The connection between the surface rigidity of the nuclear core, which has been accounted for in terms of the surface tension of liquid drop nuclei, and its proper shell structure is discussed by using the method of the quantum mechanical description of the collective motion, which has been proposed by one of the present authors and others (§ 2). On the basis of such a consideration the surface rigidity of the core can be calculated, provided that the shell model is valid for the behaviour of particles forming the core (§ 3 and § 4). The noticeable features of our results obtained are that the calculated values of rigidity of cores are, in general, considerably larger than those due to the hydrodynamical estimation, and are closely related to the proper shell structures of cores (§ 4). Such a characteristic variation of the rigidity of cores is discussed in detail by comparing with the quadrupole moments of the “core±one extra-particle type” nuclei. Theoretical quadrupole moments thus obtained finely explain the observed values which have been noticed to depend on the shell structure (§ 5). Some discussions are devoted to the rigidity of a core with large deformation (§ 6).

§ 1. Introduction

The striking success of the nuclear collective model\(^1\),\(^2\) in accounting for many features of the nuclear structure has stimulated one’s interest in some of underlying problems. One of these is the study of the relation between the surface rigidity of cores and the proper shell structures of them.

A simple hydrodynamical model\(^1\),\(^3\),\(^4\),\(^5\),\(^6\) which replaces the core by a phenomenological liquid drop gives deformations considerably larger, sometimes by one order of magnitude, than empirical values deduced from nuclear quadrupole moments. Still more critical for this model is that it can not explain the rapid variation of the rigidity of cores, which must be accepted as an empirical fact. In particular, it should be noted\(^7\) that this model is completely helpless in explaining the great variation of the surface rigidity in different isotopes of the same element, which can be seen from the quadrupole moment ratios of isotopes with the same nuclear spin.

Such circumstances make clear the phenomenological aspect of the collective model based on the simple hydrodynamical assumption. In fact, although we know that the strong coupling scheme of the collective model is successful for mass numbers between 155 and 185 and above 225, and that the weak coupling scheme applies best to the nuclei

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in the proximity of major closed shells, we can not explain in the model itself why such treatments can be successful for those nuclei. This is due to the fact that in the collective model such treatments are determined by the strength of interaction between the core and extra-particles, which depends essentially on the rigidity of the core.

On the other hand, estimates based on the simple model\(^8\)\(^{10}\) in which particles are regarded as moving independently in a deformed potential have given much smaller deformations. However, such a deformed potential model has not been discussed in connection with Bohr-Mottelson’s (B–M) formulation\(^1\) on the collective model.

Recently, the method of the quantum mechanical description of the collective motion which gives the foundation of the nuclear collective model has been proposed\(^{11}\)\(^{12}\)\(^{13}\), and it has been made clear\(^{10}\) that the difference between B–M’s and Hill-Wheeler’s descriptions of the nuclear collective model is nothing but that of the representation in the description of the nuclear states except for several trivial points.

By using this method, therefore, we can faithfully follow B–M’s formulation of the collective model and can make clear the connection between the surface rigidity of a core and the proper shell structure of the corresponding core (§ 3). Then, we can find that in B–M’s model the adiabatic approximation is used for the particles forming a core, and that the surface rigidity of a core is given by the second order coefficient of expansion of the energy eigen-value of particles forming the core, with respect to the deformation parameters of the core.

Provided that the shell model assumption is valid for the behaviour of such particles, therefore, the method of the deformed potential model can be applied to the calculation of surface rigidity of the corresponding core. Using Moszkowski’s technique\(^{10}\), the calculation of the rigidity is made in § 3 and § 4.

The striking features of our results obtained are that the calculated values of rigidity of cores are, in general, considerably larger than those of the hydrodynamical estimation, and that there is a characteristic dependence of the rigidity upon the proper shell structures of cores (Fig. III). And this feature of rigidity can explain qualitatively why the strong coupling or weak coupling scheme of the collective model is successful with specific nuclei (§ 4).

More detailed discussions on such a characteristic variation of the rigidity of cores are done in § 5 by comparing with the quadrupole moments of “core ± one extra-particle type” nuclei. Thus we can find that such a variation plays an essential part in the variation of the quadrupole moments, as shown in Table II. In particular, it should be emphasized that it can explain the great variations of the surface rigidity in the isotopes; S\(^{36}\) (neutrons: 16-shell + 1) and S\(^{38}\) (neutrons: 20-shell − 1), and Ga\(^{69}\) (neutrons: 38-shell) and Ga\(^{71}\) (neutrons: 40-shell). The differences between the quadrupole moments of pairs of these isotopes have never been explained in terms of the hydrodynamical model or of the method of configuration mixing.\(^{15}\)

Some discussions are devoted in § 6 to the rigidity of core in the case of the large deformation.
§ 2. Formulation of the collective model

In order to clarify the connection between the surface rigidity of cores and the proper shell structures of them, it is necessary first of all to establish the formulation of the collective model. On this point, the method of the quantum mechanical description of the collective motion which has been formulated by one of the present authors 11, 14 and others 12, 13 is of use. Keeping close contact with B-M's description of the collective model, we shall develop this method in this section.

The Schrödinger equation which describes a nuclear system is

\[ H\Phi = \left[ \sum_{i=1}^{A} p_i^2/2M + V(x_1, \ldots, x_A) \right] \Phi (x_1, \ldots, x_A) = E \Phi (x_1, \ldots, x_A), \]

(2.1)

where \( M \) is the nucleon mass and \( V \) the interaction potential between nucleons. Now we introduce the collective coordinates \( \alpha_{im} \) and start from the following equation,

\[ \left[ \sum_{i=1}^{A} p_i^2/2M + V(x_1, \ldots, x_A, \alpha) \right] \Phi'(x_1, \ldots, x_A, \alpha) = E \Phi'(x_1, \ldots, x_A, \alpha), \]

(2.2)

which is equivalent to (2.1) under the condition,

\[ \alpha_{im} \Phi'(x_1, \ldots, x_A, \alpha) = 0. \]

(2.2')

We transform (2.2) and (2.2') into the "collective representation" by the unitary transformation,

\[ U = \exp \left[ i \left( M/\hbar \right) \int \phi(x) \phi(x) \, dx \right] \]

\[ = \exp \left[ i \left( M/\hbar \right) \sum_{i=1}^{A} \left( x - x_i \right) \phi(x) \, dx \right] \]

\[ = \exp \left[ i \left( M/\hbar \right) \int \phi(x) \, dx \right]. \]

(2.3) **

Here \( \phi(x) \) is the velocity potential of an irrotational and incompressible carrier fluid and is represented by

\[ \phi(x) = \sum_{lm} \beta_{im} Y_{lm}(\theta, \varphi). \]

As is well known in the liquid drop model of the nucleus, \( \beta_{im} \) are related to the canonical conjugate quantities \( \pi_{im} \) of the coordinates \( \alpha_{im} \) through the relation,

\[ \pi_{im} = B_i \cdot l \cdot R_0^{i-2} \cdot \beta_{im}^*. \]

\[ B_i = l^{-1} \cdot (3/4\pi) \cdot AMR_0^2. \]

*) \( \alpha_{im} \) are the expansion parameters of the nuclear surface defined by

\[ R(\theta, \varphi) = R_0 \left[ 1 + \sum_{im} \alpha_{im} Y_{lm}(\theta, \varphi) \right] \]

where \( R_0 \) is the equilibrium radius.

