The Creation of an Electron Pair by a Fast Charged Particle

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Using the Feynman-Dyson method, the cross section for the creation of an electron pair by a fast charged particle is calculated but in a manner more precise than that of Bhabha and others. The differential cross section obtained here is valid as long as the energies of participant particles are large compared with the respective rest masses. The ambiguities in Bhabha's calculation are also examined. It is concluded that the theoretical value must be compared with the experimental results produced by high energy electrons (≥ 10 Bev).

§ 1. Introduction

The creation of an electron pair by a charged particle of spin 1/2 was investigated by Bhabha, Nishina et al. and others, and their results have been applied for analysing highly energetic electromagnetic phenomena in cosmic rays underground. The results of such analyses seem to show that quantum electrodynamics is valid even at extremely high energy, say, $10^{15}$ eV, though the comparison of the theories with the experiments is done indirectly.

However, recent developments of the experimental techniques of photographic emulsion have made it possible to measure directly the cross section of the process in question, called trident. Koshiba and Kaplon* have indicated that the experimental value of the cross section for tridents produced by a high energy electron is in disagreement with the theoretical value given by Bhabha.

It is an interesting and important problem whether there really exists a discrepancy between the experimental and the theoretical results, because it is believed that quantum electrodynamics gives a correct description of electromagnetic phenomena extensively. In order to clarify this point it will be necessary on the one hand that the experimental analyses of tridents be performed more accurately while on the other hand that the theoretical results be derived as strictly as possible following quantum electrodynamics.

As is well known, Bhabha's results have been extensively used for analyzing cosmic ray phenomena. In his approach the incident particle is regarded as a classical one moving along a straight line with uniform velocity, the field of which is replaced by a classical field. This approximation is essentially based on the same assumption as used in

* They cited the similar experimental results obtained by Freier and Naugle (unpublished).
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The Williams-Weizsäcker method and gives a correct description only when the energy transferred from the incident particle to the created electron pair is small compared with the energy of the incident particle. On the other hand, the process called "first order process" by Bhabha, i.e., the diagrams D' in the figures, is neglected in B. It is clear that this neglect may be allowed when the mass of the incident particle is large compared with that of an electron (see § 4), but is not justified when the incident particle is in fact an electron. For the latter case the estimation of the contribution from the diagrams D' and also the exchange effect are not treated quantitatively. Therefore it seems necessary to recalculate the cross section of this process more strictly over the whole range of the transferred energy by using a quantum electrodynamical treatment and estimating quantitatively the errors arising from the rather rough treatment in B.

In this paper we shall calculate the process in question using the Feynman-Dyson method, whereby the incident charged particle will be treated quantum dynamically but the target charged particle will be regarded as a fixed Coulomb field. In the following section we shall derive a general formula for the transition probability, explain the treatment used in performing our calculations, and then derive the differential cross section for the process in question. We shall derive the total cross sections in § 3; in § 4 we shall discuss the effects which must be taken into account when the incident particle is an electron, i.e., the exchange effect and the contributions from the diagrams D' which are neglected in § 2. In the final section we shall discuss our results and compare them with the experiments.

§ 2. Differential cross section

(a) General formula for the cross section
Throughout this paper we use the natural unit, \( \hbar = c = 1 \), and the following notations:

- \( P_1(p_1, iE_1), P_2(p_2, iE_2) \): The initial and the final energy-momentum four-vectors of the incident charged particle.
- \( P_{\pm}(p_{\pm}, i\epsilon_{\pm}) \): The energy-momentum four-vectors of the positron and the electron, respectively.
- \( k(k, i\epsilon) = P_1 - P_2 \): The energy-momentum four-vector of the virtual photon.
- \( m \) and \( \mu \): The rest mass of the electron and the incident particle, respectively.

The Feynman-Dyson diagrams for the present process in the lowest order consist of four diagrams as given in the figures. However, we take only the diagrams D. The reason for neglecting the matrix element \( M_D \) from the diagrams D' will be made clear in § 4.

