SH Waves in a Transversely Isotropic Medium—II
 Transradially Isotropic Cylinder
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Summary

In Part I we found the relation between phase and group velocity, in a transversely isotropic medium, of shear waves polarized perpendicular to the axis of symmetry. We showed that energy travels along rays which are not perpendicular to the wavefront and obtained the appropriate law of refraction.

Here we consider waves and rays in a cylinder which is transradially isotropic—at any point the phase velocity along the radius is $V_r$ and in directions perpendicular to a radius is $V_H$. We show in two ways that the geometry of rays can be found by an operation of 'angle-stretching'; this enables us to derive the complete disturbance due to a point source from knowledge of the corresponding disturbance in an isotropic cylinder. This method gives a neat solution, and so may be able to throw light on the problem of a transradially isotropic sphere, for which there is no such simple method available.

1. Introduction

In Part I we dealt with plane $SH$ waves in a homogeneous material, perfectly elastic and isotropic in all directions perpendicular to the axis $Oz$. Here we consider $SH$ waves in a material with axial symmetry and show that the phase velocity along a radius is $V_r$ and perpendicular to a radius is $V_H$. Thus the velocity is $V_H$ for all directions lying on circular cylindrical surfaces whose axis is the axis of symmetry. A cross-section is shown in Fig. 1. We may conveniently think of waves on a disc,

![Fig. 1. Section of transradially isotropic cylinder.](https://academic.oup.com/gji/article-abstract/16/5/461/619998)

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the displacement being always normal to the disc. Such a material and configuration may be called transradially isotropic; as compared with Part I we have introduced one curvature in the surfaces of isotropy. This is a step towards the problem of spherical symmetry, in which isotropy pertains to spherical surfaces.

We take cylindrical coordinates \((r, \theta, z)\) and corresponding components of displacement \((u, v, w)\). Then the stress–strain relations, replacing those of Part I, are

\[ p_{rr} = c_{33} u_r + c_{13} \frac{1}{r} v_\theta + c_{13} w_z, \]
\[ p_{\theta\theta} = c_{13} u_r + c_{11} \frac{1}{r} v_\theta + c_{12} w_z, \]
\[ p_{zz} = c_{13} u_r + c_{12} \frac{1}{r} v_\theta + c_{11} w_z, \]
\[ p_{r\theta} = c_{44} \left( \frac{1}{r} u_\theta + v_r \right), \]
\[ p_{\theta z} = c_s \left( v_r + \frac{1}{r} w_\theta \right), \quad c_s = \frac{1}{2}(c_{11} - c_{12}), \]
\[ p_{zr} = c_{44}(w_r + u_z). \]

We want to consider SH waves, for which \(u = 0, v = 0\) and \(w\) is a function of \(r\) and \(\theta\) only. The stress–strain relations reduce to

\[
\begin{align*}
p_{rr} &= c_{44} w_r, \\
p_{\theta\theta} &= c_s \frac{1}{r} w_\theta,
\end{align*}
\]

(1.1)

(where \(c_s = \frac{1}{2}(c_{11} - c_{12})\) as in Part I), the remaining stress and strain components vanishing.

The equation of motion is:

\[
\rho w_{tt} = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) p_{rr} + \frac{1}{r} \frac{\partial}{\partial \theta} p_{\theta\theta} \quad \text{(Love 1927, Section 59)}
\]

\[ = c_{44} \left( w_{rr} + \frac{1}{r} w_r \right) + c_s \frac{1}{r^2} w_{\theta\theta}, \]

i.e.

\[ w_{tt} = V_v^2 \left( w_{rr} + \frac{1}{r} w_r \right) + V_h^2 \frac{1}{r^2} w_{\theta\theta}. \]

(1.2)

2. The angle-stretching transformation

Let us introduce the new variable \(\vartheta = \theta/\gamma\), where \(\gamma = V_h/V_v\), which is greater than unity in most seismological applications.

