Theory of the Trojan-Horse Method

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The Trojan-Horse method is an indirect approach to determine the energy dependence of S factors of astrophysically relevant two-body reactions. This is accomplished by studying closely related three-body reactions under quasi-free scattering conditions. The basic theory of the Trojan-Horse method is developed starting from a post-form distorted wave Born approximation of the T-matrix element. In the surface approximation the cross section of the three-body reaction can be related to the S-matrix elements of the two-body reaction. The essential feature of the Trojan-Horse method is the effective suppression of the Coulomb barrier at low energies for the astrophysical reaction leading to finite cross sections at the threshold of the two-body reaction. In a modified plane wave approximation the relation between the two-body and three-body cross sections becomes very transparent. Applications of the Trojan Horse Method are discussed. It is of special interest that electron screening corrections are negligible due to the high projectile energy.

§1. Introduction

Many astrophysical models depend heavily on precise information about nuclear reaction rates that are ideally measured directly in the laboratory. However, cross sections of reactions with charged particles become very small with decreasing energy due to the Coulomb barrier and the astrophysically relevant energy range cannot be reached in direct measurements except a few cases. Therefore, the cross section \(\sigma(E)\) at low energies is obtained by extrapolating experimental data at higher energies with the astrophysical S factor

\[
S(E) = \sigma(E) E \exp(2\pi\eta),
\]

where \(E\) is the c.m. energy and \(\eta = Z_1 Z_2 e^2 / (\hbar v)\) is the Sommerfeld parameter depending on the charge numbers \(Z_1, Z_2\) of the colliding nuclei and their relative velocity \(v\). The extrapolation process introduces uncertainties and important contributions to the cross section, like resonances, can be missed. Additionally, a correction has to be applied to obtain the cross section for bare nuclei because direct laboratory measurements are affected by electron screening that enhances the measured cross sections.\(^2,3\) Independent information on low-energy cross sections is valuable for a quantitative description of electron screening that is not yet completely understood.

During the last years, several indirect methods have been developed to extract astrophysically relevant cross sections from related reactions at higher energies. For example, the Coulomb dissociation method\(^4\) and the method of asymptotic normalization coefficients (ANC)\(^5,6\) allow to extract information on low-energy radiative capture reactions. For general nuclear reactions the Trojan-Horse method (THM) can be applied. In this approach the astrophysical two-body reaction is replaced by a suitably chosen three-body reaction that is measured under special kinematical
conditions. The relation between the cross sections is established with the help of reaction theory. Without doubt, the indirect process will introduce some uncertainties, but valuable information can be obtained on the astrophysical reaction. Additionally, the errors are independent of that of the direct measurement. Of course, firm conclusions can be drawn from indirect experiments only if the methods have been validated by studying well-known reactions and if the theoretical approximations are understood.\textsuperscript{7)

A similarity between cross sections for two-body and closely related three-body reactions under certain kinematical conditions\textsuperscript{8) led to the introduction of the Trojan-Horse method,\textsuperscript{9)--11) see also Refs. 12) and 13). In this indirect approach a two-body reaction

\[ A + x \rightarrow C + c \] (1.2)

that is relevant to nuclear astrophysics is replaced by a reaction

\[ A + a \rightarrow C + c + b \] (1.3)

with three particles in the final states assuming that the Trojan Horse \( a \) is composed predominantly of clusters \( x \) and \( b \), i.e. \( a = (x + b) \). This reaction can be considered as a special case of a transfer reaction to the continuum. The energy in the entrance channel of reaction (1.3) is chosen around or above the Coulomb barrier and effects from electron screening are negligible. Nevertheless, under quasifree kinematical conditions very small energies can be reached in reaction (1.2). The essential feature of the THM is the suppression of the Coulomb barrier in the two-body reaction. The cross section of the three-body reaction remains finite when the c.m. energy in the \( A + x \) system approaches zero.