From the diagrams D, the matrix element \( M_D \) is expressed as
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\[ M_D = -\frac{Z e^2}{q^2} \sum \frac{i}{\gamma - \gamma'} \frac{\bar{u}(P_2) \gamma_{\mu} u(P_1)}{D_-} \left( \frac{\gamma'}{D_-} \right) \]

\[ + \left( \gamma' \right) \frac{i}{\gamma - \gamma'} \frac{\bar{u}(P_2) \gamma_{\mu} u(P_1)}{D_+} \left( \gamma' \frac{m}{D_+} \right) \]

\[ \equiv K \cdot 2\pi \delta_0 (P_1 - P_2 - P_3 - P_4), \]

where

\[ q = p_1 - p_2 - p_3 - p_4, \]

\[ D_\pm = (k \mp P_\pm)^2 + m^2, \]

and \( \delta_0 (P_1 - P_2 - P_3 - P_4) \) is the fourth component of the four dimensional delta function which expresses the conservation of energy.

The transition probability per unit time, \( w \), is thus

\[ w = 2\pi \sum |K|^2 \delta_0 (P_1 - P_2 - P_3 - P_4) \rho_P, \]

where \( \sum \) represents the summation over the spin directions of the final state and the average over the spin directions in the initial state, and \( \rho_P \) is the density of the final state for the three particles, i.e.,

\[ \rho_P = \frac{m^2}{\epsilon_+ \epsilon_-} dp_1 dp_2 dp_3 (2\pi)^6, \]

where spinors of the incident charged particle are normalized as \( \bar{u^*} u = 1 \), while those of the electron pair as \( \bar{w} w = 1 \).

Dividing the expression (3) by the velocity of the incident particle \( |p_1|/E_1 \), we obtain from (3) and (4) the differential cross section,

\[ d\sigma = \frac{2\pi E_1}{|p_1|} \sum |K|^2 \delta_0 (P_1 - P_2 - P_3 - P_4) \frac{m^2}{\epsilon_+ \epsilon_-} \frac{dp_1 dp_2 dp_3}{(2\pi)^6}. \]

(b) Calculation of \( \sum |K|^2 \)

In order to calculate \( \sum |K|^2 \), we introduce a new coordinate system in which the \( z \)-axis is parallel to the direction of the propagation of the virtual photon, but not to that of the incident particle, i.e.,

\[ z \text{-axis } / / k = p_1 - p_2. \]

Henceforth we shall call this coordinate system S-system. The advantage of introducing the S-system is not only that the virtual photon can be naturally separated into the transverse photon and the longitudinal and the scalar ones so that the physical meaning of the calculating procedure is made much clearer, but also that the angular integrations in the final state are more precisely carried out than was done in B. By virtue of this procedure we shall be able to obtain a differential cross section which is valid as long as the energies of participant particles are large compared with the respective rest masses.

In the following discussion we use two approximations: (i) the energies of all the particles are relativistic, i.e., large compared with their rest masses; (ii) the small angle
approximation, since by virtue of the denominators \( q^2, k^2, D_\pm \), contained in \( K \) the main contributions to the cross section come from regions where the angle of each particle is very small. Thus, for example,

\[
|p_i| \sin \theta_i \lesssim |p_i| \theta_i \equiv p_{i,1}, \quad |p_i| \cos \theta_i \equiv - (\mu^2 + p_{i,0}^2) / 2E_i.
\]

(7)

Here the magnitude of the transverse component of each momentum is, as is easily seen, of the same order as that of each rest mass (see (12)).

\( K \) consists of two factors, one of which is related with the incident charged particle and the other with the electron pair, i.e.,

\[
K = - Z e^4 / 16 \pi^2 A_\mu B_\mu,
\]

\[
A_\mu = \bar{u}(P_2) \gamma_\mu u(P_1),
\]

\[
B_\mu = \frac{1}{q^2 k^2} \bar{u}(P_-) \left[ i \frac{1}{2} \left( \gamma_{\mu} \left( P_2 - k \right) - m \right) + \left( \gamma_\mu \right) \left( k - P_+ \right) - m \right] u(-P_+).
\]

We shall calculate \( A_\mu \) separately for each case according to whether the spin of the incident particle flips or not and also whether the polarizations of the virtual photons are transverse or longitudinal. Though this round about procedure seems to be more complicated than the spur-calculation, it makes it much easier to pick out the main terms of \( \sum_k |K|^2 \), and we can easily find how much the effects of flipping of the spin of the incident particle and the polarizations of the virtual photon contribute to the cross section.