Then the wave equation (1.2) becomes, in variables \((r, \vartheta)\),

\[ w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\vartheta\vartheta} = \frac{1}{V_v^2} w_{tt}. \]

(2.2)
This is the standard wave-equation in cylindrical coordinates in an isotropic material where the velocity \( V_\nu \) is the same in all directions. The disturbance from a point source in the \((r, \theta)\) plane will spread with circular wavefront moving with velocity \( V_\nu \). The pattern of disturbance from a corresponding point source in the \((r, \theta)\) plane will be derived by means of the transformation

\[
r = r, \quad \theta = \gamma \theta,\]

in which radial distances are conserved while angles subtended at the origin are increased in the ratio \( \gamma \). Thus the pattern in the \((r, \theta)\) plane is derived from the pattern in the \((r, \theta)\) plane by what may be called 'angle stretching'. Fig. 2 shows the wave-front in the \((r, \theta)\) plane corresponding to a circle centred at \((r_0, 0)\) in the \((r, \theta)\) plane for \( \gamma = 2\cdot0 \), a value which probably exceeds those applicable in seismology and therefore exaggerates the distortion of wavefront.

Let us now consider the transformation of rays from the \((r, \theta)\) (image) plane to the \((r, \theta)\) (real) plane. In Fig. 3(b) the ray proceeds along a straight line from source \( P_0'(r_0', \theta_0) \) in the image plane to observer at \( P'(r, \theta) \). Let us denote by \( \phi \)

the angle between the ray and the radius vector. Then

\[
r \sin \phi = r_0 \sin \phi_0 \tag{2.3}
\]

and

\[
\theta - \theta_0 = \phi_0 - \phi \tag{2.4}
\]

These equations relate \( r \) and \( \theta \) at each point on the ray that sets out from \( P_0' \) in the direction specified by \( \phi_0 \). We note that the expressions \( r \sin \phi \) and \( r_0 \sin \phi_0 \) (in equation (2.3)) are both equal to \( p_0 \), the perpendicular from the origin on the ray.

The figure in the real \((r, \theta)\) plane, Fig. 3(a), is derived by angle-stretching from that in the \((r, \theta)\) plane. Hence

\[
\tan \phi = r \frac{d\theta}{dr} = \gamma r \frac{d\theta}{dr} = \gamma \tan \phi, \tag{2.5}
\]

where \( \phi \) is the angle between ray and radius vector in the \((r, \theta)\) plane. Thus the ray in the \((r, \theta)\) plane is determined by

\[
r \sin \phi = r_0 \sin \phi_0, \tag{2.6}
\]

\[
\gamma (\phi_0 - \phi) = \theta - \theta_0, \tag{2.7}
\]

where

\[
\tan \phi = \frac{1}{\gamma} \tan \phi \quad \text{and} \quad \tan \phi_0 = \frac{1}{\gamma} \tan \phi_0. \tag{2.8}
\]
From Fig. 3(b) we see that
\[ r \cos \varphi - r_0 \cos \varphi_0 = V_\nu t. \] (2.9)
This is an implicit relation between \( r \) and \( \theta \) which gives the wavefront in Fig. 3(a).

3. Disturbance due to a stress discontinuity

We now show that the short-wave approximation to the solution of the wave equation gives the same formulae for wavefront and ray as we have derived in Section 2 by a geometrical transformation. The shape of the wavefront will not depend on the type of point source employed. We choose a discontinuity in the radial component of shearing stress \( P_{rr} \).

We solve
\[ V_\theta^2 \left( w_{rr} + \frac{1}{r} w_r \right) + V_\nu^2 \frac{1}{r^2} w_{\theta\theta} = w_{tt}, \] (3.1)
with conditions, at \( r = r_0 \),
\[ [P_{rr}] = f(\theta) g(t), \] (3.2)
\[ [P_{\theta\theta}] = 0. \] (3.3)

We first expand \( f(\theta) \) in Fourier series
\[ f(\theta) = \sum_{n=-\infty}^{\infty} C_n \exp(i n\theta), \] (3.4)
where \( C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\zeta) \exp(-i n\zeta) d\zeta \), and express \( g(t) \) as
\[ g(t) = \int_{-\infty}^{\infty} \tilde{g}(\omega) \exp(-i\omega t) d\omega, \]
where
\[ \tilde{g}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) \exp(i\omega t) dt. \] (3.5)
We note that if $g(t)$ is real
\[ g(t) = 2\text{Re} \int_0^\infty \tilde{g}(\omega) \exp(-i\omega t) \, d\omega, \]
so that we need consider only positive values of $\omega$.