In \S 2 some general aspects in the theoretical description of transfer reactions into the continuum are discussed. This leads to the formulation of the THM theory. In a modified plane-wave approximation the relation between the cross section of reactions (1.2) and (1.3) becomes very transparent. For details we refer to Ref. 11). Applications of the THM are discussed in \S 3 where also a summary and an outlook are presented.

\section*{\S 2. Theory}

\subsection*{2.1. Transfer reactions into the continuum in post-form DWBA}

We assume a three-body model where the target nucleus \( A \) interacts with a projectile \( a = b + x \). The T-matrix element for the elastic breakup reaction

\[ A + a \rightarrow A + x + b \] (2.1)

is given in the post-form of the distorted-wave Born approximation (DWBA) as (see also Eq. (10) of Ref. 12))

\[ T = \langle \chi_{Bb}^{(-)}(\vec{k}_{Bb})\xi_{B}^{(-)}(\vec{k}_{Ax})\Phi_{b}|V_{xb}|\chi_{Aa}^{(+)}(\vec{k}_{Aa})\Phi_{A}\Phi_{a} \rangle, \] (2.2)

where \( B \) denotes the system \( A + x \) in the final state. \( \Phi_{a}, \Phi_{b}, \) and \( \Phi_{A} \) are the bound-state wave functions of \( a, b \) and \( A \), respectively, and \( V_{bx} \) is the potential between \( x \)
and $b$. The $\chi$’s are the scattering wave functions generated by the appropriate optical potentials. This expression for the T-matrix element is quite difficult to evaluate in general. At high beam energies eikonal methods\(^\text{14)}\) can be used to simplify it. For an intermediate model see, e.g., Ref. 15). It contains some simple limits, like the Serber model: see, e.g., Ref. 12). In the distorted waves of Eq. (2.2) the interaction of the target with the “participant” $x$ as well as the “spectator” $b$ is included to all orders in general.

It is of interest to treat also the case where the subsystem $B = A + x$ can go to other final channels $C + c$. This reaction is sketched in Fig. 1 with the relevant momenta of the nuclei. The theoretical description is especially simple when the “surface approximation” can be applied: due to Coulomb repulsion and/or strong absorption the “wave function of the transferred particle”

$$\int d\xi \Psi_B^{(-)} \Phi_A = 4\pi \sum_{lm} i^l f_l(r_{Ax}) Y_{lm}(\hat{r}_{Ax}) Y_{lm}^*(\hat{k}_C) \tag{2.3}$$

has only to be known in the nuclear exterior. The integration in Eq. (2.3) is over the nucleon variables of $A$. In this case the overlap integral is given in terms of the S-matrix element of the $C + c \rightarrow A + x$ reaction, which we denote by $S_l$, as

$$f_l(r_{Ax}) = \delta_{AxCc} j_l(k_{Ax}r_{Ax}) + \frac{1}{2} \frac{m_{Ax}k_{Ax}}{m_{Cc}k_{Cc}} (S_l - \delta_{AxCc}) h_l^{(+)}(k_{Ax}r_{Ax}) \tag{2.4}$$

for $r_{Ax} \geq R$ with a cutoff radius $R$. Here we assume spinless particles for the sake of simplicity. For charged particles $x$ the appropriate Coulomb functions have to be used in place of the Bessel (Hankel) functions $j_l$ ($h_l^{(+)}$). The validity of the surface approximation was checked by Kasano and Ichimura.\(^\text{16)}\) It was found to be quite good for the (d,p) reaction at $E_d = 26$ MeV. Inclusive breakup spectra were measured for many different systems and compared to theory. Agreement is generally good.\(^\text{12)}\)

The theory of inclusive breakup reactions was substantially generalized in a series of papers by Ichimura, Austern and Vincent (“IAV”). We give two references,
from where the full story can be traced back.\textsuperscript{17, 18} In this series of papers, also many formal aspects have been deeply elucidated and the relation of post-form to prior-form DWBA (they give identical results) has also been made very clear.