Now, taking into account (6) and (7) we obtain the expressions for the non-spin flip case:

\[
A_1(\uparrow \uparrow) = i \bar{u}_t(P_2) \gamma_\uparrow u_t(P_1) = \frac{1}{2} \left( \frac{P_+^2 + P_-^2}{E_1} \right),
\]

\[
A_2(\uparrow \uparrow) = i \bar{u}_t(P_2) \gamma_\downarrow u_t(P_1) = i \left( \frac{P_+^2 + P_-^2}{2E_1} \right),
\]

\[
A_3(\uparrow \uparrow) = i \bar{u}_t(P_2) \gamma_\uparrow u_\uparrow(P_1) = - \left[ 1 - \frac{P_+^2 (E_1 - E_2)}{8 E_1 E_2} \right] + \frac{1}{4} \left( \frac{1}{E_1^2} + \frac{1}{E_2^2} \right),
\]

(9)

\[
A_4(\uparrow \uparrow) = i \bar{u}_t(P_2) \gamma_\downarrow u_\downarrow(P_1) = i \left[ 1 - \frac{P_+^2 (E_1 - E_2)}{8 E_1 E_2} \right] - \frac{1}{4} \left( \frac{1}{E_1^2} + \frac{1}{E_2^2} \right),
\]

\[
= i A_3(\downarrow \downarrow) + \frac{1}{2} \left( \frac{\mu^2 + P_+^2}{E_1 E_2} \right),
\]

and for the spin flip case:

\[
A_1(\downarrow \uparrow) = i \bar{u}_t(P_2) \gamma_\downarrow u_\uparrow(P_1) = - \left( \frac{\mu}{2} \right) \left( E_1 - E_2 \right) / E_1 E_2,
\]

\[
A_2(\downarrow \uparrow) = i \bar{u}_t(P_2) \gamma_\uparrow u_\downarrow(P_1) = i \left( \frac{\mu}{2} \right) \left( E_1 - E_2 \right) / E_1 E_2,
\]

\[
A_3(\downarrow \uparrow) = i \bar{u}_t(P_2) \gamma_\uparrow u_\uparrow(P_1) = - \left( \frac{P^-}{2} \right) \left( E_1 - E_2 \right) / E_1 E_2,
\]

\[
A_4(\downarrow \downarrow) = i \bar{u}_t(P_2) \gamma_\downarrow u_\downarrow(P_1) = - i \left( \frac{P^-}{2} \right) \left( E_1 - E_2 \right) / E_1 E_2,
\]

(10)
where

\[ p^\pm = p_1^\pm = p_2^\pm, \quad p^\pm = p_{1z} \pm i p_{1y}, \quad p^\pm = p_{2z} \pm i p_{2y}. \]

\[ p_1^0 = p_1^\perp = p_2^\perp, \quad p_1^0 = p_{1z} + i p_{1y}, \quad p_2^0 = p_{2z} + i p_{2y}. \]

For \( A_\alpha(\downarrow\uparrow) \) and \( A_\alpha(\uparrow\downarrow) \) we can obtain similar expressions to (9) and (10).

Inserting (9) and (10) into (8), using the approximation (7) and averaging over \( \varphi_p \) we obtain

\[
\sum |K|^2 = (4\pi)^4 (Ze^2)^2 \left( \frac{e^2}{m} \right)^2 \frac{1}{q^4} D_1^1 D_2^2
\]

\[
\times \left[ \left\{ \frac{e^2}{e^2 + e^2} \right\} \left\{ e^2 + e^2 \right\} \left\{ e^2 + e^2 \right\} \left\{ e^2 + e^2 \right\} \right] D_1^1 D_2^2
\]

\[
+ 2 \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \right] D_1^1 D_2^2
\]

\[
+ \left\{ \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \right] D_1^1 D_2^2
\]

\[
+ \left\{ \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \right] D_1^1 D_2^2
\]

\[
+ 2 \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \left( \frac{e^2 + e^2}{e^2 + e^2} \right) \right] D_1^1 D_2^2
\]

(11)

where

\[ p_{\pm \perp} = (p_{\pm z}, p_{\pm y}, 0, 0), \]

\[ (p \cdot p_{\perp})_1 = (p_{1z} p_{1y}, p_{2z} p_{2y}, 0, 0). \]