To obtain representation of a point impulse, we take $g(t)$ as the Dirac function $\delta(t)$ and $f(\theta)$ as $\delta(\theta-\theta_0)$, obtaining
\[ \tilde{g}(\omega) = \frac{1}{2\pi}, \quad C_n = \frac{1}{2\pi} \exp(-in\theta_0), \]
and interpreting subsequent expressions as derived from distributions. Then
\[ f(\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \exp[\ln(\theta-\theta_0)] \]
\[ = \frac{1}{\pi} \left\{ \frac{1}{i} + \sum_{n=1}^{\infty} \cos(n(\theta-\theta_0)) \right\}. \]

Thus we may first replace equation (3.2) by
\[ [p_{nn}] = \frac{1}{2\pi^2} \cos(n(\theta-\theta_0)) \exp(-i\omega t) \quad (3.2a) \]
and then construct the full solution by superposition.

Let us write
\[ w = W(r) \cos(n(\theta-\theta_0)) \exp(-i\omega t), \quad (3.6) \]
then
\[ V_n^2 \left( W_{rr} + \frac{1}{r} W_r \right) - V_n^2 \frac{n^2}{r^2} W = -\omega^2 W, \]
or, introducing
\[ k = \omega/V_n, \quad \gamma = V_{H}/V_n, \quad (3.7) \]
\[ W_{rr} + \frac{1}{r} W_r + \left( k^2 - \frac{\gamma^2 n^2}{r^2} \right) W = 0, \quad k > 0. \quad (3.8) \]

This is Bessel's equation of order $\gamma n$ in the variable $kr$, with solution $J_\gamma(kr)$ finite at the origin, and $H_{\gamma n}(kr)$, $H_{\gamma n}(kr)$ for outgoing or ingoing waves respectively.*

Thus we choose as appropriate solutions
\[ W = A\text{H}_s\gamma_n(kr), \quad r > r_0, \]
\[ W = B\text{J}_\gamma(kr), \quad r < r_0. \]

Then by equations (3.2a) and (3.3)
\[ kc_{44}\{A\text{H}_s\gamma_n(kr_0)-B\text{J}_\gamma(kr_0)\} = \frac{1}{2\pi^2}, \]
\[ A\text{H}_s\gamma_n(kr_0)-B\text{J}_\gamma(kr_0) = 0. \]

Hence, using the relation
\[ \text{H}_s\gamma_n J_\gamma - \text{H}_s\gamma_n J_\gamma = 2i/\pi kr_0, \quad (3.9) \]
* We use Jeffreys's notation $\text{H}_s$ and $\text{H}_i$ for Hankel functions of the first and second kind.
and for $r > r_0$, $-\pi < \theta \leq \pi$, by superposition, the displacement at $(r, \theta)$ is

$$w = 2\text{Re} \int_0^\infty d\omega \exp(-i\omega t) \left(-\frac{ir_0}{4\pi c_{44}}\right) \left\{ \frac{1}{2} J_0(kr_0) H_0(kr) + \sum_{n=1}^\infty J_n(kr_0) H_n(kr) \cos n(\theta - \theta_0) \right\}. \quad (3.11)$$

This is a sum of eigenfunctions, each representing a motion whose dependence on $\theta$ is trigonometric and whose nodal lines are radii (given by zeros of $\cos n(\theta - \theta_0)$). These eigenfunctions cannot be used to express a ray solution. We therefore use the form of Watson’s transformation (Sommerfeld 1949) in which $\frac{1}{2} F(0) + \sum_{n=1}^\infty F(n)$ is replaced by

$$\frac{1}{2i} \int_C F(v) \frac{\cos v\pi}{\sin v\pi} dv, \quad (3.12)$$

where $C$ is the contour (shown in Fig. 4) which passes through the origin and encloses the poles at $v = 1, 2, \ldots$ of the integrand. This artifice is valid provided $F(v)$ has no singularities within $C$, which is true for

$$F(v) = J_{\gamma \nu}(kr_0) H_{\gamma \nu}(kr) \cos \nu(\theta - \theta_0) \quad (3.13)$$

which occurs in equation (3.11).

