Hysteresis of a Western Boundary Current Leaping across a Gap

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ABSTRACT
An idealized problem of a western boundary current of Munk thickness $L_M$ flowing across a gap in a ridge is considered using a single-layer depth-averaged approach. When the gap (of width $2a$) is narrow, $a \leq 3.12L_M$, viscous forces alone restrict penetration of the current through the gap. However, the gap is “leaky” in the linear case and some very weak flow still passes through. For larger gap width, the boundary current may leap across the gap due to inertia, characterized by the Reynolds number, completely choking off water exchange between the two basins. For $a > 4.55L_M$ the flow may be in one of two regimes (penetrating or leaping) for the same parameters, depending on previous evolution. The penetrating branch solutions become unsteady with eddies forming west of the gap between the two counterflowing zonal jets. As the boundary current slowly accelerates, transition from the penetrating to leaping regime happens when the width of a zonal jet near the gap becomes comparable with $a$, implying the Reynolds number $Re_e = (a/L_M)^3$. On the other hand, as the boundary current slowly decelerates, the leaping regime persists while the meridional advection dominates the $\beta$ effect in a wiggle of the current core within the gap, implying that the leaping regime breaks at $Re_L = L_M/a$. Thus hysteresis occurs over the range of Reynolds numbers $Re_L < Re < Re_e$. This behavior is analogous to the well-known teapot effect.

1. Introduction
The general ocean circulation is very complicated not only because of the variability of its main driving agents, wind and solar heat, but also because of irregularity of the bottom topography and coastlines. A mere look at the World Ocean bathymetric map reveals the existence of several major subbasins separated by midocean ridges with numerous gaps along fracture zones, which shape the pattern of abyssal flows. The $\beta$ effect and related westward intensification are crucial for large-scale flows.

Will the abyssal current following the ridge go through the gap or leap across it and continue? Is it intuitively clear that the narrower the gap the more difficult it is for the flow to go through. Is there a threshold below which the flow is choked off completely or is the gap always leaky? In other words, “when is a gap truly a gap?” The Stommel–Arons theory of abyssal flows (1960), which completely neglects inertia, predicts existence of zonal jets extending westward from every gap in a ridge (see also Pedlosky 1994). How does inertia change this picture?

The same questions are pertinent to the oceanic surface flows around island chains, like Hawaii (Qiu et al. 1997). Pedlosky and Spall (1999), using a linear theory, obtained an interesting result: that Rossby waves can easily penetrate (without much decrease in amplitude) through boundaries having multiple gaps. How will the flow change if inertia is included?

On larger spatial scales, the Loop Current in the Gulf of Mexico offers an example of the Gulf Stream leaping from Yucatan to the southern tip of the Florida Peninsula. A similar circumstance is the Kuroshio western boundary current in the region between the islands of Luzon and Taiwan.

Using a single layer depth-averaged approach in section 2 we formulate an idealized model of a western boundary current leaping across a gap in a ridge. Section 3 deals with the linear solution to the problem. The role of inertia, possible multiple states, and hysteresis in temporal evolution are analyzed in section 4. Finally, section 5 contains a summary and discussion.

2. Formulation of the problem
To address the questions posed in the introduction we shall consider an extremely idealized mathematical formulation of the problem that, nonetheless, retains all
essential aspects: a gap in a ridge, the $\beta$ effect, non-linearity, and viscosity.

We consider two ocean basins of constant depth separated by a straight ridge running north–south at $x = 0$ ($x$ is eastward; $y$ is northward). A gap of width $2a$, $-a < y < a$, in the ridge extends all the way from the ocean bottom to the surface. The ridge itself is assumed to be infinitely thin, while the ocean basins are infinitely large compared to $a$.

Based on the $\beta$-plane approximation, a quasigeostrophic flow in a single dynamically active layer is governed by the potential vorticity equation

$$
\frac{1}{L_R^2} \psi_x + \zeta_y + J(\psi_x, \zeta) + \beta \psi_y = A_L \nabla^2 \zeta, \tag{1}
$$

where $\psi$ is the streamfunction of a depth-averaged flow, the zonal and meridional velocity components are $u = -\psi_y$ and $v = \psi_x$, and $\zeta$ is the relative vorticity, which is related to $\psi$ by the Poisson equation

$$
\nabla^2 \psi = \zeta. \tag{2}
$$

The gradient of the Coriolis parameter is $\beta$. Dissipation is modeled by the lateral turbulent viscosity coefficient $A_L$, following Munk (1950). Here $L_R$ is the Rossby radius of deformation.

When $L_R \to \infty$, one has a barotropic model; and for finite $L_R$, a reduced-gravity model. In both cases steady solutions are the same, only the transients and stability of the flow are affected by $L_R$. Usually, the flow is somewhat more stable for smaller $L_R$. For simplicity, we shall take the limit $L_R \to 0$ when discussing time-dependent solutions, which allows us to focus on long baroclinic Rossby waves. To do so we merely have to drop the second term in (1), which is small compared to the first term.

