinear energy transfer functions \(T(\omega)\) for run 1 and six depths \(\bullet: k_p h = 36, \ast: k_p h = 10, \times: k_p h = 1, \pm: k_p h = 0.5, \triangle: k_p h = 0.3, \bigtriangleup: k_p h = 0.2\). The value of \(T(\omega)\) is given in units of dimensional constant \(C\) and normalized by factor \(\text{tgh}(k_p h)^{10}\).
Here $C = \pi g^{-1} \omega_1^4 \delta / 8$ is the dimensional constant, and the expression in brackets under the integral is the regular part of integrand presented in appendix A.

Further transformation of the algorithm for the kinetic integral calculation includes estimation of the type of singularity and analytical integration of it over the vicinity of the singular points $\omega_1 = \omega_{1m}$ and $\omega_2 = \omega_{2m}$.

A direct check shows that both singular points have a square root integrable type of singularity (see appendix A).
While integrating the kinetic integral, it is useful to consider Longuet-Higgins's diagrams for four-wave interactions (Hasselmann and Hasselmann 1981). For the case of finite depth, these diagrams have the same form of so-called figure-of-eight curves for some nondimensional interaction parameter $D$. The parameter $D$, separating the coupled ($D > 0$) and uncoupled ($D < 0$) isolines of $\Delta$, is defined in our case by the formula

$$D = \frac{\omega^2}{2} \left( 1 + 4\frac{\omega_0^2}{\omega_i^2} \right)^{1/2}. \quad (11)$$

Use of relationship (11) permits estimation of the singular point contribution by analogy with previous calculations (Polnikov 1989). Thus, the technique of calculations, described previously, is transferred to the case of finite depth. A few final expressions are given in appendix A.

3. Method of calculations

Calculations of the kinetic integral have been carried out using the numerical grid $\{\omega_i, \theta_j\}$, given by the relationships

$$\omega_i = 0.5 \times 1.12^i \quad \text{for} \quad i = 1, \cdots, 18, \quad (12a)$$

and

$$\theta_j = -\pi + (2\pi/24)j \quad \text{for} \quad j = 0, \cdots, 23. \quad (12b)$$

The choice of set $\{\omega_i, \theta_j\}$ is not optimal, from the numerical point of view, to reach accurate calculations; limited by the available technical capabilities (work station “Apollo-700”). Nevertheless, taking into account previous calculations (Polnikov 1989, 1990), the grid used provides an estimation of the kinetic integral with numerical error less than 15%; that is, it permits one to establish correctly all principal peculiarities of the NET for any spectrum shape.

With the purpose of finding the dependence of the NET parameters on the depth $h$, six variants of depths have been considered: $h = 360, 100, 10, 5, 3, \text{and} 2 \text{m}$. For a fixed value of the peak frequency $\omega_p = 1 \text{ rad s}^{-1}$, the depths considered correspond to the following six values of the nondimensional depth parameter

$$\varepsilon = \frac{k_p h}{\omega_0} = \frac{\omega_p^2}{\omega_0^2} = \frac{\omega_p h}{\omega_0}. \quad (13)$$

where $\varepsilon = 36, 10, 1, 0.5, 0.3, \text{and} 0.2$, respectively. As one can see, the set of depths used corresponds to the transition from the deep water case to the shallow water case. For each variant of $h$, the entire eight-dimensional array of matrix elements $M_{1,2,3,4}$ needs to be calculated anew, due to their mathematical structure, which makes our task more complicated technically.

The most important divergence between our calculations and those known from the literature (Herterich and Hasselmann 1980; Hasselmann and Hasselmann 1985) consists of the large variety of two-dimensional spectral shapes considered here. The representative series of shapes $S(\omega, \theta)$, presented in Table 1, covers a wide range of spectral shape parameters, defined as follows:

![Fig. 4. As in Fig. 2 but for run 4.](image-url)
the frequency width $\delta$:

$$\delta = \int S(\omega, \theta) \, d\omega \, d\theta \, S_p \omega_p = \sigma^2 S_p \omega_p \quad \text{(13a)}$$

angular narrowness $A(\omega)$:

$$A(\omega) = S(\omega, \theta_p) \, d\theta = S(\omega, \theta_p) / S(\omega) \quad \text{(13b)}$$

This choice ensures a reasonably thorough investigation of the problem.