Thus

$$w = -\frac{r_0}{4\pi c_{44}} \text{Re} \int_0^\infty d\omega \exp(-i\omega t) \int_C J_{\gamma \nu}(kr_0) H_{\gamma \nu}(kr) \cos \nu(\theta - \theta_0) \frac{\cos v\pi dv}{\sin v\pi}. \quad (3.14)$$

The product $J_{\gamma \nu}(kr_0) H_{\gamma \nu}(kr) \cos \nu(\theta - \theta_0)$ can be written as

$$\frac{1}{2}(H_{\gamma \nu}(kr_0) + H_{\gamma \nu}(kr_0)) H_{\gamma \nu}(kr) [\exp(iv(\theta - \theta_0)) + \exp[-iv(\theta - \theta_0)]]. \quad (3.15)$$

From the four terms of equation (3.15) we select the term

$$\frac{1}{2} H_{\gamma \nu}(kr_0) H_{\gamma \nu}(kr) \exp [iv(\theta - \theta_0)] \quad (3.16)$$

as giving the direct ray. The reasons for this choice are explained by Etienne (1961,
Fig. 5. Paths for the Hankel integrals.

Chap. 3), and are analogous to those given by Jeffreys & Lapwood (1957, pp. 473-4) in their discussion of a similar problem for the sphere.

We now utilize the integrals (Sommerfeld 1949)

\[
\begin{align*}
Hs_{\gamma}(kr) &= \frac{1}{\pi} \int_{W_1} \exp \left[ ikr \cos \xi + i\gamma (\xi - \frac{1}{2} \pi) \right] d\xi, \\
Hi_{\gamma}(kr_0) &= \frac{1}{\pi} \int_{W_2} \exp \left[ ikr_0 \cos \eta + i\gamma (\eta - \frac{1}{2} \pi) \right] d\eta,
\end{align*}
\]

(3.17)

where \( W_1 \) and \( W_2 \) are the paths shown in Fig. 5.

Then in the direct ray the displacement is

\[
\psi_d = -\frac{r_0}{8\pi^3 c_{44}} \text{Re} \int_0^\infty d\omega \int_0^\infty dv \int_{W_1} d\xi \int_{W_2} d\eta \left\{ \exp \left[ \chi(\xi, \eta, \nu) - i\omega t \right] \frac{\cos \nu \pi}{\sin \nu \pi} \right\},
\]

where

\[
\chi(\xi, \eta, \nu) = ikr \cos \xi + i\gamma (\xi - \frac{1}{2} \pi) + ikr_0 \cos \eta + i\gamma (\eta - \frac{1}{2} \pi) + iv(\theta - \theta_0).
\]

The short wave approximation will be found by evaluation on paths of steepest descent in each of the complex planes of \( \xi, \eta, \nu \). We have

\[
\frac{\partial \chi}{\partial \xi} = -ikr \sin \xi + i\gamma \nu
\]

and this vanishes when

\[
\sin \xi = \gamma \nu / kr.
\]

From the form of \( W_1 \) we see that if \( |\gamma \nu / kr| \) is small this gives a value of \( \xi \), say \( \xi_0 \), near zero.