We are interested in the dynamics of the western boundary current running along the eastern side of the ridge and encountering a gap and how this current may leap across the gap. The flow is generated by specifying the following kinematic boundary conditions: the streamfunction along both southern and northern parts of the ridge is constant

$$
\psi = 0 \quad \text{along } x = 0, \quad y < -a, \quad \text{and } y < a, \tag{3}
$$

while it is a different constant at the eastern boundary of the eastern basin, which is far to the east of the gap,

$$
\psi = Q \quad \text{at } x = +\infty. \tag{4}
$$

Due to the $\beta$ effect the value of $\psi = Q$ specified by (4) is carried westward along the planetary vorticity isolines $y = \text{const}$. This information is in conflict with the value $\psi = 0$ specified by (3) at the ridge. Therefore, on the eastern side of the ridge a boundary current forms with transport $Q$ per unit depth. The interior of the eastern domain is stagnant. The flow in the western basin ($x < 0$) is driven only by the penetration of the current through the gap. The western basin is assumed to be closed: the total flow through the gap is zero and the streamfunction in (3) is the same on both parts of the ridge. Therefore the current enters and then leaves the western basin. The dynamical boundary condition is no-slip at the ridge:

$$
u = 0, \quad v = 0 \quad \text{along } x = 0, \quad y < -a, \quad \text{and } y > a. \tag{5}
$$

We made an important assumption in (4) that the transport of the western boundary current $Q$ does not change or, equivalently, there is no mean zonal flow, $U(y) = 0$, in the Sverdrup interior in the eastern basin $x > 0$. Such flow can be produced by the combination of a source and a sink of equal strength $Q$ located on the eastern side of the ridge ($x > 0$) far north and south of the gap. Alternatively, one can imagine the same flow generated by a remote wind-curl distribution, details of which are not important to us so long as the total Sverdrup transport $Q$ in the vicinity of the gap does not change. That is why we do not introduce a wind forcing term in (1).

Sufficiently far north and south of the gap the flow is parallel to the boundary. Therefore the $y$ derivatives can be neglected and the nonlinear term vanishes. The dynamical balance is between the $\beta$ effect and cross-stream diffusion

$$
\beta \psi_y = A_L \psi_{xxx}, \quad |y| \to \infty, \tag{6}
$$

which gives rise to a Munk boundary layer of width

$$
L_M = \left(\frac{A_L}{\beta}\right)^{1/3} \tag{7}
$$

with the velocity profile

$$
\nu(\xi) = \frac{Q}{L_M} \frac{2}{\sqrt{3}} \exp(-\xi/2) \sin\left(\frac{\sqrt{3}}{2} \xi\right), \tag{8}
$$

where $\xi = x/L_M$.

There are four parameters in the problem: $Q$ the transport (per unit depth) of the western boundary current, $\beta$, $A_L$ the turbulent viscosity coefficient, and $a$ the half-width of the gap. If we use $L_M$ as a horizontal scale, then a steady solution of the problem will depend on two nondimensional parameters. It is convenient to use the nondimensional half-width of the gap

$$
\gamma = \frac{a}{L_M} \tag{9}
$$

and the Reynolds number

$$
\Re = \frac{Q}{A_L} \tag{10}
$$

characterizing the inertia of the western boundary current. Our goal is to describe quantitatively the dependence of the flow on $\gamma$ and $\Re$. The central question is
under what conditions can the western boundary current with the Munk velocity profile leap across the gap?

3. Linear solution: Munk boundary layer and $\beta$ plume

First we study a steady linear solution (Re = 0) of the problem (1)–(5), which is characterized by the balance between the $\beta$-effect term and the lateral friction term in (1). Since the lateral friction is represented by a biharmonic operator, the horizontal variables cannot be separated, and no analytic solution is available even in the linear case. In the linear problem the pattern of flow depends only on a single nondimensional parameter $\gamma$, and the flow pattern is unique. Also there is symmetry with respect to the central latitude $y = 0$, which means that the direction of the flow can be reversed.

In the limit of vanishing gap $\gamma \ll 1$ we have to recover a result that no flow can penetrate through the gap into the western basin. The western boundary current just runs along the ridge and conforms to the Munk velocity profile (8). On the other hand, when the viscous boundary-layer thickness is much smaller than the gap width, $L_M \ll 2a$, the flow will penetrate into the western basin, according to Stommel and Arons (1960), as two zonal jets originating from the southern and northern tips of the ridge gap. The transition between these two limits happens at $\gamma = O(1)$. To get a quantitative description of this transition, we have to adopt a numerical approach. A finite-difference approximation of the problem (1)–(5) was solved by time stepping until a steady state (or a quasi-steady) state was reached (details can be found in the appendix).