Note that the normalizing coefficient $C(\omega)$ for the angular spreading function $\Psi(\theta)$ of spectrum $S(\omega, \theta) = S(\omega) \Psi(\theta)$ [in that case $C(\omega) = A(\omega) / \Psi(\theta_p)$] is not accounted for here because, in our choice of $\Psi(\theta)$, $C(\omega) = \text{const}$ for each variant, and it simply changes the magnitude of $S_p$ but does not influence the shape of the two-dimensional function of NET $T(\omega, \theta) = \partial S(\omega, \theta) / \partial \theta$.

As in previous studies, the spectral shape $S(\omega)$ is taken in the form typical for a deep water case. For this reason, in the present work we investigate only the influence of finite depth on the NET due to the matrix elements. The impact of shoaling on changing the spectral shape with the depth (as pointed out by Kitaigorodskii et al. 1975) should be considered separately.

With the goal of finding objective relationships of the two-dimensional NET function $T(\omega, \theta)$ with the parameters of spectral shape, we introduce the following informative parameters of function $T(\omega, \theta)$:

1) $T^+$ the absolute positive extremum (maximum) of $T(\omega, \theta)$
2) $T^-$ the absolute negative extremum (minimum) of $T(\omega, \theta)$
3) $\omega_1$ the frequency coordinate of $T^+$
4) $\omega_2$ the frequency coordinate of $T^-$
5) $\omega_0$ the frequency coordinate of zero point of $T(\omega, \theta)$ on the frequency axis
6) $T_l$ the local positive extremum of $T(\omega, \theta)$
7) $\omega_3$ the frequency coordinate of $T_l$
8) $\theta_l$ the angular coordinate of $T_l$.

Formally, the dependencies of all of these parameters of $T(\omega, \theta)$ on the parameters of spectrum $S(\omega, \theta)$, that is, on $\omega_p$, $\theta_p$, $S_p$, $\delta$, and $A(\omega)$ must be determined. The problem posed is analogous to that considered earlier for deep water (Polnikov 1989), but it is new for the case of finite depth.

4. Results of the calculations

Results of the calculations are presented in Figs. 1–6 and in appendix B for the six variants of spectral shapes given in Table I.

In the figures, the one-dimensional functions of NET [i.e., $T'(\omega) = \partial S(\omega) / \partial \theta = \int [\partial S(\omega, \theta) / \partial \theta] \, d\theta$] are given in units of dimensional constant $C$ presented above. With the aim to compare curves of $T(\omega)$ for different magnitudes of depth, the values of $T(\omega)$ are normalized by multiplication of $\partial S(\omega) / \partial \theta$ to a certain power function of the kind $\tanh k_0 h_0$) (using the simplified relationship $k_0 = \omega / g$). The fitting of parameter $n$ has shown that...
the values of normalized extrema $T^+, T^-$ become comparable, for different depth $h$, when $n$ has values in the range $n \approx 4–10$. For this reason, the functions $\partial S(\omega)/\partial \tanh^n(k_p h)$ are presented in Figs. 1–6.

In appendix B, as an example of direct numerical results, the two-dimensional NET functions are presented in a tabulated form for the first run given in Table 1. The values of $T(\omega, \theta)$ are given at the grid points $(\omega_i, \theta_j)$ in percentage of the difference $R = T^- - T^+$.

From these numerical data, the representative Tables 2–7 have been constructed, describing the dependencies of the main parameters of $T(\omega, \theta)$ on the shape parameters of the spectrum under the integral.

### 5. Analysis and parameterization

Analysis of the results presented above demonstrates the principal properties of the NET for the finite depth case. The most important properties are as follows:

1) The extremal values of the NET, $T^+$ and $T^-$, grow with diminishing depth $h$ approximately as function $\tanh^n(k_p h)$. The value of $n$ changes within the range $n \approx 4–6$ depending on the spectral shape. The smaller the parameters $\delta$ or $\Lambda(\omega)$, the greater $n$.

2) The parameters $\omega_0, \omega_1, \omega_2, \omega_3$ of the two-dimensional NET function $T(\omega, \theta)$ are complicated functions of depth and spectral shape.

3) The one-dimensional curve $T(\omega)$ and two-dimensional surface $T(\omega, \theta)$ shift to lower frequency as depth approaches zero ($k_p h \to 0$).