Again \( \frac{\partial \chi}{\partial \eta} = -ikr_0 \sin \eta + i\gamma \nu \), which vanishes when

\[
\sin \eta = \gamma \nu / kr_0,
\]

(3.20)

and from the form of \( W_2 \) we see that we must select a value \( \eta_0 \) of \( \eta \) near \( \pi \).
Finally, remembering that \(\xi_0\) and \(\eta_0\) are functions of \(v\),

\[
\frac{\partial \chi}{\partial v} = iy(\xi + \eta - \pi) + i(\theta - \theta_0) - \frac{\partial \chi}{\partial \xi} \frac{\partial \xi}{\partial v} - \frac{\partial \chi}{\partial \eta} \frac{\partial \eta}{\partial v}
\]

\[
= i(y(\xi + \eta - \pi) + \theta - \theta_0)
\]

(3.21)

at the multiple saddlepoint.

Let us write \(\xi_0 = \varphi\), \(\eta_0 = \pi - \varphi_0\), then equations (3.19), (3.20) and (3.21) give

\[
r \sin \varphi = r_0 \sin \varphi_0,
\]

(3.22)

\[
\gamma(\varphi - \varphi_0) + \theta - \theta_0 = 0.
\]

(3.23)

If we are given \((r_0, \theta_0)\) and \((r, \theta)\) the equations (3.22) and (3.23) must be solved to give \(\varphi, \varphi_0\) for the direct ray. They are clearly identical with those found geometrically in equations (2.6) and (2.7). Equation (3.22) is Snell's Law.

The wavefront is the surface of constant phase at time \(t\), namely

\[
\chi(\xi_0, \eta_0, \nu_0) - i\omega t = \text{const.}
\]

This is

\[
k(r \cos \varphi - r_0 \cos \varphi_0) - \omega t = \text{const.}
\]

(3.24)

Since \(\omega = kV_t\), this is the same as equation (2.9) obtained from geometrical relations.

Again, for the ray, if \(\phi\) is the angle between tangent and radius vector,

\[
\tan \phi = rd\theta/dr = -\gamma r d\varphi/dr,
\]

by equation (3.23). But from equation (3.22)

\[
dr \sin \varphi + r \cos \varphi d\varphi = 0.
\]

Hence

\[
\tan \phi = \gamma \tan \varphi
\]

(3.25)

as in equation (2.8).

We do not proceed to the evaluation of the displacement \(w\), since our concern in seismology is with the disturbance at the surface of a sphere; the corresponding problem here is solved in Section 4.

### 4. Displacement at the surface of a transradially isotropic cylinder

In Section 3 we imposed no outer boundary condition, so that in effect we were examining a ray from a source in an infinite medium in order to obtain the modified Snell's law. We now add the condition that the solid is bounded by a free surface at \(r = a\), i.e.

\[
p_{rs} = 0 \text{ at } r = a.
\]

(4.1)

Taking the same source as in Section 3, we follow the same analysis as far as equation (3.8). We then select solutions of the form needed by the source and boundary conditions:

\[
W = AHs_m(kr) + BHi_m(kr), \quad a > r > r_0, \quad k > 0,
\]

(4.2)

\[
W = CJ_m(kr), \quad r < r_0, \quad k > 0.
\]

(4.3)
The three conditions
\[
\begin{align*}
\frac{c_{44}}{r^2} \frac{\partial w}{\partial r} &= 0 \quad \text{at } r = a, \\
\frac{1}{2\pi^2} \frac{\partial w}{\partial r} &= 0 \quad \text{at } r = r_0, \\
\frac{c_2}{r^2} \frac{\partial w}{\partial \theta} &= 0 \quad \text{at } r = r_0,
\end{align*}
\] (4.4)
determine \(A, B\) and \(C\). Then the displacement at \(r = a\) is given by

\[
AHs_m(ka) + BH_l(ka) = \frac{r_0}{2\pi} J_m(kr_0) \frac{1}{2\pi^2}. \tag{4.5}
\]

This replaces the partial solution (3.10). Superposing such terms we obtain instead of equation (3.11) the complete solution

\[
w = -\frac{r_0}{\pi^2ac_{44}} \Re \int_0^\infty \frac{d\omega}{k} e^{-\omega t} \left\{ \frac{1}{J_0'(ka)} + \sum_{n=1}^\infty \frac{J_m'(ka)}{J_m'(ka)} \cos n(\theta - \theta_0) \right\}. \tag{4.6}
\]