**) According to Eden and Francis 16 this corresponds to the "model operator" into the collective model. Here note that \( \sum_{i=1}^{A} \phi(x_i) \) is a symmetric function of \( x_i \), so that the "individual particles" in the "collective representation" obey the Fermi statistics in the same manner as nucleons.
In this "collective representation", (2·2) and (2·2') are expressed as

\[ \sum_{i=1}^{A} p_i^2/2M + (M/2) \cdot \int \rho(x) (\text{grad } \phi(x))^2 dx + V(x_1, \ldots, x_A; \alpha) + M \int \phi(x) \cdot \text{grad } \phi(x) dx \cdot \phi(x_1, \ldots, x_A; \alpha) = E \phi(x_1, \ldots, x_A; \alpha), \]  

where

\[ \sum_{i=1}^{A} F_{i,m}(x_i) = \left( \frac{4\pi}{3A} \sum_{i=1}^{A} \frac{r_i}{R_0} \right)^{\frac{3}{2}} Y_{lm}(\theta, \phi) \]  

and

\[ \mathbf{v}(x) = (1/2M) \sum_{i=1}^{A} \left[ p_i \delta(x - x_i) + \delta(x - x_i) p_i \right], \]

which is the velocity operator of the individual particles.

Eq. (2·4) with (2·4') is equivalent to B-M's except for several trivial points as discussed in the following. We can easily ascertain that this representation is used in B-M's description, by calculating any physical operator in this representation.***)

1) Collective oscillations of particles forming a shell structure

The relationship of the motion of individual particles and the collective motion, which is described by eq. (2·4) with (2·4'), is especially simple if the frequencies for particle excitation are large compared with the frequencies of the collective type of motion. In this case, the adiabatic approximation can be used and the wave function of (2·4) is represented approximately by

\[ \Phi(x_1, \ldots, x_A; \alpha) = \psi(x_1, \ldots, x_A; \alpha) \chi(\alpha), \]  

where \( \chi(\alpha) \) describes the oscillation of a nucleus as a whole. Here \( \psi(x_1, \ldots, x_A; \alpha) \) is the solution of

\[ \sum_{i=1}^{A} p_i^2/2M + V(x_1, \ldots, x_A; \alpha) \psi(x_1, \ldots, x_A; \alpha) = E \psi(x_1, \ldots, x_A; \alpha), \]

and must satisfy approximately

\[ \int \rho(\alpha) \psi(x_1, \ldots, x_A; \alpha) dx_1 \cdots dx_A = \begin{cases} \rho_0 (= 3A/4\pi R_0^3) & \text{inside the deformed nucleus} \\ 0 & \text{outside the deformed nucleus} \end{cases} \]

*) Strictly speaking, the \( \alpha \)-dependence of \( V \) is not uniquely defined by this transformation, but this dependence can be approximately given by Tomonaga's procedure or Miyazima's transformation.

**) This function is just the expression for the collective parameters in terms of the individual particles.

*** It is clear that Tolhoek's and Coester's analyses of B-M's model are equivalent to solving (2·4) with (2·4') by use of some assumptions; if we assume that \( \Psi(x_1, \ldots, x_A; \alpha) \) has a form, \( \delta(\alpha - \Sigma F(x_i)) \psi(x_1, \ldots, x_A) g(\alpha) \) and calculate an equation which \( g(\alpha) \) satisfies, we obtain the same results as Tolhoek's and Coester's.
because of the condition (2.4'). (2.6) ensures

\[ \left\{ \psi^* (x_1, \ldots, x_A; \alpha) \sum_{i=1}^A F_{im} (x_i) \psi (x_1, \ldots, x_A; \alpha) \right\} dx_1 \cdots dx_A = \alpha_{im} \]

so that the subsidiary condition for \( \chi (\alpha) : \)

\[ [\alpha_{im} - \left\{ \psi^* (x_1, \ldots, x_A; \alpha) \sum_{i=1}^A F_{im} (x_i) \psi (x_1, \ldots, x_A; \alpha) \right\} dx_1 \cdots dx_A ] \chi (\alpha) = 0 \]

becomes an identical equation. Therefore the collective motion is described by only one equation,

\[ \left( \frac{M}{2} \right) \left( \langle \rho (x) \rangle (\text{grad} \phi (x))^2 + E'' (\alpha) \right) \chi (\alpha) = E \chi (\alpha). \quad \text{(2.7) \*3} \]

If the particle structure prefers spherical symmetry, \( E'' (\alpha) \) may be expanded around the equilibrium state (\( \alpha_{im} = 0 \)):

\[ E'' (\alpha) = E'' (0) + \sum_{im} E'' (0) |\alpha_{im}|^2. \]

Then (2.7) is reduced to

\[ \left[ \sum_{im} (1/2B_i) |\pi_{im}|^2 + \sum_{im} E'' (0) |\alpha_{im}|^2 \right] \chi (\alpha) = (E - E'' (0)) \chi (\alpha). \quad \text{(2.7') \*1} \]

This equation is nothing but that for the surface motion given by B–M. (cf. 1), II. 5)

2) Coupling to particle motion

If the particles possess modes of excitation with frequencies smaller than or comparable with the collective frequencies, the simple separation of collective and particle motion such as (2.5) is no longer possible. Since the particle structure can be non-adiabatically excited in this case, the nucleus must be described in terms of a coupled system of collective and particle degrees of freedom.

In the treatment of this case, B–M have adopted the picture of “core + extra-particles.” Namely, the degrees of freedom of the extra-particles represent the low frequency modes of excitation of the particle structure, associated with the particles in the last unfilled levels. And the particles forming a closed shell (core) manifest themselves only through the collective motion of the core.