The streamfunction of a steady flow for $\gamma = 2.56$ is shown in Fig. 1a. The ridge is marked by unevenly spaced dots along the $y$ axis. They correspond to the grid nodes of the finite-difference model. The resolution is increased in the vicinity of the tips of the ridge. Only a small portion of a computational domain is shown. The horizontal coordinates are scaled by $L_M$. One can see that, when the gap has approximately the same width as the boundary current or less, the viscous forces prohibit the core of the current from penetrating into the western basin. Most of the streamlines make only a small wiggle near the gap. Nonetheless, some weak flow is induced in the western basin. The streamlines $\psi = 0$ separating from the tips of the ridge extend westward to infinity. They bound the so-called “$\beta$-plume” region in analogy with Stommel (1982). A steady $\beta$-plume solution for a larger gap, $\gamma = 5.39$, is shown in Fig. 1b. In this case all the streamlines originating in the Munk boundary layer enter the western basin.

Two counterflowing zonal jets extend from the tips of the ridge into the western basin. As one moves westward, the jets spread and interact; therefore the streamlines make a U-turn and leave the western basin. For large gaps (Fig. 1c, $\gamma = 17.86$) the two zonal jets extend westward from the tips but do not interact. Along the zonal strip between the jets, the value of the streamfunction comes from the Sverdrup interior of the eastern basin $\psi(y) = Q$. $|y| < a$. The jets start to interact much farther to the west when their width becomes comparable to $a$.

We need to describe quantitatively the penetration of the current into the western basin. Since the extent of penetration of different streamlines may be significantly different, we choose the streamline $\psi = Q/2$, which lies in the core of the boundary current. Within the Munk boundary layer (8) the streamline $\psi = Q/2$ is located at $x/L_M = 1.2838$, while the maximum of velocity is observed at $x/L_M = 2\pi/3\sqrt{3} = 1.2092$. Figure 2 shows the penetration measure $X_p$, defined as the farthest westward extent of the streamline $\psi = Q/2$, plotted against the gap half-width $a$. Both quantities are scaled by $L_M$. Due to the symmetry, in the linear case $X_p$ is the $x$ coordinate of the point where the streamline $\psi = Q/2$ intersects the line $y = 0$. For $\gamma = 0$ the extent $X_p/L_M = 1.2838$ as in the Munk boundary layer. No flow can penetrate through the zero-width gap. For small $\gamma$ viscous forces essentially prevent the core of the current from flowing through the gap, thus $X_p$ changes only slightly. The rapid increase of the extent of penetration occurs near $\gamma \approx 3$ when $X_p$ exceeds both the Munk scale $L_M$ and the gap half-width $a$. Around the same $\gamma$ all the streamlines start to weave through the gap (see Fig. 1b). The circulation in the $\beta$-plume region becomes strong. For large $\gamma$ the extent $X_p/L_M$ is proportional to $\gamma^4$, as we shall see below.

In the $\beta$-plume region away from the ridge, where the flow is confined in latitude and zonally elongated, the main balance is between the $\beta$ effect and the zonal diffusion

$$\beta \psi_x = A_L \psi_{yxyy},$$

Equation (11) is sometimes called a generalized heat equation. If the variable $-x$ is identified with time, then (11) would describe spreading of a heat anomaly $\psi$ caused by hyperdiffusion (a fourth-order operator, hence the name generalized) as compared with regular diffusion (described by a second-order operator). An extensive analysis of this equation can be found in Gill and Smith (1970) and Cleary (1987).

There is an asymptotic similarity solution of (11) depending on

$$\eta = \frac{y/L_M}{(-x/L_M)^{\gamma/4}},$$

which indicates a general algebraic spread of a jet or a plume westward. The transverse (meridional) structure of the flow can be expressed in terms of the functions $J_0(\eta)$ and $J_{\gamma-1}(\eta)$ (see Gill and Smith 1970), which behave like the error function and its derivative, with oscillatory tails decaying exponentially rapidly at infinity. For example, in the case of two overlapping jets.
Fig. 1. Streamfunction patterns $\psi(x, y)$ of steady linear solutions for several values of gap width illustrating three flow regimes: (a) $\gamma = 2.56$, the boundary current does not penetrate through the gap; (b) $\gamma = 5.39$, two zonal jets emerge from the gap but immediately start to overlap; (c) $\gamma = 17.86$, two zonal jets emerge from the gap and do not interact for some distance. The dots indicate grid nodes corresponding to the ridge. The contour interval is $0.1Q$. All the streamlines far east of the gap are $\psi = Q$ meaning very weak flow. The horizontal coordinates $x$ and $y$ are scaled by $L_M$. 

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4. Role of inertia, multiple states, and hysteresis

According to the linear theory there are two limiting flow regimes: the penetrating regime for large $\gamma$, when the flow enters the gap and extends westward, and the leaping regime for small $\gamma$, when the western boundary current leaps across the gap without much penetration into the western basin.