For the successful application of the method of steepest descent we try to obtain terms of exponential type. We therefore write

\[
\frac{J_m'(kr_0)}{J_m(kr_0)} = \frac{Hs_m(kr_0) + Hi_l(kr_0)}{Hs_m(kr_0) + Hi_l(kr_0)} \frac{Hs_m(kr_0) + Hi_l(kr_0)}{Hi_l(kr_0)} \left\{ \sum_{n=1}^m (-\rho)^n + \frac{(-\rho)^{m+1}}{1 + \rho} \right\}, \tag{4.8}
\]

where

\[
\rho(n) = \frac{Hs_m'(ka)}{Hi_l'(ka)}.
\]

We have used here the technique of Debye (1908) and Van der Pol & Bremmer (1937) to obtain successive reflected rays. \(s\) can be identified as the number of reflections (Etienne 1961). Since in this paper we concentrate on the direct ray to the surface we evaluate only the term with \(s = 0\).

We can apply the Watson transformation to equation (4.8) since no poles of the integrand lie on the real axis (Sommerfeld 1949), and then obtain the optical ray solution by evaluation at a saddlepoint. It will be seen later (equations (4.13) and (4.17)) that the saddlepoint lies on the real axis of \(v\) if \(ka\) is real. But our analysis arises from the ideal case in which there is no dissipation in the mechanical system. If dissipation exists, \(\omega\) in \(\exp(-i\omega t)\) must be taken complex with negative imaginary part, and this implies that \(ka\) is complex with negative imaginary part and the saddlepoint lies below the real axis of \(v\). We have chosen \(\rho\) in equation (4.8) such that, for \(\Im(v) < 0, |\rho| < 1\) and \(\sum (-\rho)^n\) converges. Again, for \(\Im(v) < 0, \cos v/n/(\sin v/n)\) can be expanded as

\[
[1 + \exp(-i\nu)] \sum_{m=0}^\infty \exp(-i\nu m),
\]

and \(m\) is identified as the number of times that a ray has encircled the origin (Jeffreys & Lapwood (1957)). Thus we select for the direct ray arriving at the surface (assuming \(\theta > \theta_0\)) the term containing

\[
\{Hi_l(kr_0)/Hi_l'(ka)\} \exp[i\nu(\theta - \theta_0)] = \frac{Hi_l(kr_0)Hs_m'(ka)}{G_0'(ka)} \exp[i\nu(\theta - \theta_0)].
\]
where \( G_v(ka) = H_0^0(ka) H_{1,v}(ka) \). \( G_v(ka) \), which becomes the denominator when both original numerator and denominator are multiplied by \( H_{1,v}(ka) \), is introduced because it varies comparatively slowly with \( v \), being asymptotically equal to \( 2/\pi ka \) for large \( ka \). So we wish to evaluate at the saddlepoint in the \( v \)-plane the displacement

\[
 w_d = -\frac{r_0}{2\pi^2 ac_{44}} \Re \int_0^\infty \frac{dv}{k} \exp \left( -i\omega \right) \int \frac{H_{1,v}(kr_0) H_{1,v}(ka)}{G_v(ka)} \exp \left[ iv(\theta - \theta_0) \right] dv.
\]

(4.9)

We now introduce integrals for the Hankel functions, and have

\[
 \pi H_{1,v}(ka) = \int \exp \left[ ikr_0 \cos \eta + iv(\eta - \frac{1}{2} \pi) \right] d\eta \sim \frac{(2\pi)^\frac{1}{2} e^{i\gamma}}{\left| kr_0 \cos \eta_v^* \right|^\frac{1}{2}} \exp \left[ ikr_0 \cos \eta_v + iv(\eta_v - \frac{1}{2} \pi) \right],
\]

(4.10)

where \( \eta_v \) is the solution of

\[
 kr_0 \sin \eta = \gamma v
\]