Conforming to this picture, our description is reformulated as follows. We perform the unitary transformation (2.3) only for the quantities which are concerned with the particles constituting the closed shell, so that the Schrödinger equation in the “collective representation” is given by

\[ \left[ \sum_{i=1}^N p_i^2 / 2M + (M/2) \left\{ \rho \phi \right\} \left( \text{grad} \phi (x) \right)^2 + V (x_1, \ldots, x_N, \alpha) \right] + M \left\{ \phi \cdot \text{grad} \phi (x) \right\} dx + \sum_{j=N+1}^A p_j^2 / 2M + V (x_{N+1}, \ldots, x_A) \]

\* The term including \( \psi (x) \) in (2.4) vanishes, when we take its expectation value in \( \phi \).
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\[ + V^{(3)}(x_{13}, \cdots, x_{N}, x_{N+1}, \cdots, x_{A}; \alpha) ] \mathcal{F}(x_{13}, \cdots, x_{N}, x_{N+1}, \cdots, x_{A}; \alpha) \]
\[ \equiv E \mathcal{F}(x_{13}, \cdots, x_{N}, x_{N+1}, \cdots, x_{A}; \alpha), \]
\[ \left[ \alpha_{lm} - \sum_{i=1}^{N} F_{lm}(x_i) \right] \mathcal{F}(x_{13}, \cdots, x_{N}, x_{N+1}, \cdots, x_{A}; \alpha) = 0, \]

where \( x_{13}, \cdots, x_N \) denote the coordinates of the particles which constitute the closed shell (core) and \( x_{N+1}, \cdots, x_A \) denote those of the particles in the last unoccupied levels (extra-particles). Prior to the unitary transformation, we divided the interaction potential between the particles into three parts. The first part is the interaction potential between the particles forming the core, and the second is that between the extra-particles. The third is the interaction potential between the particles forming the core and the extra-particles. These in the "collective representation" are denoted by \( V^{(1)}, V^{(2)}, \) and \( V^{(3)} \) respectively. *)

In the present case, the adiabatic approximation is possible only for the core part. Therefore, the approximate wave function which satisfies eq. (2·8) with (2·8') may be of the form:

\[ \mathcal{F}(x_{13}, \cdots, x_{N}, x_{N+1}, \cdots, x_{A}; \alpha) = \psi_{\text{core}}(x_{13}, \cdots, x_N; \alpha) \chi(x_{N+1}, \cdots, x_A; \alpha). \]

\[ \chi(x_{N+1}, \cdots, x_A; \alpha) \] is the solution of the equation,
\[ \left[ \sum_{i=1}^{N} p_i^2/2M + V^{(1)}(x_{13}, \cdots, x_N, \alpha) \right] \chi(x_{N+1}, \cdots, x_A; \alpha) = E_{\text{core}}(\alpha) \psi_{\text{core}}(x_{13}, \cdots, x_N; \alpha), \]

(2·10)

and must satisfy approximately in the same manner as (2·6)
\[ \int \psi_{\text{core}}^*(x_{13}, \cdots, x_N; \alpha) \rho(x) \psi_{\text{core}}(x_{13}, \cdots, x_N; \alpha) dx_{13} \cdots dx_N \]
\[ = \begin{cases} \rho_0 \left( = 3N/4\pi R_0^3 \right) & \text{inside the deformed core} \\ 0 & \text{outside the deformed core} \end{cases} \]

(2·11)

because of the condition (2·8'). In this case \( \chi(x_{N+1}, \cdots, x_A; \alpha) \) is no longer subject to the subsidiary condition and satisfies the equation;
\[ \left[ H_{\text{surf}}(\alpha) + H_p(x_p) + H_{\text{int}}(x_p, \alpha) \right] \chi(x_{N+1}, \cdots, x_A; \alpha) \]
\[ = (E - E_{\text{core}}) \chi(x_{N+1}, \cdots, x_A; \alpha), \]

(2·12)

where
\[ H_{\text{surf}}(\alpha) = (M/2) \left\{ \langle \rho(x) \rangle \psi_{\text{core}}(\text{grad} \ \phi(x) )^2 dx + \langle E_{\text{core}}(\alpha) - E_{\text{core}} \rangle, \right. \]
\[ H_p(x_p) = \sum_{j=1}^{N} \left( p_j^2/2M + \langle V^{(3)}(x_{13}, \cdots, x_N, x_{N+1}, \cdots, x_A; \alpha) \rangle_{|a=0} + V^{(3)}(x_{N+1}, \cdots, x_A), \right. \]
\[ H_{\text{int}}(x_p, \alpha) = \frac{\partial}{\partial \alpha} \langle V^{(3)}(x_{13}, \cdots, x_N, x_{N+1}, \cdots, x_A; \alpha) \rangle_{|a=0}. \]

---

*) The \( \alpha \)-dependences of \( V^{(1)} \) and \( V^{(3)} \) can also be given approximately in the same way as that of \( V \) in (2·4), but we shall not touch this point here.
is nothing but the Hamiltonian which was adopted by B–M as the starting point of the collective model, and eq. (2·12) is just the Schrödinger equation describing the collective model.

3) **Surface rigidity of the core and its proper shell structure**

From the above formulation of the collective model, we can easily find the relationship between the surface rigidity of the core and the proper shell structure of it. The eq. (2·10) describes the particle structure of the core, provided that the wave function \( \psi_{\text{core}}(x_1, \cdots, x_N; \alpha) \) satisfies (2·11) approximately. And the surface rigidity of the core is given by the second order coefficient of expansion of \( E_{\text{core}}(\alpha) \) with respect to \( \alpha \). The problem to obtain the surface rigidity of the core is thus focused to calculate \( E_{\text{core}}(\alpha) \).

In the calculation of \( E_{\text{core}}(\alpha) \), the specific type and form of the interaction potential, \( V^{(i)}(x_1, \cdots, x_N, \alpha) \), come at first into question. As a rule, a concrete form of \( V^{(1)}(x_1, \cdots, x_N, \alpha) \) can be obtained approximately by Tomonaga’s procedure \(^{12} \) or Miyazima’s transformation \(^{13} \) when the interaction potential between nucleons is given. In the present stage of the study of nuclear forces in a many nucleon system, however, it may not be so much significant to introduce the concrete form of \( V^{(1)}(x_1, \cdots, x_N, \alpha) \). It seems rather natural to assume the properties of \( V^{(1)}(x_1, \cdots, x_N, \alpha) \) through the model which gives an important guide in the interpretation of nuclear phenomena. The striking success of the nuclear shell model suggests that the particles may be considered as occupying states of binding characteristic to the independent particle motion in an averaged potential. Therefore, we assume here that \( V^{(1)}(x_1, \cdots, x_N, \alpha) \) has the following properties:

i) In the zero order approximation, \( V^{(1)}(x_1, \cdots, x_N, \alpha) \) can be replaced by an averaged potential with the deformation described by \( \alpha. \)

ii) The form of this averaged potential is assumed to be of the square well, because the wave function \( \psi_{\text{core}}(x_1, \cdots, x_N; \alpha) \) must satisfy the condition (2·11) approximately.

iii) Moreover, if necessary, we can add the amendments such as the \( I\cdot s \)-coupling term and others to it. These amendments to the deformed averaged potential may be considered as a residual part of \( V^{(1)} \).

§ 3. **Method of calculations**

Since we are calling the low energy states in the vicinity of the ground state into question, we shall adopt only \( \alpha_{lm} \) of \( l=2 \) for the deformation of a core. Moreover, since the observed rotational spectra indicate that the many nuclei prefer the cylindrical symmetry,
we shall perform the calculation by using a spheroidally deformed, averaged potential.

In this case, the wave function in (2.10) is represented by the Slater determinant of single particle wave functions in the spheroidal square well potential, and the energy eigen-value, $E^{\text{core}}(\alpha)$, in (2.10) is expressed by the sum of energy eigen-values, $E'(\alpha)$, with respect to the single particle wave functions:

$$E^{\text{core}}(\alpha) = \sum_s E'(\alpha). \quad (3.1)$$

Therefore, our problem to obtain $E^{\text{core}}(\alpha)$ is reduced in calculating the energy eigen-values of the Schrödinger equation with respect to the single particle in the spheroidal square well potential.