2.2. Cross section in modified plane wave approximation and THM

The appearance of the S-matrix element of the two-body reaction in (2.4) allows to establish a relation between the cross sections of reaction (1.2) and (1.3), see Ref. 11) for details. Replacing the distorted waves in Eq. (2.2) by plane waves and applying the surface approximation, the cross section for the three-body reaction

\[
\frac{d^2\sigma}{dE_{C_c}d\Omega_{C_c}d\Omega_{B_b}} = KF \left| W(\vec{Q}_{B_b}) \right|^2 \frac{d\sigma^{TH}}{d\Omega} \tag{2.5}
\]

factorizes into a kinematical factor

\[
KF = \frac{\mu_{Aa}\mu_{Bb}\mu_{C_c}}{(2\pi)^5h^6} \frac{k_{B_b}k_{C_c}}{k_{Aa}} \frac{16\pi^2}{k_{Aa}Q_{Aa}v_{Aa}}, \tag{2.6}
\]

a momentum distribution \(|W|^2\) and the so-called TH cross section \(d\sigma^{TH}/d\Omega\) (see Ref. 11) for the definition of reduces masses, momenta etc.). The momentum amplitude

\[
W(\vec{Q}_{B_b}) = -\left( \mathcal{E}_a + \frac{\hbar^2Q_{B_b}^2}{2\mathcal{E}_{xb}} \right) \langle \exp(i\vec{Q}_{B_b} \cdot \vec{r}_{xb}) \Phi_x \Phi_b | \Phi_a \rangle \tag{2.7}
\]

is related to the wavefunction of the Trojan horse \(a\) with binding energy \(\mathcal{E}_a(>0)\) in momentum space. It depends on the momentum

\[
\vec{Q}_{B_b} = \vec{k}_{B_b} - \frac{m_b}{m_b + m_x} \vec{k}_{Aa}. \tag{2.8}
\]

Neglecting the Fermi motion of \(b\) inside the Trojan Horse the second term is the momentum of the incoming spectator \(b\) with respect to \(A\), and \(-\vec{Q}_{B_b}\) corresponds to the momentum transfer to the spectator \(b\). The momentum distribution essentially describes the Fermi motion of \(b\) and \(x\) inside the Trojan horse \(a\). The TH cross section

\[
\frac{d\sigma^{TH}}{d\Omega} = \frac{1}{4k_{C_c}^2} \sum_l (2l + 1) P_l(\vec{k}_{C_c} \cdot \vec{Q}_{Aa}) \left| S_lJ_l^{(+)} - \delta_{AxCc}J_l^{(-)} \right|^2 \tag{2.9}
\]

with Legendre polynomials \(P_l\) and the S-matrix elements \(S_l\) looks very similar to the cross section for the inverse of the astrophysical reaction (1.2) except for the TH integrals

\[
J_l^{(\pm)}(R, \eta_{Ax}, k_{Ax}, Q_{Aa}) = k_{Ax}Q_{Aa} \int_{-\infty}^\infty dr \ r \ u_l^{(\pm)}(\eta_{Ax}; k_{Ax}r) j_l(Q_{Aa}r) \tag{2.10}
\]

with the Coulomb wave functions \(u_l^{(\pm)} = e^{\mp\sigma_l}(G_l \pm iF_l)\). The TH integrals depend on the cutoff radius \(R\) of the surface approximation, the c.m. momentum \(k_{Ax}\) in the \(A + x\) relative motion and

\[
\vec{Q}_{Aa} = \vec{k}_{Aa} - \frac{m_A}{m_A + m_x} \vec{k}_{B_b}, \tag{2.11}
\]
that reduces to $-k_x$ for target mass $m_A \to \infty$. The properties of the TH integrals are discussed extensively in Ref. 11).

The expression (2.5) resembles the form of the cross section in a plane-wave impulse approximation\,\textsuperscript{19} that has been used in the past in order to extract information on the momentum distribution of nuclei. However, only the DWBA with the surface approximation explains the effective reduction of the Coulomb barrier for small c.m. energies in the $A + x$ system.