4) The size of the negative domain $V^2$ of $T(\omega, \theta)$ (along the frequency axis) becomes smaller as $k_p h \to 0$. Due to this, extremum $T^-$ grows faster than extremum $T^+$.

5) The local positive extremum $T^l$ becomes relatively large (with respect to $T^+$) while $k_p h \to 0$.

6) The angular spreading function of $T(\omega, \theta)$ is less changeable than the frequency dependence and is...
similar to the form typical of the deep water case.

Some of these properties can be seen, to some extent, in the figures in a previous paper (Hasselmann and Hasselmann 1985), but they are formulated here in physical terms for the first time.

The relatively wide variety of spectral shapes considered here permits construction of a more or less reasonable parameterization of the two-dimensional function $T(\omega, \theta)$.

The main idea of such parameterization resides in the following representation:

$$T(\omega, \theta) = C_n C \Phi(\omega, \theta) \tanh(\kappa_d h). \quad (14)$$

Here $C_n$ is the nondimensional fitting constant, $C = \frac{\pi g^{-1/2} \omega_0^3 S/8}{\delta}$ is the dimensionless constant of kinetic integral, $\Phi(\omega, \theta)$ is the angular spreading function of the NET, and the last factor in (14) is the main dependence of $T(\omega, \theta)$ on the depth [with the appropriate parameterization of the function $n(\omega_0, k_d h, \delta, A(\omega))$].

Taking into account only the last list of parameters, one can propose the following parameterization of the different parts of representation (14).

For parameter $n$,

$$n = \frac{(25 + 3\bar{e})/[(3.5 + 3\bar{\delta})A_p + 0.31]}{(15)}$$

coresponds to variation of $n$ in the range $4 \leq n \leq 6$ for $0.3 \leq \delta \leq 0.7, 0.6 \leq A_p = A(\omega_0) \leq 1$, and $\bar{e} = k_d h < 1$.

For function $\Phi(\omega, \theta)$, taking into account the previous experience of parameterization of the angular spreading function for deep water

Table 5. Dependence of function $T(\omega, \theta)$ parameters on depth for run 4.

<table>
<thead>
<tr>
<th>$k_d h$</th>
<th>$T^*$</th>
<th>$\omega_0$ for $T^*$</th>
<th>$\omega_0$ for $T^*$</th>
<th>$\omega_0/\omega_0$</th>
<th>$\omega_0/\omega_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>1.0</td>
<td>1</td>
<td>1</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>10</td>
<td>1.5</td>
<td>1.2</td>
<td>1.1</td>
<td>1.2</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>0.5</td>
<td>4.0</td>
<td>4.0</td>
<td>4.0</td>
<td>4.0</td>
<td>1.0</td>
</tr>
<tr>
<td>0.3</td>
<td>6.0</td>
<td>6.0</td>
<td>6.0</td>
<td>6.0</td>
<td>1.0</td>
</tr>
<tr>
<td>0.2</td>
<td>8.0</td>
<td>8.0</td>
<td>8.0</td>
<td>8.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

for $\omega < \omega_0$:

$$\Phi(\omega, \theta) = \frac{1.5 B + \delta}{A_p} \left[ \frac{\omega}{\omega_0} \Psi(\theta) \left( \frac{1}{\epsilon + \delta} \right) \right]$$

for $\omega_0 < \omega < \omega_0$:

$$\Phi(\omega, \theta) = \frac{1.5 B + \delta}{A_p} \left[ \frac{\omega - \omega_0}{\omega_0 - \omega_0} \Psi(\theta) \left( \frac{1}{\epsilon + \delta} \right) \right]$$

for $\omega > \omega_0$:

$$\Phi(\omega, \theta) = \frac{B + B_2 + \delta (0.5 + A_p)}{(\epsilon + 2\delta)(\omega_0 - \omega)^2 + 0.1 \epsilon^{-2} \Psi(\theta)}$$

where

$$\Psi(\theta) = [S(\omega, \theta)/S(\omega, \theta_p)]^{1/2(A_p)} \Psi(\theta)$$

$$\Psi(\psi) = \max \left\{ 0, \cos \left[ A_p (\theta - \theta_p) + 0.12 \pi \right] \times \frac{(\omega - \omega_0)}{(\epsilon + 0.1 \delta)} \right\}$$

$$\Psi(\psi) = \max \left\{ 0, \cos(0.12 \pi) + \cos(0.12 \pi) \right\}$$

and

$$\theta = [\theta - \theta_p] / \pi A(\omega)$$

The fitting coefficients are

$$B_1 = 60, \quad B_2 = 0.2, \quad B_3 = 3, \quad B_4 = 1 \times 10^{-4}$$

For points $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4$, the following expressions have been found:
6. Discussion