1) Moszkowski's transformation\(^6\)

The good quantum numbers of particle states in a spheroidal well are the parity and the component of angular momentum about the axis of deformation. Hereafter we shall denote this symmetry axis of deformation as $z$ and the component of angular momentum about it as $m$.

According to Moszkowski's method, we shall calculate the energy eigen-values of single particle in the spheroidal square well. First, the coordinate transformation:

$$x = D^{-1/2} x', \quad y = D^{-1/2} y' \quad \text{and} \quad z = Dz' \quad (3.2)$$

is carried out. Here $D$ is defined for the small deformation by

$$D = 1 + d, \quad d = \sqrt{4\pi} \frac{\beta}{\beta} \quad (3.3)$$

where

$$\beta = \sum_m |\alpha_m|^2.$$

In this new coordinate system, the spheroidal square well potential becomes

$$V(r') = \begin{cases} 0 & \text{for } r' > R_0 \\ -V_0 & \text{for } r' < R_0 \end{cases}$$

which means the square well potential. And the Schrödinger equation of a single particle in this coordinate system becomes

$$(1/2M) (DP_x^2 + DP_y^2 + D^{-2} P_z^2) \psi (x') + V (r') \psi (x') = E' \psi (x'). \quad (3.4)$$

Because we are concerned with the surface rigidity of the core (closed shell) which prefers the spherical symmetry, it is sufficient to obtain $E'(\alpha)$ to the second order with respect to $d$ by using, as the zero order functions, the functions $\phi_0 (x')$ for the Hamiltonian (3.4) when $D = 1$. To avoid confusion, we shall hereafter drop the “prime” of the newly transformed coordinates $x'$.

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\* This method has the same meaning as Miyazima's transformation in a sense.
2) **Determinations of the level scheme and the property of potential**

In our problem, the ordering and the distances of the single particle energy levels in the zero order approximation are of decisive importance. Therefore, we shall add the \( l \cdot s \)-coupling\(^*\) and \( l^2 \)** terms to \( V(r) \) so that the level scheme for the zero order approximation reproduces approximately Klinkenberg's one\(^{16} \) which is based on the shell model. Thus (3.4) is rewritten as follows:

\[
(H^0 + H') \psi(x) = E' \psi(x),
\]

\[
H^0 = p^2/2M + V(r) - (b^2/2MR^2_0) k \cdot s + (b^2/2MR^2_0) g F,
\]

\[
H' = (p^2/2M) d^2 - 2(d - d^2) T_{l0}(p),
\]

\[
T_{l0}(p) = (1/2M) \cdot (3p^2 - p^2)/2, \quad H' = dH^{(1)} + d^2H^{(2)},
\]

\[
dH^{(1)} = -2d T_{l0}(p), \quad d^2H^{(2)} = d^2(p^2/2M + 2T_{l0}(p)).
\]

The \( F \)-term may mean a correction to the square well potential \( V(r) \). Namely, this pushes up the states of higher angular momenta, so that it may be considered to play a role of interpolation to the oscillator potential. In this sense it seems natural to set up this term in the transformed system. The constant \( g \) of \( F \)-term and the constant \( k \) of \( l \cdot s \)-coupling term are treated as parameters which are adjusted suitably so as to reproduce Klinkenberg's level scheme. The values of \( g \) and \( k \) are, therefore, chosen as dependent on energy levels, as shown in Fig. I.***

For the sake of simplifying calculations, we shall hereafter replace \( V(r) \) with an infinite square well and the effects of the finite wall height will be taken into account later.

3) **Expression of \( E' \)**

The eigen-functions for \( H^0 \), which are used as the zero order functions, are

\[
\psi_{l \pm m}^{(0)} = c_j (\omega_{ab} \cdot r/R_0) \left[ \sqrt{(j + m)/2j} \alpha Y_{l+1/2}^{m+1/2} \pm \sqrt{(j - m)/2j} \beta Y_{l-1/2}^{m+1/2} \right] \quad \text{for} \quad j = l + 1/2
\]

\[
\psi_{l \pm m}^{(0)} = c_j (\omega_{ab} \cdot r/R_0) \left[ \sqrt{(j - m + 1)/(2j + 2)} \alpha Y_{l+1/2}^{m+1/2} \mp \sqrt{(j + m + 1)/(2j + 2)} \beta Y_{l+1/2}^{m+1/2} \right] \quad \text{for} \quad j = l - 1/2,
\]

in which \( c \) is a normalization constant, \( \alpha \) and \( \beta \) are Pauli spin functions, and \( j_n (\omega_{ab} \cdot r/R_0) \) is the usual spherical Bessel function. And the eigen-values of \( H^0 \) are given by

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\(^*\) Whether or not the adoption of the \( l \cdot s \)-coupling term in such a transformed coordinate system is legitimate may depend on its origin. Since the \( l \cdot s \)-coupling term used in the shell model is set up under the assumption of the "spherically symmetric" averaged potential, the adoption of \( l \cdot s \)-coupling term in such a transformed coordinate system would not always be nonsense.

\(^**\) After the work was finished, it has come to our notice that S. G. Nilsson [Kgl. Danske Videnskab Selskab, Mat.-fys. Medd 29, No. 16 (1955)] used this \( F \) term for the same purpose as ours. The authors are indebted to Dr. T. Tamura for his kind information of this point.

\(^***\) According to the physical meaning of \( F \) term, we adopted here the same values of \( g \) for the degenerated states in an oscillator potential.
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Fig. 1 Level scheme.

\[
W_{nl\ell} = \begin{cases} 
\frac{\hbar^2}{2MR_0}\omega_{nl}^2 + (\frac{\hbar^2}{2MR_0}) g l (l + 1) + (\frac{\hbar^2}{2MR_0}) k (l + 1) & \text{for } j = l - 1/2 \\
(\frac{\hbar^2}{2MR_0}) \omega_{nl}^2 + (\frac{\hbar^2}{2MR_0}) g l (l + 1) - (\frac{\hbar^2}{2MR_0}) kl & \text{for } j = l + 1/2
\end{cases}
\]

where \( \omega_{nl} \) denotes the \( n \)-th root of the spherical Bessel function of order \( l \). Therefore, \( E_{nl\ell} \) which is calculated to the second order with respect to \( d \) by using \( H' \) as a perturbation is given as follows:

\[
E_{nl\ell} = W_{nl\ell} + E_{nl\ell}^{(1)} d + E_{nl\ell}^{(2)} d^2,
\]

\[
E_{nl\ell}^{(1)} = (n l m | H^{(1)} | n l m) = -2 (n l m | T_{30} (p) | n l m)
\]
\[
= -2 (\frac{\hbar^2}{2MR_0}) \omega_{nl}^2 (l | m | P_2 (\theta) | l m),
\]