The general forms of the terms involved in the outer curly brackets of \( \sum |K|^2 \) are const. \( \times 0 \left( (\mu/e)^{2k} (m/e)^{3j} \right) \), here \( k \) or \( l \) is zero or positive integer. In (11) we pick out only the terms satisfying \( k + l = 2 \). By this procedure we can derive an expression which always involves the corresponding main terms to all values of the transferred energy. The three square brackets in (11) correspond to the contributions from the following three processes, respectively; (i) the spin of the incident particle does not flip and the virtual photons are transverse; (ii) the spin of the incident particle does not flip but the virtual photons are longitudinal and scalar ones; (iii) the spin of the incident particle flips and the virtual photons are transverse. The contribution from the process that the spin flips and the virtual photons are longitudinal and scalar ones is neglected on account of the smallness of its order of magnitude. The interference terms between the transverse and the
longitudinal photons vanish by the average over $\varphi_\perp$. Using the approximation (7), we can write down $q^2, k^2$ and $D_\pm$ as follows:

$$q^2 = (p_\perp + p_-)_\perp^2 + \frac{1}{4} \left( \frac{1}{\epsilon_+} (m^2 + p_\perp^2) + \frac{1}{\epsilon_-} (m^2 + p_-^2) + \frac{\epsilon_e}{E_1 E_2} (\mu^2 + p_{\perp 1}^2) \right)^2,$$

$$k^2 = \frac{\epsilon_e^2}{E_1 E_2} (\mu^2 + p_\perp^2),$$

$$D_\pm = \frac{\epsilon_e}{\epsilon_\pm} \left( M^2 + p_{\perp \pm \perp}^2 + \frac{\epsilon_\pm - \epsilon_{\perp}}{E_1 E_2} p_{\perp \perp}^2 \right), \quad M^2 = m^2 + (\epsilon_+ - /E_1 E_2) \mu^2. \tag{12}$$

(c) Integrations over angles

In order to perform angular integrations, we must transform the density of the final state $\rho_\perp$ into the S-system (see Appendix). Its form is

$$\rho_\perp = \frac{m^2}{\epsilon_+ \epsilon_-} \left( \frac{\epsilon_e}{E_1} \right)^2 d(p_{\perp \perp} dp_{\perp \perp} dp_{\perp \perp} d\epsilon_+ d\epsilon_- d\epsilon_{\perp} d\epsilon_{\perp}, \tag{13}$$

or if we introduce a new set of variables instead of $p_{\perp \perp},$

$$\zeta = (p_+ + p_-)_\perp, \quad \eta = (p_+ - p_-)_\perp, \tag{14}$$

the final form of $\rho_\perp$ is given by

$$\rho_\perp = \frac{m^2}{\epsilon_+ \epsilon_-} \left( \frac{\epsilon_e}{E_1} \right)^2 \int d(\zeta^2) d(\eta^2) d(\rho_{\perp \perp}) d\epsilon_+ d\epsilon_- d\epsilon_{\perp} d\epsilon_{\perp} dE_2, \tag{15}$$

where $\varphi_\zeta$ and $\varphi_\eta$ are azimuthal angles of $\zeta$ and $\eta$, respectively. By the transformation (14), (12) becomes

$$q^2 = \zeta^2 + \left( \frac{\epsilon_e}{2 \epsilon_+ \epsilon_-} \right)^2 \left[ M^2 + \frac{1}{4} (\zeta^2 + \eta^2) + \frac{\epsilon_\pm - \epsilon_{\perp}}{\epsilon_e} (\zeta \eta) + \frac{\epsilon_\pm - \epsilon_{\perp}}{E_1 E_2} p_{\perp \perp}^2 \right]^2,$$

$$D_\pm = \frac{\epsilon_e}{\epsilon_\perp} \left[ M^2 + \frac{1}{4} (\zeta^2 + \eta^2) + \frac{\epsilon_\pm - \epsilon_{\perp}}{E_1 E_2} p_{\perp \perp}^2 \right]. \tag{16}$$