It is intuitively clear that inertia may help the boundary current to leap across the gap. To illustrate this effect a sequence of steady streamfunction patterns for $\gamma = 4.17$ and increasing $Re$ is shown in Fig. 4. The boundary current flows from south to north. Figure 4a shows the linear solution for $Re = 0$ with a strong well-developed $\beta$-plume. All streamlines from the Munk boundary layer penetrate into the western basin. The westernmost extent of the $\psi = Q/2$ streamline, $X_p/L_M = -42.51$, is marked by an arrow on the plot. For $Re = 20$ (Fig. 4b) the symmetry is broken, the circulation pattern in the $\beta$-plume is shifted northward. The turning point of the $\psi = Q/2$ streamline is no longer located at $y = 0$ but shifted northward too. The penetration scale itself is reduced to $X_p/L_M = -17.22$.

In Fig. 4c for $Re = 26$ we see dramatic reduction of circulation west of the gap. The $\beta$-plume region consists of two weak circulation cells of opposite sign. The westward extent of the $\psi = Q/2$ isosline does not exceed the gap half-width $a$. The recirculation near the edge of the ridge is somewhat stronger. Note that, as the inertia is increased, the main $\beta$-plume tongue shifts northward and disappears, being displaced by weak circulation of the opposite sign west of the gap.

For $Re = 40$ in Fig. 4d the flow through the gap is completely choked off. The core of the current makes only a small wiggle as the current leaps across the gap. The streamline $\psi = 0$ separates from the southern ridge edge ($y = -a$) and reattaches to the northern ridge. No
Fig. 4. A sequence of streamfunction patterns $\phi(x, y)$ for $\gamma = 4.17$ illustrating the transition from the penetrating regime in the linear case to the leaping regime, as the inertia ($Re$) of the boundary current is increased for (a) $Re = 0$; (b) $Re = 20$, $\beta$ plume modified by advection; (c) $Re = 26$, close to the stage where $X_P = -a$; (d) $Re = 40$, leaping boundary current: $X_P$ is indicated by an arrow. The contour interval is 0.1 $Q$. The dashed contour is negative. All the streamlines separating from the ridge are $\psi = 0$, there is no flow through the ridge.
water parcels are allowed to flow into the western basin; they all proceed northward with the boundary current. In the $\beta$-plume region circulation is weak and has a sense opposite that of the linear case. For larger Re the current leaps with less wiggle.

Also note the development of flow separation and recirculation near the edge of the northern ridge. In analogy to the flow past a flat plane this is a front-edge separation. It gradually appears and reaches its maximum approximately when the pattern of the flow rapidly changes from the penetrating to the leaping state. For larger Re, in the leaping regime, the separation tends to disappear.

Thus, as the inertia (Re) is increased, the boundary current can leap across the gap and the pattern of the flow changes from the penetrating to the leaping state. To quantitatively describe this effect we shall use the westward extent $X_P$ of the streamline $\psi = Q/2$ as a function of Re for several values of $\gamma$. Starting with $\gamma = 4.55$ (shown by stars) the curves are multivalued. In order to avoid crowding, the leaping branch (located near $X_P = 0$) is not continued to large Re. The termination point of the leaping branch at Re is marked by a cross for $\gamma = 4.96$ and by an open circle for $\gamma = 5.39$. Beyond the first instability the penetrating branch curve splits in four (dashed and dot-dashed) indicating the limits of variability of $X_P$ in the time-dependent solution (see text). Transitions along a hysteresis loop are marked by arrows.

![Figure 5](http://example.com/f5.png)

**Fig. 5.** The westward extent $X_P$ of the streamline $\psi = Q/2$ as a function of Re for several values of $\gamma$. Starting with $\gamma = 4.55$ (shown by stars) the curves are multivalued. In order to avoid crowding, the leaping branch (located near $X_P = 0$) is not continued to large Re. The termination point of the leaping branch at Re is marked by a cross for $\gamma = 4.96$ and by an open circle for $\gamma = 5.39$. Beyond the first instability the penetrating branch curve splits in four (dashed and dot-dashed) indicating the limits of variability of $X_P$ in the time-dependent solution (see text). Transitions along a hysteresis loop are marked by arrows.

For larger gap half-width $\gamma = 4.55$ we notice that the curve becomes multivalued, which means that there are at least two steady states for the same Re within a certain range of Re. These different steady states can be achieved by integrating from different initial conditions. In fact, in our numerical experiments we used two kinds of initial conditions. The first one is the $\beta$-plume linear solution. For the second kind we specified the Munk boundary current in the eastern basin and no flow in the western basin: as if the gap were closed and we let it open at $t = 0$. The majority of data points in Fig. 5 was obtained by starting from one of these initial conditions. For small Re, starting from the first initial condition inevitably led to steady flow patterns penetrating the gap ($\beta$-plume solutions slightly modified by non-linearity). On the other hand, for large Re starting from the closed gap initial condition led to the leaping regime.