(3.8)
The second term in the right hand side of (3·9) can be evaluated in reference to the well-known properties of spherical Bessel functions:

\[ \sum_k |(n\,l\,j\,m|H^{(1)}|n_k\,l_k\,j_k\,m_k)^2|/(W_{nlj} - W_n^0) \]

\[ = 4(\frac{\hbar^2}{2MR^2_0}) \sum_{n'} \left[ 4(2l+1)^2a_{nl}^0 \frac{a_{n'l'+l+2}^0}{(a_{nl}^0 - a_{n'l'+l+2}^0)^2} \frac{|(l\,j\,m|P_n(\theta)|l+2, j+2, m)^2}{(W_{n'l'} - W_n^0, l+2, j+2, l+2) / (\hbar^2/2MR^2_0)} \right. 

\[ + \left. 4(2l+1)^2a_{nl}^0 \frac{a_{n'l'+l-2}^0}{(a_{nl}^0 - a_{n'l'+l-2}^0)^2} \frac{|(l\,j\,m|P_n(\theta)|l-2, j-2, m)^2}{(W_{n'l'} - W_n^0, l-2, j-2, l-2) / (\hbar^2/2MR^2_0)} \right] \]

for \( j=\frac{l}{2} \).

\[ \sum_k |(n\,l\,j\,m|H^{(1)}|n_k\,l_k\,j_k\,m_k)^2|/(W_{nlj} - W_n^0) \]

\[ = 4(\frac{\hbar^2}{2MR^2_0}) \sum_{n'} \left[ 4(2l+1)^2a_{nl}^0 \frac{a_{n'l'+l+2}^0}{(a_{nl}^0 - a_{n'l'+l+2}^0)^2} \frac{|(l\,j\,m|P_n(\theta)|l+2, j+2, m)^2}{(W_{n'l'} - W_n^0, l+2, j+2, l+2) / (\hbar^2/2MR^2_0)} \right. 

\[ + \left. 4(2l+1)^2a_{nl}^0 \frac{a_{n'l'+l-2}^0}{(a_{nl}^0 - a_{n'l'+l-2}^0)^2} \frac{|(l\,j\,m|P_n(\theta)|l-2, j-2, m)^2}{(W_{n'l'} - W_n^0, l-2, j-2, l-2) / (\hbar^2/2MR^2_0)} \right] \]

for \( j=\frac{l-1}{2} \). (3·11)

4) **Approximations in numerical calculations**

From (3·8), (3·9), (3·10) and (3·11), the following facts are easily understood:

i) The coupling between \( \psi_{nljm}^0 \) and \( \psi_{n'l'j'm'}^0 \) by a spheroidal distortion can arise only when \( m=m'=0 \) and \( l-l'=0 \) or \( \pm 2 \).

ii) The matrix elements of \( H^{(1)} \) between the states having the different values of \( n \) vanish if the orbital angular momenta are the same.

iii) The matrix elements \( |(n\,l\,j\,m|H^{(1)}|n'\,l'\,j'\,m)^2| \), in which \( l-l'=\pm 2 \) and \( |n-n'| \geq 2 \), are very small because of the large energy denominator \( |a_{nl}^0 - a_{n'l'}^0| \). Therefore, the main contributions to \( E_{nljm}^{(0)} \) result from the coupling of \( \psi_{nljm}^0 \) with states \( \psi_{n'l'j'm'}^0 \) in which \( l'=l \) or \( l\pm 2 \) and \( n'=n \) or \( n\pm 1 \).
iv) Of these main contributions to $E^{(3)}_{\text{shell}}$, the contributions resulting from the coupling with the states in the proximity of the $n,l,j$-level are especially large. And the values are strongly dependent on the level spacing $A_{n_{\text{adj}}-n_{\text{adj}}}$. 

According to the above considerations, we shall adopt the following approximation and treatment in calculating the surface rigidity of a neutron or proton core (closed shell) under consideration,

$$ C = \frac{\partial^2 E^{\text{core}}}{\partial \alpha^2} = \frac{5}{2\pi} \sum_{n_{\text{adj}}} E^{(3)}_{n_{\text{adj}}}, $$

(3.12)

(a) The sums over $n'$ in (3.10) and (3.11) are performed only for $n'=n$ and $n\pm 1$.

(b) According to the consideration iv), we separate the surface rigidity of the neutron or proton core into two parts:

$$ C = C^{(1)} + C^{(2)}, $$

(3.13)

where $C^{(2)}$ depends strongly on the level spacing (the strongly shell dependent part), and $C^{(1)}$ depends rather weakly on it:

$$ C^{(1)} = \frac{5}{2\pi} \sum_{n_{\text{adj}}} E^{(0)}_{n_{\text{adj}}}, \quad C^{(2)} = \frac{5}{2\pi} \sum_{n_{\text{adj}}} E^{(2)}_{n_{\text{adj}}}, $$

(3.14)

Here $E^{(0)}_{n_{\text{adj}}}$ is the term which does not depend on the near-lying levels, as given by

$$ E^{(0)}_{n_{\text{adj}}} = \sum_{n',l',j',m'} \left\{ (n l j m | H^{(1)} | n' l' j' m')^2 / (W_{n_{\text{adj}}} - W_{n_{\text{adj}}}^{(0)}) \right\}, $$

(3.15)

where the "prime" of $\sum'$ means the exclusions of coupled terms with the near-lying levels. $E^{(2)}_{n_{\text{adj}}}$ is the part which depends strongly on the near-lying levels, as given by

$$ E^{(2)}_{n_{\text{adj}}} = \sum_{n',l',j',m'} \left\{ (n l j m | H^{(1)} | n' l' j' m')^2 / (A_{n_{\text{adj}}-n_{\text{adj}}}^{(2)}) \cdot (\beta^2/2MR_0^2) \right\}, $$

(3.16)

where $\sum''$ means the sum of only the coupled terms with the near-lying levels.

§ 4. Results of calculation and discussions

As discussed above, $C^{(1)}$ does not depend on the spacing between the near-lying levels. In its numerical calculation, therefore, we used for $W_{n_{\text{adj}}}$ the expression (3.7) and for $g$ and $k$ in it the values adopted in Fig. I. The result thus obtained is given in Fig. IIa and IIb.

The level ordering of $3s 1/2$, $1h 11/2$, and $2d 3/2$ states and of $2f 5/2$, $3p 3/2$, $3p 1/2$ and $1i 13/2$ states are not definite because of the pairing effect, so that, in these regions, the neutron or proton numbers which constitute the closed subshells are uncertain. However, the values of $C^{(1)}$ do not so much depend on the level order of these states. Therefore, we adopted here, according to Mayer and Jensen, the level orderings; $3s 1/2,$


As is shown in Fig II, the values of $C^{(2)}$ of the closed shells lie on a fairly smooth curve.

Contrary to $C^{(1)}$, $C^{(2)}$ is just the term which results from the coupling between the finally filled levels of the closed shell under consideration and their near-lying levels, so that it depends very strongly on the spacing between these levels. The fairly smooth curve plotted by the values of $C^{(2)}$ for the closed shells is considerably transfigured by adding $C^{(2)}$ to it.