From (16) we can see that the main contribution to the integration over $\zeta$ and $\eta$ comes from the domain where the order of magnitude of $\eta$ is $0(M)$ and the order of magnitude of $\zeta$ is $0(M^2/\epsilon)$ (not $O(M)$). Therefore in the common denominator $1/q^4 D_\perp$, we neglect $\zeta$ in comparison with $\eta$ and $p_{\perp \perp}$, i.e.,

$$q^2 = \zeta^2 + \left( \frac{\epsilon_e}{2 \epsilon_+ \epsilon_-} \right)^2 \left[ M^2 + \frac{1}{4} \left( \zeta^2 + \eta^2 \right) + \frac{\epsilon_\pm - \epsilon_{\perp}}{\epsilon_e} (\zeta \eta) + \frac{\epsilon_\pm - \epsilon_{\perp}}{E_1 E_2} p_{\perp \perp}^2 \right]^2,$$

$$D_\perp = \frac{\epsilon_e}{\epsilon_\perp} \left[ M^2 + \frac{\eta^2}{4} + \frac{\epsilon_\pm - \epsilon_{\perp}}{E_1 E_2} p_{\perp \perp}^2 \right]. \tag{17}$$

Here the azimuthal angles $\varphi_\zeta$ and $\varphi_\eta$ disappear. The neglect of $\varphi_\zeta$ and $\varphi_\eta$ may be justified by introducing the S-system even when the transferred energy is large. But in the S*-system (see Appendix) used in B we can not neglect $\varphi_\eta$, because the expressions corresponding to (16) contain a term like $(\epsilon_+ - /E_1 E_2) (p_\eta)_\perp$ which can not be neglected.
in the case of the large transferred energy. For this reason the calculation procedure using the $S'$-system will be very complicated for large transferred energy. After averaging over $\varphi_z$, we pick out only the terms of the lowest order of $\zeta$ which are proportional to $\zeta^2$ and obtain the next transformed expression of (11),

$$\sum|K|^2 = (4\pi)^4 (Ze)^2 \left(\frac{e^2}{m}\right)^2 \frac{\zeta^2}{q^4}$$

$$\times \left[ \left\{ m^2 \epsilon_+ \epsilon_- \eta^2 - \frac{1}{Dk^2} \frac{\epsilon_+ \epsilon_- (\epsilon_+^2 + \epsilon_-^2) \eta^2 + \frac{1}{k^2D^2} (\epsilon_+^2 + \epsilon_-^2) \right\} \left( \frac{1}{E_1^2} + \frac{1}{E_2^2} \right) \rho_\perp^2 

+ \frac{4}{Dk} \left( \frac{\epsilon_+ \epsilon_-}{\epsilon_0^2 - \epsilon_0^2} \right) \eta^2 

+ \left( \frac{\epsilon_+ \epsilon_-}{E_1E_2} \left\{ m^2 \epsilon_+ \epsilon_- \eta^2 - \frac{1}{Dk^2} \frac{\epsilon_+ \epsilon_- (\epsilon_+^2 + \epsilon_-^2) \eta^2 + \frac{1}{k^2D^2} (\epsilon_+^2 + \epsilon_-^2) \right\} \right) \right] \right], \quad (18)$$

where

$$D = M^2 + \eta^4/4 + (\epsilon_+ \epsilon_-/E_1E_2) \rho_\perp^2. \quad (19)$$

By the above transformation, integrations over the angles have been replaced by $\zeta^2$, $\eta^2$ and $\rho_\perp^2$. Since (18) has been averaged over the variables $\varphi_\eta$, $\varphi_\zeta$ and $\varphi_z$, integrations over these variables will give only $(2\pi)^3$. We take the integration domain of $\zeta^2$ from zero to $(\alpha M)^2$, where $\alpha$ is a number of order unity, and for $\eta^2$ and $\rho_\perp^2$ from zero to infinity as a good approximation.

When the incident particle collides with a neutral atom, it is necessary to consider the screening effect. This effect may be easily taken into account if we replace $\zeta^2/q^4$ in (18) by

$$\zeta^2/q^4 \cdot [1 - F(q^2)]^3, \quad (20)$$

where $F(q^2)$ is the atomic form factor. The minimum value of $q^2$ is given from (17) by

$$q_{\text{min}}^2 = (\epsilon/2\epsilon_+ \epsilon_-) M^4, \quad (21)$$

hence the screening is effective when

$$(\epsilon/2\epsilon_+ \epsilon_-) M^4 \ll Z^2/m/137. \quad (22)$$

In this case, integration over $\zeta^2$ can be done in the same manner as in B. As to pair creation by $\gamma$-rays, more precise calculations of the screening effect were made by Bethe$^\dagger$, but since the treatment of B is in principle the same as Bethe's calculation, we will follow B.