Alternatively, each (penetrating or leaping) branch of the steady solutions can be traced by gradually varying parameters of the problem, in this case Re, using the so-called continuation method. We had to use the continuation method in order to obtain data points for Fig. 5 only near the values Re, Re, where the corresponding penetrating or leaping branches terminate (or more precisely turn around). Bifurcation theory suggests that there may be a third middle branch of steady solutions connecting the two overlapping branches (Re < Re), with the steady states of the middle branch being unstable and hence impossible to achieve by time integration (Gilmore 1981, chap. 9). Outside the overlapping interval of Re the steady solution is unique and can be achieved by integration from any initial condition.

As the Reynolds number is increased, the flow may become unsteady. For example, for $\gamma = 4.96$, the penetrating branch of steady solutions can be traced only for $\gamma < 41$. For larger Re eddies appear west of the gap (Fig. 6a) making the flow around the two zonal jets time dependent. The following evidence suggests that the formation of eddies is a result of shear instability rather than non-existence of a steady solution. First, the penetrating branch of the steady solution does not have any wiggles near the critical Reynolds number Re, which otherwise would indicate termination or turning of the branch. Second, varying $L_M$ changes Re, but the branch of steady solutions remains unaffected. Third, the increase in amplitude of eddies with the supercriticality parameter Re - Re, and equilibration by non-linearity indicate a Hopf bifurcation (Gilmore 1981, chap. 19).

When the solution becomes time dependent, the measure $X_P$ becomes ambiguous. Figure 6b shows $\max_{-\pi < x < \pi} \psi(x, y)$ as a function of $x$. The two undulating curves (solid and dashed) correspond to two moments in time separated by one-half period $T/2$ of the eddy formation cycle. The solution appears to be periodic in
FIG. 6. A snapshot (a) of the streamfunction \( \psi(x, y, t) \) of the periodic solution on the penetrating time-dependent branch for \( \gamma = 4.96, \Re = 45 \). The contour interval is 0.1\( Q \), negative streamlines are dashed.

(b) Plot of \( \max_{x \leq x \leq L_M} \psi(x, y) \) vs \( x \) (solid line). The dashed line is a similar curve for a moment in time half a period earlier (or later). The intersections with the dotted horizontal line \( Q/2 \) give multiple possible values for \( X_P \).

FIG. 7a. Expanded view of the solution branches for \( \gamma = 5.39 \).

The stable leaping branch and steady part of the penetrating branch are shown by solid lines with open circles indicating \( \Re_L \) and \( \Re_C \). Beyond the first instability \( \Re_C \) the penetrating branch curve splits in four (dashed and dot-dashed) indicating the limits of variability of \( X_P \) in the time-dependent solution (see text). \( \Re_C \) corresponds to termination of the time-dependent penetrating branch. Transitions along the hysteresis loop are marked by arrows. The letters (A through E) mark the different stages along the hysteresis loop with the corresponding patterns of streamfunction shown in (b).

In this case of \( \gamma = 4.96 \) and \( \Re = 45 \), with \( T_e = 15.93 \), \( L_M/(\beta L_M^2) \). The peaks of the curve correspond to anticyclonic eddy centers moving westward at the speed of \( O(\beta L_M^2) \) somewhat decreasing westward as the jets spread. Intersections of the solid curve with the horizontal dotted line give locations where the streamline \( \psi = Q/2 \) can be found at one particular instant. The existence of multiple intersections is what makes the measure \( X_P \) ambiguous.

The first intersection closest to the gap (e.g., at \( x/L_M = -13.8 \) for the solid curve) indicates the westward extent \( X_P \) of the boundary current core. As the eddies form and pinch off from the main stream, \( X_P \) varies with time. Note that the first intersection of the dashed curve is located farther west than the solid curve. Over an eddy formation cycle, \( X_P/L_M \) varies between \(-27 \) and \(-13 \) in this example. The intersection farthest from the gap indicates the position \( X_{PE} \) of the \( \psi = Q/2 \) streamline associated with an eddy disturbing the steady flow in the \( \beta \)-plume; \( X_{PE} \) varies between \(-62 \) and \(-50 \).

In Fig. 5 (and Fig. 7a) to mark the time-dependent part of the leaping branch, we use four curves indicating variability limits of \( X_P \) (two dashed curves) and \( X_{PE} \) (two dash-dotted curves). Thus a single solid curve corresponding to steady solutions of the penetrating branch for \( \Re < \Re_C \) splits into four curves at \( \Re_C \). In such a way we indicate the time-dependent part of the penetrating branch.

To trace the penetrating branch in the time-dependent regime, in our numerical experiments we increased \( \Re \) by small steps (usually \( \Delta \Re = 1 \)) and integrated in time until a periodic or quasi-stationary solution (one with steady variability) is reached. For \( \gamma = 4.96 \) the periodic (or quasi-stationary) regime persists until \( \Re_P = 47 \), beyond which the solution abruptly evolves toward the steady leaping state (that looks similar to Fig. 4d). This transition apparently happens because time-dependent perturbations kick the solution into the basin of attraction of the leaping branch. The leaping solutions are steady and this branch can be traced by the continuation method to yet higher \( \Re \). At these \( \Re \), starting the integration from the second kind of initial condition (closed gap) also brings the solution to the steady leaping state. The leaping branch of solutions can also be traced by decreasing \( \Re \) down to \( \Re_L = 39 \) (for \( \gamma = 4.96 \)). For smaller \( \Re \), flow rapidly evolves toward the penetrating branch state, which is steady for such values of \( \Re \).