(4.11)

which lies near \( \eta = \pi \), and

\[
 \pi H_{1,v}(ka) = \int \cos \xi \exp \left[ ika \cos \xi + iv(\xi - \frac{1}{2} \pi) \right] d\xi, \]

\[
 \sim \frac{(2\pi)^\frac{1}{2} i e^{-i\gamma}}{\left| ka \cos \xi_v^* \right|^\frac{1}{2}} \cos \xi_v \exp \left[ ika \cos \xi_v + iv(\xi_v - \frac{1}{2} \pi) \right],
\]

(4.12)

where \( \xi_v \) is the solution of

\[
 ka \sin \xi = \gamma v
\]

(4.13)

which lies near \( \xi = 0 \). Again

\[
 G_v(ka) \sim -\frac{2}{\pi ka} \frac{\cos \xi_v \cos \xi_v^*}{|\cos \xi_v \cos \xi_v^*|^\frac{1}{2}} \exp \left[ ika (\cos \xi_v + \cos \xi_v^*) + iv(\xi_v + \xi_v^* - \pi) \right],
\]

(4.14)

where \( \zeta_v \) is the solution of

\[
 ka \sin \zeta = \gamma v
\]

which lies near \( \zeta = \pi \).

Thus

\[
 w_d \sim -\frac{r_0}{\pi^2 ac_{44}} \Re \int_0^\infty \frac{dv}{\epsilon} \int \frac{i \cos \xi_v \exp \left[ \psi(v) - i\omega \right]}{k^2 (ar_0)^\frac{1}{2} |\cos \xi_v \cos \eta_v^*|^\frac{1}{2}} G_v
\]

(4.15)

where

\[
 \psi(v) = ika \cos \xi_v + iv(\xi_v - \frac{1}{2} \pi) + ikr_0 \cos \eta_v + iv(\eta_v - \frac{1}{2} \pi) + iv(\theta - \theta_0).
\]

(4.16)

We determine the saddlepoint from

\[
 \psi'(v) = i(\gamma(\xi_v + \eta_v - \pi) + \theta - \theta_0) = 0.
\]

(4.17)

To satisfy equations (4.11), (4.13) and (4.17) we write

\[
 \xi_v = \varphi_v, \quad \eta_v = \pi - \varphi_v, \quad v = kp/\gamma,
\]
so that
\[
\gamma(\varphi_a - \varphi_0) + \theta - \theta_0 = 0,
\]
\[
a \sin \varphi_a = r_0 \sin \varphi_0 = p,
\]
and \(p\) is identified as the ray parameter in the \((r, \theta)\) space.

In \(G_s(ka)\) we put \(\zeta_v = \pi - \varphi_a\), so that in evaluating equation (4.15) we use
\[
G_s(ka) \sim 2 \cos \varphi_a \pi ka.
\tag{4.18}
\]

Differentiating equation (4.17) we have
\[
\psi''(v) = i \gamma \frac{\partial}{\partial v} (\xi_v + \eta_v)
\]
\[
= (i \gamma / v) (\tan \xi_v + \tan \eta_v)
\tag{4.19}
\]
from equations (4.11) and (4.13).

At the saddlepoint
\[
\psi(v) = ik(a \cos \varphi_a - r_0 \cos \varphi_0)
\]
\[
= i \omega (a \cos \varphi_a - r_0 \cos \varphi_0) / V_v.
\tag{4.20}
\]

If we denote \(a \cos \varphi_a - r_0 \cos \varphi_0\) by \(V_v t_d\) then \(t_d\) is the time of travel from \(P_o'\) to \(P_a'\) in the image plane \((r, \theta)\), and this is equal to the time of travel from \(P_o\) to \(P_a\) in the \((r, \theta)\) plane.