First, the properties of the NET described above reflect only the main features of the two-dimensional function \( T(\alpha, \theta) \) topological dependence on the spectral shape parameters. The description of fine details [such as dependence of coordinates of the local extremum \((\omega_1, \theta_1)\) and its intensity \( T_1 \) on values of \( \varepsilon, \delta, A(\alpha) \), or the specification of property 6] needs additional, more exact calculations of \( T(\alpha, \theta) \).

Regarding numerical accuracy, some attention should be paid to the remarkably anomalous results for the values \( \varepsilon = k_{pl}h \sim 1 \). For these values, the error of approximation (4) is greatest. This fact warrants caution in conclusions regarding the range \( \varepsilon \sim 1 \).

Second, it is interesting to understand why the high-frequency local extremum, \( T_1 \), becomes greater than that for low frequency \( T_\omega \) for the case of small values of \( k_{pl}h \). One may expect that for such small values of \( k_{pl}h \), the four-wave kinetic equation approximation is not valid for a description of the nonlinear wave dynamics, and a more accurate approximation needs to be derived. This point is to be investigated theoretically and numerically in more detail.

Further theoretical investigations should also clarify the role of three-wave interactions in the description of nonlinear properties of waves in finite depth and shallow water cases. Present papers dealing with this issue (Abrue et al. 1992; Madsen and Sorensen 1993) are rather questionable.

Nevertheless, the results obtained here give additional understanding of the role of nonlinearity in wave dynamics for the case of finite depth. They permit one to construct an appropriate analytical parameterization of the NET. In our proposal, the error of the parameterization, presented by Eqs. (14)–(17), is less than 15% in the description of extremal values of the numerical function \( T(\alpha, \theta) \).

At present, this parameterization is being used for numerical experiments with a wind wave model for shallow water, constructed recently by the author and colleagues (Polnikov and Sychov 1996). We have found that for rather shallow water \( (k_{pl}h \lesssim 0.3) \), the influence of nonlinearity becomes very strong and exceeds the effects of refraction and shoaling. This means that a reasonable numerical result cannot be obtained by modeling without correct use of the nonlinear evolution term in the source function. Investigation of this term is currently in progress.

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APPENDIX A

Some Referenced Formulas

Expressions for \( k_a, \theta_a \) as the functions of variables \( \omega_1, \theta_1, \omega_a, \theta_a \) are as follows:

\[
\begin{align*}
  k_a &= [k_1^2 + 2k_1\cos(\theta_1 - \theta_a)]^{1/2}, \quad (A1)
  \\
  \theta_a &= \arctan \left[ \frac{k_1 \sin \theta_1 + k_4 \cos \theta_1}{k_3 \cos \theta_1 + k_4 \cos \theta_1} \right], \quad (A2)
\end{align*}
\]

where \( k_1, k_4 \) are to be replaced by \( \omega_1, \omega_t \) via relationship (4).

Equations for singular points \( \omega_{\text{ima}} \) and \( \omega_{\text{lma}} \), following from Eqs. (9a,b), are of the kind

\[
\begin{align*}
  (\omega_1 + \omega_0 \omega_0^2)^{1/2} + gk_a &= 0, \quad (A3a)
  \\
  [(\omega_a - \omega_1)^2 + \omega_3^2(\omega_a - \omega_1)^2]^{1/2} &= 0, \quad (A3b)
\end{align*}
\]

Just these equations are solved numerically with respect to \( \omega_1 \) at each grid point \( \{\omega_1, \theta_1\} \).

A regular part of integrand in (10) is of the kind

\[
[M_{1,2,3,4}]^2 \times \frac{\{S_4 S_5 (\omega_1^2 S_1 + \omega_0^2 S_2) - S_1 (\omega_1 S_5 + \omega_0 S_2)\}}{\omega_1 \omega_2 \omega_3 \omega_4 [(k_1 + k_2 + k_3)(k_2 + k_4 - k_1)]^{1/2}},
\]

where \( S_i = S(\omega_1, \theta_1)S_i \) and \( \omega_0 = \omega_1^2 + \omega_0^2/2 \).

