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DISCUSSION

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Having seen a clarification and simplification of our understanding of the turbulence models the authors "compare," we are troubled to see apparent differences in calculated results where no differences would be expected to exist.

First we note that the case of mixing length variation $l = 0.4y$ is valid for much of the turbulent core; but for about a decade it has been clear that it will lead to serious errors in calculated results if applied through the sublayer region. In the low turbulent Reynolds number range this error can be as much as fifty percent in calculated velocities for a given wall shear stress. The models of van Driest and Reichardt represent two closely comparable approaches that correct the eddy diffusivity in the sublayer.

They differ in form (Reichardt's reduces to a cubic relationship for l or ν near the wall) but they each will permit the generation of velocity profiles comparable to one another to two significant figures. These two approaches differ from using $l = 0.4y$ only in the region $Y^+ < 10$ which in the present paper is less than 1/60 to 1/110 of the channel width. From the brief description of the actual calculation it appears that this region could not even be resolved in the numerical procedure. We must suggest that the difference between the predictions of the Reichardt and van Driest formulations are undoubtedly artifacts of the numerical procedure used.

One might suggest that one approach involves mixing length and the other eddy viscosity; but careful studies of pressure flow and Couette flow by both formulations show that the similarities of predictions remain whether or not shear stress is constant across the channel or varies as in pressure flow.

The authors should state in equation (13) that it is valid only at the wall as written. In the approach of Elrod and others, $\mu\, \partial u/\partial y$ is replaced by the sum of viscous and Reynolds shear stress in the body of the flow.

In referring to equation (11) we note that the stated relationship was used by Goldstein to provide an arbitrary match for core flow data from the experimental measurements of Dönch[8] and Nikuradse.[9] It is a curiosity that later others used the same approximate relationship through the sublayer to accomplish the correction for mixing length suppression near the wall and that this, properly done, will give reasonable bulk flow/shear stress predictions.

We suggest that it appears the authors own results show that they have not provided an analysis which sensitively permits comparison of the several approximation approaches. Rather these unexplainable differences in prediction highlight that the calculations as done give only rough estimates of what the formulations would properly predict.

In the numerical solution of hydrodynamic flow equations, the choice of variables is between the stream function-vorticity ($\Psi - \omega$) system and the primitive variable (u, v, p) system.

The choice of which system to use always depends on how many dimensions are being considered and whether or not a pressure solution is needed. In a two-dimensional flow problem, equation (6)

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[8] Forschungsarbeitan des Ver. deutsch. Ing., No. 282, 1926.
[9] Ibid., No. 289, 1929

specifies the two equations for the $\Psi - \omega$ system. The equation in Ψ is of the Poisson form and the other in ω is a nonlinear parabolic equation with advection and diffusion terms.

There are a variety of methods which give convergent solutions for the Ψ equation. The parabolic equation in ω because of, among other things, the advection term is harder to solve. A treatise is given in the book by P. J. Roache.[10] If the pressure solution is desired, it can either be obtained by solving a Poisson's equation for pressure or directly integrating the pressure gradient expressions in the momentum equations.

The equation for pressure in this case need only be solved once after a convergent solution has been obtained for the equations in Ψ and ω. The u, v, p system consists of two parabolic equations (for u and v) and the equation for pressure. However in this system, the equation for pressure has to be solved at every time step of the calculation or after every iteration of the finite-difference momentum equations.

Clearly then, if the pressure solution is not needed as in the author's work, the obvious one to prefer is the $\Psi - \omega$ system. One advantage of the $\Psi - \omega$ formulation over that of the primitive variables is that implicit methods can be used. One that is increasingly popular is the Alternating Direction Implicit Method. Azzez and Hellums referenced by the authors have shown that implicit methods do not give convergent results for the primitive variable equations.

In three-dimensional flows, the stream function Ψ does not exist with constant Ψ-lines being streamlines. However it is possible to define a vector-potential Ψ' (not to be confused with the velocity potential in irrotational flow theory) such that

$$\mathbf{V} = \nabla x \Psi'$$

with

$$\nabla^2 \Psi' = -\omega\cdot$$

and ω defined as

$$\omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$$

where

$$\omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = -\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)$$

$$\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

There are correspondingly three vorticity transport equations (one for each of the three components). The primitive variable system (u, v, w, p) in this case consist of three parabolic equations and the pressure equation. The feeling here is mixed. Each system has its advantages and disadvantages. The primitive variable system requires the storage of only four three-dimensional arrays and the (Ψ', ω) system requires seven (or at best six if the pressure solution is not wanted) such arrays.

There are four equations to solve in the primitive variable system and six or seven in the ($\Psi' - \omega$) system. Our experience in solving two-dimensional problems however, indicates that it is faster to solve the three vector potential equations

$$\nabla^2 \Psi' = -\omega$$

with Dirichlet boundary conditions than it is to solve one Poisson's equation

$$\nabla^2 p = f(x, y, z)$$

with Neumann boundary conditions. In this system therefore, the choice depends on the user but the primitive variable system may be preferred.

Returning to details, we ask:

(a) If the authors used the upwind differencing method as they report, did they make a correction on the diffusion terms since the method introduces artificial or numerical viscosity effects?

(b) What kind of a mesh did the authors use? Is the 12 × 18 grid uniform over the domain? Which way is it oriented? Can this mesh effectively give a good resolution in the sublayer?

[10] *Computational Fluid Dynamics*, Hermosa Press, Albuquerque, N. M., 1972.