Now we can give the qualitative discussions on the surface rigidity of the closed shells. As to the more detailed discussions, however, some considerations are still more necessary, because of the following two reasons:

a) Up to here, our calculations have been performed under the condition of the infinite well potential.

*) $C^{(2)}$ depends also very sensitively on the level ordering. Therefore, in the regions of $3s 1/2$, $1h 11/2$, $2d 3/2$ and of $2f 5/2$, $3p 3/2$, $3p 1/2$, $1i 13/2$, where the order of these levels is not definite because of the pairing effect, we must check up general tendency of the variation of $C^{(2)}$ by making up the various possible choices of the order of these levels.
b) The values of $C^{(\Omega)}$ depend considerably on the values of level spacing between the finally filled levels of the closed shell under consideration and their near-lying levels.

For these two problems, we took the following steps:

a') For the "spherical" square well potential, the effects of the "finite" wall height can be given approximately by:

$$E_{v_0, r_0} \approx E_{v_0, r_0}(1 + \lambda_0/R_0)^{-2}$$

$$\lambda_0 = \left[ \frac{\hbar^2}{(2MV_0)} \right]^{1/3},$$

as obtained by Feenberg-Hammack. Namely, in this case, we can take into account the effect of the "finite" wall height by replacing only the nuclear radius $R_0$ by

$$R_0' = R_0 + \lambda_0.$$  

(4.2)

In the case of the spheroidal potential, this procedure might also be possible*). As a matter of convenience, we adopt here this procedure as the first approximation, since our calculations have been performed in particular under the condition that the deformation of the potential is small.

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Table 1.** The values of level spacings in $\hbar^2/2MR^3$ unit used for our calculation of $C^{(\Omega)}$.

<table>
<thead>
<tr>
<th>Level</th>
<th>$2s_{1/2}$</th>
<th>$1d_{3/2}$</th>
<th>$2p_{3/2}$</th>
<th>$1f_{5/2}$</th>
<th>$2p_{1/2}$</th>
<th>$2d_{5/2}$</th>
<th>$1g_{7/2}$</th>
<th>$3s_{1/2}$</th>
<th>$2d_{3/2}$</th>
<th>$2f_{7/2}$</th>
<th>$1h_{9/2}$</th>
<th>$2f_{5/2}$</th>
<th>$3p_{3/2}$</th>
<th>$3p_{1/2}$</th>
<th>$2g_{9/2}$</th>
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<tr>
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</table>

*) Strictly speaking, it is necessary to investigate to what extent this approximation is possible in the deformed potential.

**) In this table, the value of $\Delta_{2s_{1/2} - 1d_{3/2}}$ is larger than that used ordinarily. In our calculation, however, we used the values which are deduced from the separation energy, because in our case $\Delta$ means the energy difference between the ground state of the corresponding closed shell and the state of the single particle excitation.
b') The values of level spacing between the finally filled levels of the closed shell under consideration and their near-lying levels, $\Delta_n^m$, are very important in evaluating $C^{(n)}$. So we should take those as close as possible to real values rather than those obtained from the level scheme given in Fig. I. Hence, we adopt those (in unit of $b^2/2MR_0^{(n)}$) which are deduced from the separation energy in the vicinity of nuclei having the corresponding neutron or proton core. And in the case where this deduction is not possible, we adopt those (in unit of $b^2/2MR_0^{(n)}$) used by Horie-Arima. The values of level spacing thus obtained are listed in Table 1. Here we used as the nuclear radius, $R_0 = 1.4 \times 10^{-13} \text{cm}$, and as the depth of potential, $V_0 = 36\text{Mev}$. And for the values of mass number $A$ we adopted those of the nuclei having the corresponding (neutron or proton) cores. The surface rigidity of the closed shell thus calculated, $C = C^{(1)} + C^{(2)}$, is shown in Fig. IIIa and IIIb.

In this figure the Coulomb correction is not included. Here $R'_0 = R_0 + \lambda_0$, $\lambda_0 = (b^2/MV_0)^{1/2}$, where $R_0$ is the nuclear radius and $V_0$ is the depth of the potential. The dotted lines are used for the region in which the assignment of subshells is uncertain because of the pairing effect. The lines joining the points for the closed shells have no meaning. In these regions, the rigidity may be reduced very much because of the crossing of the particle levels due to the large deformation of the core which is caused by the coupling of the core and extra-particles. The line indicated by "hydr." is calculated by using the formula

$$C_{n}^{hydr.} = (4R_0^2S) \times N/A,$$

where $S$ is the hydrodynamical surface tension,

$$4\pi R_0^2S = 15.4 \cdot A^{1/3} \text{Mev.}$$
Here the dotted lines are used for the region in which the assignment of subshells is uncertain because of the pairing effect. In this region, we calculated $C$ by making up the three possible choices of level ordering.

From Fig. III one can observe a rapid variation of $C$ from nucleus to nucleus, which has never been obtained with the hydrodynamical model. Its qualitative feature are summarized as follows:

i) The values of the surface rigidity of the closed shells corresponding to the magic numbers are very large except for $28$ shell.

ii) The values of the surface rigidity of the subshells between the neutron or proton numbers $50$ and $82$ become very small, irrespective of the various possible choices of the level ordering.

iii) The values of the surface rigidity of the subshells between the neutron or proton numbers $82$ and $126$ are relatively small except for a particular choice of the level ordering.

The conclusion i), which has been asserted intuitively in connection with the shell model, means that the weak coupling scheme of the collective model applies best to the nuclei in the vicinity of the major closed shells. The conclusions ii) and iii) mean that the strong coupling scheme of the collective model is successful in these regions.

In addition to this general tendency, there is a relatively irregular variation of the rigidity of each closed shell. In comparison with the electric quadrupole moments of the "core ± one extra-particle type" nuclei, we shall discuss this variation in the next section in detail.

§ 5. Nuclear quadrupole moments and variations of surface rigidity of cores

The magnitudes of the electric quadrupole moments directly reveal their collective origin and the empirical values of quadrupole moments give the direct information about the rigidity of nuclear cores. Therefore, the characteristic variation of the rigidity of cores, which is shown in Fig. III, should be reflected in the quadrupole moments of the "core ± one extra-particle type" nuclei.