2.3. Threshold behaviour of cross sections

The energy dependence of the two-body cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k_{Ax}^2} \left| \sum_l (2l + 1) P_l \langle \hat{k}_C \cdot \hat{k}_{Ax} \rangle S_l \right|^2 \propto k_{Ax}^{-2} \exp(-2\pi\eta_{Ax}) \tag{2.12}$$

for the inelastic reaction (1.2) is governed by the $k_{Ax}^{-2}$ factor and the energy dependence $S_l \propto \exp(-\pi\eta_{Ax})$ of the relevant S-matrix element. This motivates the introduction of the astrophysical S factor (1.1) for the extrapolation of experimental data to low energies. In the TH cross section (2.9) the factor $k_{Ax}^{-2}$ is replaced with $k_{Cc}^{-2}$ and the TH integrals $J_l^{(\pm)}$ appear. Their energy dependence for small $k_{Ax}$ is determined by $k_{Ax}\sqrt{v_{Ax}} \exp(\pi\eta_{Ax})$ from the contribution of the irregular Coulomb wave function. This leads to a $k_{Ax}$ dependence of the three-body cross section (2.5) according to

$$\frac{d^3\sigma}{dE_C d\Omega_C d\Omega_c} \propto k_{Ax}^{-2}v_{Ax}^{-1} \exp(-2\pi\eta_{Ax})k_{Ax}^2v_{Ax} \exp(2\pi\eta_{Ax}) = \text{const} \tag{2.13}$$

in the lowest order of $k_{Ax}$. As a result the cross section does not vanish at the threshold but takes on a finite value. Also in the case of neutron transfer, like in a (d,p) stripping reaction, it is well known that the cross section is finite at the threshold $E_n = 0\,\text{,}^{12, 13}$ The reason is the same as in the case of charged particles: the momentum dependence of the S-matrix element is cancelled by the corresponding Trojan-Horse enhancement factor.

In a similar way, the threshold behaviour can be studied in the elastic breakup case. In this case we have three contributions, the pure Coulomb, the nuclear and the interference term, see Eqs. (70)–(73) of Ref. 11). All three terms show the same threshold behaviour, the cross section behaves as $k_{Ax} \exp(-2\pi\eta)$ close to threshold. In contrast, the Coulomb term dominates in the direct two-body elastic scattering of the $A + x$-system. The $d + p \to p + p + n$ breakup reaction was studied recently in the relevant kinematical region in Ref. 20).

2.4. Kinematical conditions

In most experiments so far nuclei with a dominant s-wave contribution in their ground state have been employed as Trojan horses. Then, the momentum amplitude $W(\hat{Q}_{BB})$ has a maximum at zero. Correspondingly, the equation $\hat{Q}_{BB} = 0$ defines the so-called quasi-free condition in the three-body phase space where the cross section for the quasi-free reaction reaches a maximum. From this condition the
corresponding quasi-free c.m. energy

\[ E_{Ax}^{qf} = E_{Aa} \left( 1 - \frac{\mu_{Aa}}{\mu_{Bb}} \frac{\mu_{Ba}^2}{m_x^2} \right) - \varepsilon_a \]  

in the initial channel of the two-body reaction (1.2) is derived from energy conservation. The relation between \( E_{Ax}^{qf} \) and \( E_{Aa} \) is purely a kinematical consequence. It is obvious that even with a large c.m. energy \( E_{Aa} \) in the entrance channel of the three-body reaction (1.3) a small energy \( E_{Ax} \) can be reached. The width of the momentum amplitude \( W(\vec{Q}_{Bb}) \) determines the range of energies around \( E_{Ax}^{qf} \) that can be explored due to the Fermi motion of \( b \) and \( x \) inside the Trojan horse \( a \). In an actual experiment a cutoff in the momentum \( \vec{Q}_{Bb} \) is chosen to select the region where the quasi-free process dominates the cross section over all processes.