J. M. Robertson[11]

This study of flow development in a belt-type, channel-flow apparatus is of relevance to several technical areas, as noted in the authors' introduction. There have been few experimental studies of belt-driven flows; reference to the recent one of L. J. Huey and J. W. Williamson [37][12] should also be included. For some time the writer has been interested in such flows as leading to turbulent plane-Couette flow [38], the simplest turbulent shear flow—homogeneous but non-isotropic.

The experimental set-up used, with an upstream length of uncovered belt ahead of the channel entrance, is novel. The efficacy of this open belt in increasing the centerline velocity to values within $0.08V_P$ of the Couette expectation is significant. It is unfortunate that the authors conclusion that $0.5V_P$ could be achieved via a longer open belt was not verified via tests, or at least that one other open length was not studied. As an indication of the other extreme with no length of open belt upstream, the flow pattern found by the writer is indicated in the accompanying figure. The centerline velocity near the end of the channel then varied between 0.28 and $0.33V_P$ in various tests. This apparatus differed from the authors' in being 3.5 and 4 times in cross-sectional dimensions but only 2.3 times as long. A considerable fraction of the length was taken up by the flow development. Uncrowned pulleys were successfully used at belt speeds up to 26 mps. Originally the belt was unsupported (in contrast to the authors') but in later tests [38] a support plate was added to eliminate possible flapping which had been suggested—no change in flow was found, however. Is the "wooden platform" mentioned as being "installed flush with the uppermost surface of the moving belt" the item shown to the left of the upstream pulley in Fig. 1? The writer found it propi-

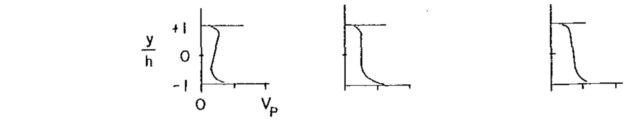

Fig. 12 Pressure and velocity variation in belt-channel apparatus

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[12] Numbers 37–38 in brackets designate Additional References at end of discussion.

tious to add one such at the downstream end of his channel.

Mention is made of an axial pressure gradient—favorable near the entrance and unfavorable in the fully-developed region—along the channel as associated with the need to overcome the side-wall channel friction. Indication of this would be desirable. The writer found (see figure) only a small negative pressure coefficient with axial variations which do not seem to follow the authors rationalle. The variations near the downstream end are attributed to the proximity of the pulley.

In regard to the numerical turbulent-flow analysis, the writer finds it difficult to accept mixing-length models based on Nikuradse's fully-developed pipe flow experiments (equation (10)), or Taylor's vorticity transfer in the core region of a conduit (equation (11)), etc. as appropriate for developing flows. For fully-developed turbulent Couette flows, his studies suggest $L = y/0.41$ in the wall region and constant in the core region as quite satisfactory.

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NOTE: There is some variation in the definition of y and h employed at different places in the paper. The authors should clear these up and make the paper consistent. Presumably $2h$ is the belt-upper plate spacing, so that there is an error in Fig. 1. Now in Figs. 4, 8, 9, 10, and 11 the spacing seems to be h and one is left to wonder what definition has been employed for Figs. 3, 6, and 7 in regard to $x/2h$. Also y has a different origin in Figs. 4, 8, 9, 10, and 11 than in body of paper and Figs. 3, 5, 6, and 7.

Authors' Closure

The authors would like to thank Dr. Salimonu, Prof. Burton and Prof. Robertson for their interest in the paper.

The appearance of "false diffusion" is common to all methods using upwind differences. Its magnitude is related to the magnitude of the velocity, to the mesh size and to the angle the mesh makes locally with the velocity vector (reference [31] of the paper). Some of the best ways of correcting for this error is by choosing the mesh parallel to the streamlines and by decreasing the mesh size. We have made the mesh parallel to the channel walls. An indication that this corrected for much of the problem can be seen perhaps from the fact that in laminar flow—where false diffusion leads to greatest error—we have good agreement among results of experiment, analysis and numerical work. For turbulent flow the importance of false diffusion is expected to decrease as μ_{eff}/μ is large here. We have also varied the size of our graded mesh and found little change in mean velocity profile in the neighborhood of 12 × 18. (We were unable to obtain good agreement with experiment when using a uniform mesh, even for larger number of nodal points).

With reference to Professor Robertson's question about the "wooden platform", observation of the hot wire anemometer output during the development of the experimental apparatus showed random velocity fluctuations even at low speeds. It was determined that most of the flow disturbance was created by the upstream pulley and the return section of the belt. The platform, which was contoured to fit close to the pulley, (Fig. 1 of the paper) and extended along each side of the upstream open belt, eliminated the disturbances.

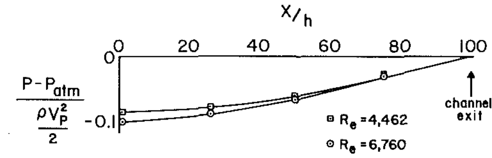

Fig. 13

Although extensive pressure surveys were not made, sufficient data were obtained to verify the statement made about the pressure gradients in the channel. These results, available for the higher Reynolds numbers of 4,460 and 6,760 are shown in the figure.

Errata: In Figs. 6 and 7 the "distance from entrance" (horizontal axis) should be in x/h and the "position relative to channel center" in $\eta = y/\bar{h}$ (where $h = 2\bar{h}$ is the channel height).