According to the collective model, the quadrupole moments of the "core ± one extra-particle type" nuclei are given by

$$Q=Q_{s.p}+Q_{o},$$

(3·1)

of which the first part is due to the extra particle outside the core. The second part is due to the deformed core and is given by:

$$Q_{o}=P_{q}(x)Q_{o},$$

$$Q_{o}=-\frac{3}{4\pi}\frac{2I-1}{2(I+1)}\frac{K}{C}ZR_{o}^{2}$$
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\[ P_Q(x) = 1 - \frac{3}{2I+1} \frac{2I+1}{(I+1)(2I+3)} \frac{x^2}{\sqrt{x^2+4/9}} \]  

(for \( I = j \geq 1 \)),

where \( Q_0 \) is the intrinsic quadrupole moment of the core relative to the axes fixed in the core. Here \( K \) is the expectation value of the interaction term \( H_{\text{int}}(x, \alpha) \) of (2·13) with respect to the radial part of particle coordinates. Correctly, its value depends on the shell structure, but here we adopt 40 MeV according to B–M's*). \( P_Q(x) \) is referred to as the projection factor. \( x \) is a dimensionless parameter** which is related to the strength of the particle-surface coupling and is given explicitly by

\[ x = \sqrt{\frac{5}{6\pi}} \frac{1}{\sqrt{j}} \frac{K}{\sqrt{b\omega C}} \]

where

\[ \omega = \sqrt{C/B} \]

\[ B = \frac{1}{2} \frac{3}{4\pi} AMR_s^2. \]

By using the values of the surface rigidity \( C \) given by Fig. III, we can calculate the quadrupole moments of the "core± one extra-particle type" nuclei from (5·1) and (5·2). Table II shows the comparison of the calculated values of such quadrupole moments with observed ones and hydrodynamical ones. Here the Coulomb correction for \( C \) is assumed to be

\[ C_{\text{coul}} = - \frac{3}{2\pi} \frac{1}{5} \frac{Z^2e^2}{R_0}. \]

It is very interesting to see whether or not the variation of the rigidity of neutron or proton cores in Fig. III can explain the qualitative tendency of quadrupole moments of the "core± one extra-particle type" nuclei.

\begin{itemize}
  \item[a)] Rigidity of neutron or proton cores between 8 and 50
\end{itemize}

In this region, the major closed shells are given at the proton or neutron numbers 8, 20, 28 and 50. As is shown in Fig. III, the values of the rigidity of these cores are remarkably large except for 28-shell. Therefore, the weak surface-particle coupling scheme may be valid for \( ^{17}_{\text{O}} \), \( ^{39}_{18}\text{K} \) and \( ^{88}_{40}\text{Zr} \), and the values of the quadrupole moments of these nuclei may be much smaller than those of the hydrodynamical estimation. There are no experimental values of the quadrupole moments of \( ^{18}_{18}\text{K} \) and \( ^{88}_{40}\text{Zr} \), but the facts that the observed value of that of \( ^{17}_{\text{O}} \), \( Q_{\text{obs}}(O^{17}) \), is \(-0.005\) and the hydro-

*) If an extra-particle is exchanged with a hole having the same single particle state, the sign of \( K \) is reversed.

**) For \( x\sqrt{j} \ll 1 \), the weak coupling scheme is valid, but for \( x \gg 1 \), the strong coupling scheme is valid.

***) Recently, the quadrupole moment of \( ^{39}_{18}\text{K} \) was measured by Ritter and Series (Proc. Phys. Soc. 68A, 450). Its value is \( 0.14 \times 10^{-26} \text{cm}^2 \pm 25\% \), which is much smaller than \( Q_{\text{hydr}} \) and is very close to \( Q_{\text{calc}} \). The authors are indebted to Profs. Townes and Murakawa for bringing this work to their attention.
Table II.

The values of surface rigidity \( C \) and of quadrupole moments for the “core ± one extra-particle type” nuclei.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Configuration</th>
<th>I</th>
<th>( C )(Mev)</th>
<th>( Q \times 10^{-24} \text{cm}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{11}\text{B} )</td>
<td>((1p_\text{s})^-1)</td>
<td>3/2</td>
<td>117 24.1</td>
<td>0.036 0.044 0.093 0.025</td>
</tr>
<tr>
<td>(^{27}\text{Al} )</td>
<td>((1d_{5/2})^-1)</td>
<td>5/2</td>
<td>26 38.8</td>
<td>0.16 0.37 0.24 0.062</td>
</tr>
<tr>
<td>(^{37}\text{Cl} )</td>
<td>(1d_{3/2})</td>
<td>3/2</td>
<td>348 45.6</td>
<td>-0.062 -0.087 -0.25 -0.051</td>
</tr>
<tr>
<td>(^{39}\text{K} )</td>
<td>((1d_{5/2})^-1)</td>
<td>3/2</td>
<td>505 45.6</td>
<td>0.14 0.089 0.33 0.035</td>
</tr>
<tr>
<td>(^{60}\text{Co} )</td>
<td>((1f_{7/2})^-1)</td>
<td>7/2</td>
<td>299 54.8</td>
<td>0.5 0.30 0.90 0.12</td>
</tr>
<tr>
<td>(^{69}\text{Ga} )</td>
<td>((2p_{3/2})^-1)</td>
<td>3/2</td>
<td>323 58.6</td>
<td>0.23 0.20 0.52 0.08</td>
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<tr>
<td>(^{74}\text{Ge} )</td>
<td>((2p_{1/2})^-1)</td>
<td>3/2</td>
<td>691 60.2</td>
<td>0.14 0.14 0.52 0.08</td>
</tr>
<tr>
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<td>((2p_{3/2})^-1)</td>
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<td>736 64.5</td>
<td>0.14 0.17 0.68 0.09</td>
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<td>(2d_{5/2})</td>
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<td>973 67.0</td>
<td>-0.05 -0.35 -1.7 -0.18</td>
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<tr>
<td>(^{100}\text{Bi} )</td>
<td>(1h_{9/2})</td>
<td>9/2</td>
<td>1580 58.0</td>
<td>-0.4 -0.55 -5.2 -0.30</td>
</tr>
</tbody>
</table>

The dynamical estimation \( Q_{\text{hydr}}(0_{17}^-) \) is \(-0.14\) support the above argument. The hydrodynamical estimation, \( Q_{\text{hydr}}(0_{17}^-) = -0.14 \), differs from the empirical value by one order of magnitude. Our calculated value \( Q_{\text{cal}}(0_{17}^-) = -0.031 \) is better than \( Q_{\text{hydr}}(0_{17}^-) \), but is too large in its absolute value.

Fig. III indicates that the values of rigidity of 14-shell in both protons and neutrons are smaller than those of closed shells in this region. This suggests that the empirical value of the quadrupole moment of \(^{27}\text{Al} \) may be fairly different from \( Q_{s.p.}(\text{Al}^{17}) \). In fact, we can easily understand that the deformation of \(^{27}\text{Al} \) is fairly large, by the facts that \( Q_{\text{obs.}}(\text{Al}^{17}) = 0.16 \), \( Q_{s.p.}(\text{Al}^{17}) = 0.062 \) and \( Q_{\text{hydr.}}(\text{Al}^{17}) = 0.24 \). Our calculated value, \( Q_{\text{cal}} = 0.37 \) is worse than the hydrodynamical value. The examples for \(^{17}\text{O} \) and \(^{27}\text{Al} \) seem to indicate that the collective treatment of light nuclei is not legitimate.