At the saddlepoint
\[
\psi''(v) = i \gamma \left( \frac{\gamma}{ka} \sec \varphi_a - \frac{\gamma}{kr_0} \sec \varphi_0 \right)
\]
\[
= \frac{i \gamma^2}{kar_0} (r_0 \sec \varphi_0 - a \sec \varphi_0)
\]
\[
= - \frac{i \gamma^2 V_v t_d}{kar_0 \cos \varphi_0 \cos \varphi_a},
\tag{4.21}
\]

\[\text{(a)}\]
\[\text{(b)}\]

Fig. 6. Ray from source \(P_o\) to \(P_a\) on surface, (a) real space (b) image space.
and the approximation by the method of steepest descent gives

\[ w_d = -\frac{r_0}{2\pi\gamma c_4} \left( \frac{2}{\pi t_d} \right)^\frac{1}{2} \text{Re} \int_0^\infty \omega^{-\frac{1}{2}} \exp \left[ -i\omega(t-t_d) + \frac{1}{2}i\pi \right] d\omega \]

\[ = -\frac{r_0}{2\pi\gamma c_4} \left( \frac{2}{t_d(t-t_d)} \right)^\frac{1}{2} H(t-t_d). \quad (4.22) \]

5. Displacement in direct ray derived from energy spread

We now show that we may interpret the expression (4.22) for the displacement at \((a, \theta)\) in terms of the spread of energy from the source at \(P_0\).

The wavefront around \(P_0\) is an oval, half of which is shown in Fig. 2. We examine the rate of flow of energy in the curvilinear sector shown in Fig. 7(a), with vertex at \(P_0\) and cutting the circle \(r = a\) in an arc \(P_0Q\) of length \(a\delta\theta\). Energy travels along rays with group velocity \(U\); \(U\) is given, as shown in Part I, by the formula

\[ \frac{1}{U^2} = \frac{1}{V_H^2} \cos^2 \phi + \frac{1}{V_V^2} \sin^2 \phi \]

by equation (2.5) and \(V_H = \gamma V_V\). Thus, at \(P_0\), \(U_0 = V_V \cos \phi_0 \sec \phi_0\) and, at \(P_a\), \(U_a = V_V \cos \phi_a \sec \phi_a\). Let \(I_o\) denote the energy-density on the wavefront at small time \(\alpha\) after the disturbance leaves \(P_0\). Then equating rates of energy flow across the arc \(U_0 a\delta\phi_0\) and the arc \(PQ\) we have

\[ I_o U_0 a\delta\phi_0 U_0 = I_o a\delta\theta \cos \phi_a U_a. \quad (5.2) \]

Fig. 7. Spread of energy (a) real space, (b) image space.
But

\[ \delta \theta = \gamma (\delta \varphi_0 - \delta \varphi_\alpha). \]  
(5.3)

\[ \cot \varphi_\alpha \delta \varphi_\alpha = \cot \varphi_0 \delta \varphi_0, \]  
(5.4)

\[ \sec^2 \varphi_0 \delta \varphi_0 = \gamma \sec^2 \varphi_0 \delta \varphi_0, \]  
(5.5)

and so

\[ \frac{I_\alpha}{I_\alpha} = \frac{U_0 a \delta \varphi_0}{a \cos \varphi_\alpha \delta \theta} \frac{U_0}{U_\alpha} = \frac{\alpha V_\nu}{a \cos \varphi_\alpha - r_0 \cos \varphi_0} = \frac{\alpha}{l_\alpha}, \]  
(5.6)

in the notation of Section 4. The ratio of amplitudes equals the square root of the ratio of intensities, and the factor \( t_\alpha^{-1} \) is thus accounted for.

If instead of considering the real space we calculate the amplitude from the image space the expression (5.6) is immediately obtained. The value of working in the image space is thus confirmed.

Since by angle-stretching \( EP \) is greater than \( E' P' \), where \( E \) is the epicentre and \( E' \) its image (Fig. 7), the amplitude at \( P \) is greater than it would be at the same epicentral distance in an isotropic disc, as long as \( P_0 P > P_0' P' \).

6. Multiply reflected rays

The terms given by values of \( s \) greater than zero in equation (4.8) will correspond to arrivals after \( s \) reflections: amplitudes may be calculated as for the direct ray. But by use of the image space we may obtain all amplitudes very economically from the known expressions for rays in an isotropic cylinder (Shimamura & Sato 1965).

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