Since the penetrating (steady or quasi-stationary) branch and leaping branches overlap, \( \Re_L < \Re_P \), a hysteresis in temporal evolution may occur. This happens if the Reynolds number of the boundary current varies slowly, with a timescale longer than the transition
Fig. 7b. Stage a: Re = 50, steady penetrating branch solution; B: Re = 55, time-dependent penetrating branch periodic solution; C: Re = 66, transient flow from the penetrating to leaping regime; D: Re = 50, steady leaping branch solution; E: Re = 45, transient flow from leaping to penetrating regime. The contour interval is 0.25Q. Negative streamlines are dashed. The streamlines in the eastern basin away from the boundary current are $\psi = Q$ meaning very weak flow. The horizontal coordinates $x$ and $y$ are scaled by $L_M$ as in Fig. 1.
time from one branch to another, and also if the range of variability of $Re_p$ spans both $Re_p$ and $Re_{p'}$.

The hysteresis for $γ = 5.39$ (in this case $Re_L = 46$, $Re_p = 52$, $Re_{p'} = 65$) is illustrated in Fig. 7a with different stages marked by letters A through E along the solution branches. Corresponding flow patterns are shown in the five panels of Fig. 7b marked by the same letters. For example, starting from small $Re$, as $Re$ is increased the solution follows the penetrating branch (stage A). Then it becomes unsteady for $Re < Re_c$, for $Re = 55$ the solution appears to be periodic (stage B). Then, as $Re$ exceeds $Re_{p'}$, the solution rapidly evolves toward the leaping state. Stage C shows a snapshot of the transient stage: the boundary current leaps the gap, previously formed eddies are moving westward, while no new eddies pinch off. The solution then stays on the steady leaping branch as $Re$ is increased further. On the backward swing, as $Re$ is decreased, the solution follows the steady leaping branch down to $Re_L$ (see stage D showing the steady flow pattern for $Re = 50$). Then, as $Re$ is decreased just below $Re_L$, the leaping regime breaks and flow starts to penetrate through the gap into the western basin. Stage E shows the growth of the loop current with formation of some eddies. When transient motions have died out, the solution settles onto the steady penetrating branch (with flow pattern similar to one in stage A) and remains on it for smaller $Re$. The cycle repeats. Note that stages A and D show two different steady solutions for exactly the same parameters ($Re = 50$). Thus when the Reynolds number of the boundary current is within the range $Re_p < Re < Re_L$ the flow may penetrate or leap the gap depending on the history of evolution.

The phenomenon of inertia leading to appearance of multiple (steady) states is quite common in nonlinear problems. In the context of wind-driven circulation in an oceanic subtropical gyre, multiple equilibria were analyzed in Ierley and Sheremet (1995) and Sheremet et al. (1997). A more familiar example is the so-called teapot effect. If you want to pour tea into a cup and start to tilt a teapot rapidly, then tea will flow around the lip of the spout and spill all over (this is analogous to our penetrating branch). However, if you tilt the teapot rapidly, then the flow will separate from the lip of the spout and go right into the cup (in analogy with our leaping branch). You can also spill the tea by not returning the teapot to the horizontal position rapidly enough (by letting the flow stay on the penetrating branch). Some poorly designed teapots are exceptionally good at making spills. With those, one can easily produce two different kinds of flow for the same angle of tilt depending on whether the flow rate has been increased or decreased. A recent paper containing mathematical analysis and laboratory experiments with the teapot effect is by Kistler and Scriven (1994: see also references given therein). In civil engineering concerned with the design of expanding spillways and dam overflows, a similar phenomenon is known under the name Coanda effect, and it has a long history of research. One can argue about what forces (viscous, surface tension, $β$ effect, etc.) are crucial in governing the separation of the flow in a particular problem; however, it is curious to note that using the theory of potential flow of inviscid fluid Vanden-Broeck and Keller (1986) have constructed solutions with the separation located at the lip or at an arbitrary point along the spout, thus reproducing both regimes.

As the gap width is increased, the leaping of the boundary current happens at larger $Re$. Figure 8 summarizes the onset of transitions between penetrating and leaping branches for varying $γ$. For narrow gaps, $γ < 3.12$, flow cannot penetrate the gap. The current leaps even in the linear case $Re = 0$. For larger $γ$ up to 4.17 the transition to leaping is single-valued: there is no distinction between $Re_p$ and $Re_L$ and there is no hysteresis. For $γ = 4.55$ multiple steady states are possible: the penetrating and leaping branches overlap. $Re_p$ and $Re_L$ are different, hysteresis occurs. For larger $γ$ the transition between the penetrating and leaping flow regimes tends to happen at larger $Re$, with a larger difference between $Re_p$ and $Re_L$.