Fig. III shows that there is a rapid variation of the surface rigidity between 16-shell and 20-shell. This must be reflected in the quadrupole moment ratio of \(^{33}\text{S} \) (neutrons: 16-shell+1) and \(^{35}\text{S} \) (neutrons: 20-shell−1) which have same nuclear spin. It is obvious that we obtain

\[
\left| Q_{\text{hydr.}}(\text{S}^{33}) \right| \left| Q_{\text{hydr.}}(\text{S}^{35}) \right| = 1
\]

and

\[
Q_{s.p.}(\text{S}^{33}) = Q_{s.p.}(\text{S}^{35}) = 0.
\]

Contrary to this, the observed values give a ratio

\[
\left| Q_{\text{obs.}}(\text{S}^{33}) \right| \left| Q_{\text{obs.}}(\text{S}^{35}) \right| = 1.45.
\]
We may suppose that this value different from unity is due to the rapid variation of the surface rigidity. The ratio given by our calculation is

\[ \frac{|Q_{\text{cal.}}(S^{55})|}{|Q_{\text{cal.}}(S^{57})|} = 1.64. \]

This is the major success of our theory.

The fact \( Q_{\text{ob.}}(\text{Cl}^{55})/Q_{\text{ob.}}(\text{Cl}^{57}) = 1.23 \) may also imply that the rigidity of the core of \( \text{Cl}^{57} \) is larger than that of \( \text{Cl}^{55} \) because of the 20-neutron shell, though the framework of our method forbids to calculate \( Q(\text{Cl}^{55}) \).

Fig. III shows that between 38-shell and 40-shell there is a large variation of the rigidity. This should also be reflected in the quadrupole moment ratio of \( ^{51}\text{Ga}^{69} \) and \( ^{51}\text{Ga}^{71} \). The single particle model and the hydrodynamical one give

\[ \frac{|Q_{\text{sp.}}(\text{Ga}^{69})|}{|Q_{\text{sp.}}(\text{Ga}^{71})|} = \frac{|Q_{\text{hydr.}}(\text{Ga}^{69})|}{|Q_{\text{hydr.}}(\text{Ga}^{71})|} = 1. \]

According to our calculation, \( |Q_{\text{cal.}}(\text{Ga}^{69})|/|Q_{\text{cal.}}(\text{Ga}^{71})| = 1.4 \) because of the variation of the rigidity in these two isotopes. This is compared with the observed one; \( |Q_{\text{ob.}}(\text{Ga}^{69})|/|Q_{\text{ob.}}(\text{Ga}^{71})| = 1.6 \). This again shows the success of our theory. Moreover, the absolute values of \( Q_{\text{ob.}}(\text{Ga}^{69}) = 0.23 \) and \( Q_{\text{ob.}}(\text{Ga}^{71}) = 0.14 \) are explainable by our theory, whereas both the single particle and the hydrodynamical models fail to explain them. This fact supports that the rigidity of 40-shell is very large as indicated in Fig. III.

The 28-shell is assigned to a major closed shell. Fig. III indicates, however, that the rigidity of the 28-shell is relatively smaller than that of other major closed shells. Therefore we may suppose that the quadrupole moments of the nuclei having this core may fairly deviate from the single particle moments. Namely, the deformations of such nuclei may be larger than those of the nuclei lying near other major closed shells. In fact, \( Q_{\text{ob.}}(\text{Co}^{59}) = 0.5 \pm 0.2 \), which is significantly larger than \( Q_{\text{sp.}}(\text{Co}^{59}) = 0.12 \), while our calculation gives \( Q_{\text{cal.}}(\text{Co}^{59}) = 0.3 \) in agreement with \( Q_{\text{ob.}} \) within the experimental error.

\[ b) \] The rigidity of neutron or proton cores between 50 and 82 and between 82 and 126

In these regions, the numbers of the neutrons or protons which compose the closed subshells are not clear because of the pairing effect. In Fig. III, therefore, we took the several possible cases of the level ordering.

The following tendency is maintained in these regions, irrespective of the choice of level ordering. The rigidity of the 50, 82 and 126 (neutron or proton) shells is extremely large and the rigidity of the subshells between the 50 and 82 major closed shells is very small. And the values of the surface rigidity of the subshells between the 82 and 126 major closed shells are relatively small as compared with that of the major closed shells.

This explains why the strong coupling scheme of the collective model is successful for nuclei of mass numbers between 155 and 185. On this point it should be noted that the rigidity of nuclei with "core + several extra-particles" may be considerably reduced compared with the rigidity of the "spherically symmetric core" because of the crossing of the particle levels due to the deformation of the core which is caused by the extra-particles.
The fact that the rigidity of the 82- and 126-shell is very large can be inferred, as is well known, from $Q_{ob}(\text{Bi}^{209})$ which is very close to $Q_{s.p.}(\text{Bi}^{209})$.

Fig. IIIb indicates that there is an abrupt change between the values of the rigidity of 80- and 82-proton shell. The fact that the quadrupole moment of $^{197}\text{Au}$ ($=0.56$), is much larger than $Q_{s.p.}(\text{Au}^{197})$ might reflect this situation.

§ 6. Discussions

We have tried to make clear the relation between the rigidity of the core and the proper shell structure of it, and have calculated the rigidity concretely by using a simplified picture, and further have discussed the qualitative tendency in the variation of the rigidity of cores, in comparison with the quadrupole moments of the "core±one extra-particle type" nuclei. Then, we have found the results obtained can qualitatively explain the characteristic variation of the rigidity of cores, which must be accepted as an empirical fact. A similar feature is also obtained by Horie and Arima in their method of configuration mixing. Its relation to ours will be discussed in a forthcoming paper.

Of course, the numerical values of our calculation are not quantitative enough, because of the use of the simplified picture where the effects of the direct inter-particle interactions are neglected, and of the adoption of various approximations in the course of calculations. As the general tendency, however, the calculated values of surface rigidity are much larger than those obtained from the hydrodynamical surface tension. On this point it should be emphasized that our calculation is performed only for the closed shells which are of spherical symmetry.

When the core has a large deformation owing to extra-particles, there may occur the crossing of particle levels, which causes the rearrangement of the level filling of particles. Therefore, the surface rigidity of such a deformed core may be very much different from that of the spherically symmetric core. Preliminary discussion on this problem was given by Gallone and Salvetti. They showed, by using an extremely simplified model, that the surface rigidity is much reduced by the large deformation and on the average it approaches to the hydrodynamical one.

This can easily be inferred from the fact that the rigidity of the hydrodynamical model is due to a statistical feature of the level distributions. The increment of the crossing of particle levels as the deformation increases may mean that the level distribution characterized by the shell model becomes more and more the statistical one.

In the regions where the strong particle-surface coupling scheme is successful, therefore, the hydrodynamical rigidity may be better in the case of several extra-particles, because in these regions the deformations of cores are considerably large.

More detailed discussions on this point is being prepared together with the analyses of the quadrupole moments of the "core+several extra-particles type" nuclei.

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Note added in proof Recently, J. M. Araújo has calculated the nuclear deformability by using the generalized independent particle model (time dependent well potential) [Nuclear Physics, Vol. 1, No. 4 (1956)]. The relation between Araújo's description and ours will be discussed in later occasion.