In the vicinity of point \( \omega_{\text{ima}} \), the denominator under the integral in Eq. (10) is of the form

Table 7. Dependence of function \( T(\alpha, \theta) \) parameters on depth for run 6.

<table>
<thead>
<tr>
<th>( k_{pl}h )</th>
<th>( T )</th>
<th>( \omega_1/\omega_0 ) for ( T )</th>
<th>( T )</th>
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\[
\omega_i = (0.55 + e/3 + \delta/6)\omega_i, \quad (17a)
\]

\[
\omega_2 = (\omega_i/\omega_p + 0.5\delta(\delta + e/2))\omega_p, \quad (17b)
\]

\[
\omega_0 = (\omega_i + \omega_2)/2, \quad (17c)
\]

\[
\omega_t = (\omega_i/\omega_p + 0.8(\delta e/\omega_p)^{1/3})\omega_p, \quad (17d)
\]
\[ D_1 = (b_j \delta)^{1/2} d_2^{1/2} |_{\omega - \omega_{\text{im}}}, \]  
(A5a)

where
\[ b_1 = \frac{2(\omega_{\text{im}}^2 + \omega_i^2/2)}{\omega_{\text{im}}^2 + \omega_i^2} + \frac{2((\omega_u - \omega_{\text{im}})^2 + \omega_i^2)}{((\omega_u - \omega_{\text{im}})^2 + \omega_i^2)^{1/2}}, \]  
(A5b)

and \( \delta = (\omega_i - \omega_{\text{im}}) \geq 0 \) is the small positive value.

In the vicinity of point \( \omega_{\text{im}} \), the same denominator has the form
\[ D_2 = \left( -\frac{\Delta}{2} + b_2 \delta^2 \right)^{1/2} d_2^{1/2} |_{\omega - \omega_{\text{im}}}, \]  
(A6a)

where \( b_2 \) is given by the relationship (11) and
\[ b_2 = 2 \left( 1 + \frac{6\omega_i^2}{\omega_u^2} \right) \left( 1 + \frac{4\omega_i^2}{\omega_u^2} \right)^{1/2}, \]  
(A6b)

with \( \delta = (\omega_{\text{im}} - \omega_i) \geq 0 \).

Note that due to changeability of the sign of \( \delta \), the singularity at point \( \omega_{\text{im}} \) is removed only after threefold integration. Owing to this technique, the singularity is integrated absolutely by the same way for both finite and deep water cases.

**APPENDIX B**

**Example of Numerical Results for Two-Dimensional NET**

Here we give as an example tables of the two-dimensional function of the NET \( T(\omega, \theta) \) for the first run presented in Table 1. Values of \( T^P \) and \( T^- \) are given in units of dimensional constant \( C \) (see text), but values of \( T(\omega, \theta) \) are given in percentage of difference \( R = T^P - T^- \). Vertical (angular) axis is given in units of \( \theta \) subindex \( j \) (\( \theta_{j,12} = 0^\circ \)), and horizontal (frequency) axis is given in units of \( \omega \) subindex \( i \). Due to angular symmetry of the NET, half of function \( T(\omega, \theta) \) is presented.

<p>| Table B1. Two-dimensional NET function ( T(\omega, \theta) ) for run 1 and ( k_{\text{eff}}h = 36. T^P = 31.8; T^- = -73.9 ). |
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<p>| Table B2. Two-dimensional NET function ( T(\omega, \theta) ) for run 1 and ( k_{\text{eff}}h = 10. T^P = 37.9; T^- = -91.4 ). |
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<p>| Table B3. Two-dimensional NET function ( T(\omega, \theta) ) for run 1 and ( k_{\text{eff}}h = 1. T^P = 276; T^- = -458 ). |
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### Table B4. Two-dimensional NET function $T(\omega, \theta)$ for run 1 and $k_p h = 0.5$. $T^+ = 780$; $T^- = -2370.$

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zero elsewhere

### Table B5. Two-dimensional NET function $T(\omega, \theta)$ for run 1 and $k_p h = 0.3$. $T^+ = 22 200$; $T^- = -37 100.$

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### Table B6. Two-dimensional NET function $T(\omega, \theta)$ for run 1 and $k_p h = 0.2$. $T^+ = 254 000$; $T^- = -535 000.$

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### REFERENCES