Based on Fig. 8 we hypothesize that leaping occurs when a combination of the viscous boundary layer width $L_M$ and some inertial length scale, $L_Q$, exceeds the gap width

$$C_1L_M + C_2L_Q > a,$$  \hspace{1cm} (14)

where $C_1$ and $C_2$ are $O(1)$. The appropriate inertial length scales for the leaping-to-penetrating ($L_{\omega M}$) and for penetrating-to-leaping ($L_{\omega P}$) regime transitions are apparently different as are the respective $Re_L$ and $Re_p$. They can be obtained from balancing the dominant terms (advection and $β$ effect) in (1).
When the leaping regime is about to break, we can assume that the meridional advection balances the β effect in (1):

$$V \frac{Z}{a} = \beta V,$$

where $V = Q/L_M$ is the meridional velocity scale and $Z = Q/L_M$ is the relative vorticity scale in the western boundary current. As the leaping current wiggles within the gap, $\zeta$ varies by $O(Z)$ over the distance $a$. Neglecting $L_M$ in (14) when $L_M \ll a$, this leads to

$$L_{op} = \frac{Q}{\beta L_M^2} \approx a,$$

which translates to a nondimensional form

$$\text{Re}_L \equiv \gamma, \quad \gamma \gg 1.$$  

Thus to maintain the leaping regime, the velocity of the boundary current should linearly increase with the gap width.

On the other hand, starting from the penetrating regime, a transition to leaping may occur if the two zonal jets begin to overlap within the gap. The meridional width of the inertial jet flowing toward the tip of the ridge is controlled by the balance between zonal advection and the β effect in (1):

$$UZ \approx \beta Q,$$  

where $U = Q/L_M$ is the zonal velocity scale and $Z = Q/L_M^2$ is the scale for relative vorticity. Condition (18) can be interpreted as the hydraulic control on a β plane with respect to Rossby waves (see Sheremet, 2000, manuscript submitted to *J. Phys. Oceanogr.*). It follows from (14), again neglecting $L_M$ when $L_M \ll a$, that

$$L_{op} = \left( \frac{Q}{\beta a} \right)^{1/3} \approx a,$$  

and assuming that the southern, westward flowing, jet has similar or smaller width, transition to leaping should occur for

$$\text{Re}_p \equiv \gamma^3, \quad \gamma \gg 1.$$  

That is why the Re curve in Fig. 8 tends to be straight, while the Re curve bends upward for large $\gamma$. The curve Re $p$ is quite accurately fit by the cubic $\text{Re} = (\gamma - C_1)^3/C_2$ [corresponding to the nondimensional form of (14) with the equality sign] with $C_1 = 1.28$ and $C_2 = 1.0$, which is shown by dots in Fig. 8.

It is an open question whether the boundary current can leap an arbitrarily wide gap provided that the inertia (or Re) is sufficiently high. The difficulty stems from the instability of the laminar Munk boundary current itself for $\text{Re} = O(100)$. Stability analysis of parallel flow with the Munk velocity profile by Ierley and Young (1991) indicates that the first instability occurs at $\text{Re}_p = 21.574$. This result is valid for a barotropic model with $L_n = \infty$. Introducing a finite Rossby radius of deformation shifts the stability limit to a somewhat higher Re. The stability is also affected by the meridional extent of the domain and grid stretching in the numerical model. Anyway, beyond the critical Re, eddies appear in the boundary current, which enhance turbulent diffusion, therefore the effective Re of the boundary current is reduced. Understanding the leaping of the boundary current in a turbulent regime requires further work, possibly involving laboratory experiments.

5. Summary and discussion

We have considered an idealized problem of a western boundary current of Munk thickness $L_M$ flowing across a gap in a ridge. When the gap (of width $2a$) is narrow, $a \leq 3.12L_M$, viscous forces alone restrict penetration of the current through the gap. However, the gap is “leaky” in the linear case; some very weak flow still passes through. For larger gap width, the boundary current may leap across the gap due to inertia. In the nonlinear case flow through the gap may be completely choked off. For $a \approx 4.55L_M$, the flow may be in one of two regimes (penetrating or leaping) for the same parameters depending on previous evolution.

As the boundary current slowly accelerates, transition from the penetrating to the leaping regime happens when the width of a zonal jet near the gap becomes comparable to $a$, implying $\text{Re}_p \approx \gamma^3$. On the other hand, as the boundary current slowly decelerates, the leaping regime persists while the meridional advection dominates the β effect in a wiggle within the gap, implying that the leaping regime breaks at $\text{Re}_L \equiv \gamma$. Thus the hysteresis occurs over the range of Reynolds numbers $\text{Re}_L < \text{Re} < \text{Re}_p$. This behavior is analogous to the well-known teapot effect.