§ 3. Applications of the Trojan-Horse method, summary and outlook

Several reactions have been studied with the TH method recently\(^{20,22-29}\) with \( ^2\text{H} \) and \( ^6\text{Li} (= \alpha+d) \) as typical “Trojan Horses”. These nuclei allow to study the transfer of protons, neutrons, deuterons and \( \alpha \)-particles, which covers most of the cases of astrophysical interest for the two-body reaction.

In nuclear astrophysics, transfer reactions (like (d,p) or \( ^3\text{He,d} \), or \( \text{Li,}\alpha \)) are used to study resonant states. For example, in the \( ^{22}\text{Na}(^3\text{He,d})^{23}\text{Mg} \) reaction states near the proton threshold were studied.\(^{21}\) This is relevant for the hydrogen burning of \( ^{22}\text{Na} \). In principle, also the continuum can be studied. For example, the “parallelism” of (d,p) and (n,n) reactions has been beautifully shown already in 1971, see Ref. 8). The d+\( ^6\text{Li} \) reaction was investigated in Ref. 28) in this indirect way. Another recent application is given in Ref. 29) to the \( ^7\text{Li}(\alpha,\alpha)^4\text{He} \) reaction. An especially interesting case would be the indirect study of the \( ^{12}\text{C}(\alpha,\gamma)^{16}\text{O} \) reaction by means of a \( (^7\text{Li},t) \) or \( (^6\text{Li},d) \) reaction. Quite recently\(^{30}\) the sub-Coulomb \( \alpha \)-transfer reaction \( (^6\text{Li},d) \) and \( (^7\text{Li},t) \) to the bound \( 2^+ \) and \( 1^- \) states in \( ^{16}\text{O} \) has been used to obtain information on the astrophysical S-factor.

In this contribution, the basic theory of the Trojan-Horse method was reviewed starting from a distorted wave Born approximation of the T-matrix element. The essential surface approximation allows to find the relation between the cross section of the three-body reaction and the S-matrix elements of the astrophysically relevant two-body reaction. In the modified plane wave approximation the relation between the three-body and two-body cross sections becomes very transparent. The three-body cross section is a product of a kinematical factor, a momentum distribution and a so-called “Trojan-Horse” two-body cross section. The energy dependence of the appearing Trojan-Horse integrals leads to a finite cross section of the three-body reaction at the threshold of the two-body reaction without the suppression by the Coulomb barrier. This allows to extract the energy dependence of astrophysical cross sections from the three-body breakup reaction to very low energies without the problems of electron screening and extremely low cross section. A comparison of results for S factors from direct and indirect experiments can improve the information on the electron screening effect, see also Ref. 31). However, dedicated
Trojan-Horse experiments are necessary in order to achieve a precision comparable to direct measurements.

The validity of the Trojan-Horse method can be tested by comparing the cross sections extracted from the indirect experiment with results from direct measurements of well studied reactions. In principle it is possible to assess systematic uncertainties of the Trojan-Horse method by studying various combinations of projectile energies, spectators in the Trojan Horse and scattering angles. Furthermore, different theoretical approximations can be compared, e.g. full DWBA calculations with and without the surface approximation and simpler modified plane wave approximations.

One may also envisage applications of the Trojan-Horse method to exotic nuclear beams. An unstable projectile hits a Trojan-Horse target allowing to study specific reactions on exotic nuclei. We mention the d(\(^{56}\text{Ni},p\))\(^{57}\text{Ni}\) reaction studied in inverse kinematics in Ref. 32). In this paper stripping to bound states was studied; extension to stripping into the continuum would be of interest for this and other reactions of this type.

A study of low-energy elastic scattering with the Trojan-Horse method opens another application which can lead to improved information relevant to the theoretical description of nuclear reactions at low energies.

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References

7) S. Austin, nucl-th/0201010.