In the problem formulation, we have omitted other effects that may influence flow leaping in the real ocean, such as finite curvature of the ridge tip, tilt of the ridge relative to meridian, bottom slope, and bottom drag. We have assumed that the boundary current has the Munk velocity profile, which naturally develops for uniform boundary currents of constant transport. However, if the boundary current accelerates downstream, the nonlinearity will determine not only the typical velocities but also the velocity profile. In this regard it is important to note that the expression for $L_{op}$ (19) does not contain $L_n$ and depends entirely on inertial characteristics of the current, while in the expression for $L_{op}$ (16) $L_M$ sets the scale for the current width. We also note that due to the symmetry of the β-plane approximation to the coordinate transformation $\gamma \rightarrow -\gamma$ the occurrence of hysteresis is invariant of whether the boundary current flows northward or southward.

An interesting application of this problem may be to the flow in Luzon Strait. It was long ago recognized (Nitani 1972) that the Kuroshio may either leap across the strait or penetrate well into the South China Sea as a loop current, possibly depending on the season. The
half-width of the strait is \( a = 170 \) km, \( \beta = 2 \times 10^{-11} \) m\(^{-1}\) s\(^{-1}\). An average volume transport of the Kuroshio just east of the northern part of Luzon is estimated as \( V = 30 \times 10^6 \) m\(^3\) s\(^{-1}\) from hydrographic sections, with the vertical distribution having an \( e \)-folding depth scale of \( H = 250 \) m ( Nitani 1972). This gives \( Q = 1.2 \times 10^4 \) m\(^2\) s\(^{-1}\), and therefore \( L_{op} = 182 \) km according to (19). Analysis of the potential vorticity (Nitani 1972) suggests that the Kuroshio at its beginning may have an inertial character like the Gulf Stream (Charney 1955); hence \( L_{op} \ll a \), which is a necessary requirement for (19) to be valid. This means that normally the Kuroshio can leap across Luzon Strait \((L_{op} \simeq a)\). However, during periods when its strength is substantially reduced, it may penetrate into the South China Sea. Thus multiple states and hysteresis are likely to occur.

Because of the possible hysteresis, in analyzing the observational data, it is important to correlate the Kuroshio penetrations not only with the parameters describing the present state of the current, but also to take into account its history. For example, Farris and Wimbush (1996) found a relationship between the loop current stage (derived from satellite infrared images) and the wind stress history: the Kuroshio penetrations occur when the time-integrated strength of the northeast monsoon exceeds a threshold value. This is in qualitative agreement with the present theory in the sense that the penetrations occur when the Kuroshio is weakened by the monsoon blowing in the opposite direction.

There is increased interest in oceanographic observations in this region with several major observational programs planned or under way having a goal of understanding mesoscale processes at the continental slope along the Chinese coastline and the role of the Kuroshio penetrations in them. It would be worthwhile looking for hysteresis of the flow in results of numerical simulations with the realistic bottom topography of the Luzon Strait. Also, laboratory experiments should be conducted to study the current leaping phenomenon in the turbulent flow regime.

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APPENDIX

Important Details of the Numerical Model

The problem (1)–(5) was solved numerically using a finite difference approximation. In order to obtain high resolution in the Munk boundary layer, near the tips of the ridge, and to allow the solution to reach an asymptote at large distance, both the \( x \) and \( y \) coordinates were transformed and all partial derivatives replaced according to

\[
x = x(\xi): \quad \frac{\partial}{\partial x} f(x, y) = A(\xi) \frac{\partial}{\partial \xi} f(\xi, \eta),
\]

\[
A(\xi) = \left(\frac{dx(\xi)}{d\xi}\right)^{-1},
\]

\[
y = y(\eta): \quad \frac{\partial}{\partial y} f(x, y) = B(\eta) \frac{\partial}{\partial \eta} f(\xi, \eta),
\]

\[
B(\eta) = \left(\frac{dy(\eta)}{d\eta}\right)^{-1},
\]

where

\[
A(\xi) = \frac{A_0}{1 + (\xi/L)^2},
\]

\[
B(\eta) = B_0 \left( \frac{1}{1 + (\eta - \eta_0)^2/L^2} + \frac{1}{1 + (\eta - \eta_0)^2/L^2} \right).
\]

The normalization constants were chosen so that \( A(0) = 1 \), \( B(\eta_0) = B(\eta_{-0}) = 1 \), where \( \eta \) and \( \eta_{\pm 0} \) correspond to the southern and northern tips of the ridge. Typically we used \( L = 4L_{M} \) and horizontal resolution \( \Delta x = \Delta y \equiv L_{M}/4 \) in the vicinity of the gap.

All spatial derivatives were then approximated by standard centered differences. A vorticity-conserving scheme was used for the advective term everywhere except for the five grid points adjacent to the tips of the ridge where upstream differences were utilized in order to avoid spurious jigsaw oscillations at high Re. A semi-implicit time-stepping scheme was implemented for the advective and \( \beta \)-effect terms, which prevents nonlinear numerical instability.

REFERENCES