(d) \textbf{Differential cross section}

Inserting (17), (18), (19), and (15) into (5), and after performing the integrations over $\zeta^2$, $\eta^2$, $\rho_\perp^2$, $\varphi_\eta$, $\varphi_\zeta$ and $\varphi_z$, we get finally, as the differential cross section for the creation of a pair, the electron of which has an energy between $\epsilon_-$ and $\epsilon_- + d\epsilon_-$, and the positron of which has an energy between $\epsilon_+$ and $\epsilon_+ + d\epsilon_+$. 

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\[ \sigma = \frac{2}{\pi} \cdot (Ze)^2 \left( \frac{e^2}{m} \right)^2 d\epsilon_+ d\epsilon_- L \]

\[ \times \left[ \epsilon_+^2 + \epsilon_-^2 \left( \frac{1}{3} + \frac{4}{3} x \right) \text{log} \left( \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \right) - \frac{2}{3} \epsilon_+ \epsilon_- \left( \frac{1 + 2x}{x} \right) \text{log} \left( \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \right) - 2 \right] \]

\[ \times \frac{E_1^3 + E_2^3}{E_1^3} \]

\[ + \frac{8}{3} \epsilon_+ \epsilon_- \frac{E_2}{\epsilon_1} \frac{E_1^3}{1 + x} \]

\[ + \left[ \frac{\epsilon_+^2 + \epsilon_-^2}{\epsilon_1^3} \left( \frac{1}{3} \frac{1}{1 + x} + \frac{1}{x} - \frac{4}{3} \text{log} \left( \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \right) \right) \right] \]

\[ + \frac{2}{3} \epsilon_+ \epsilon_- \frac{1}{\epsilon_1^3} \frac{1 + x}{x} \text{log} \left( \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \right) + \frac{\epsilon_+^2}{E_1^3} \right], \quad (23) \]

where

\[ x = (\mu^2 / E_1 \epsilon_1) \cdot (\epsilon_+ \epsilon_- / m^2), \quad (24) \]

and

\[ L = \begin{cases} \log \left( 2\alpha \epsilon_+ \epsilon_- / \epsilon M \right) - 1 & \text{for non-screening,} \\ \log \left( \alpha \cdot 137Z^{-1/3} \cdot M / m \right) & \text{for complete screening.} \end{cases} \quad (25) \]

In the expression (23), the terms proportional to \((E_1\epsilon + E_2^2)/E_1^3\) and \(E_2/E_1\) correspond respectively to the process in which the spin of the incident particle does not flip and the direction of polarization of the virtual photon is transverse, and to the process in which the spin of the incident particle does not flip but the direction of polarization of the virtual photon is longitudinal. The term proportional to \(E_2^4 / E_1^3\) corresponds to the process that the spin flips and the virtual photon polarizes transversely.

In order to get approximate expressions to (23) for each of both cases that the transferred energy is large or small, it is convenient to introduce a set of quantities instead of \(\epsilon_\pm\), i.e.,

\[ u = (\epsilon_+ + \epsilon_-) / E_1 \]

\[ v = (\epsilon_+ - \epsilon_-) / \epsilon. \quad (26) \]

We say the transferred energy is small when

\[ u \ll m / \mu, \quad (27) \]

and large when

\[ u \gg m / \mu. \quad (28) \]

Hereafter we shall call the former the domain I and the latter II. If we express \(x\) by \(u\) and \(v\), we obtain from (24) and (26)

\[ x = \frac{1}{2} (u \mu / m)^2 \cdot (1 - v^2) / (1 - u). \quad (29) \]
Since in the domain I always $x \ll 1$, we obtain as the approximate formula of the differential cross section in this domain

$$
\sigma_1 = \frac{4}{\pi} (Z^2)^{\frac{3}{2}} \frac{e^3 \gamma^3}{m^3} \frac{d \epsilon_+ d \epsilon_-}{e^4} \left[ \log \frac{2 \alpha e_+ e_-}{\epsilon m} \right] \left( \epsilon_+^3 + \epsilon_-^3 + \frac{8}{3} e_+ e_- \right) \frac{m^2 E_1^2}{\mu^2 e_+ e_-} 
$$

$$
= \frac{4}{3\pi} (Z^2)^{\frac{3}{2}} \frac{e^3 \gamma^3}{m^3} \frac{d \epsilon_+ d \epsilon_-}{e^4} \left[ \log \frac{\alpha E_1}{2m} u \left( 1 - v^2 \right) \left( 1 + \frac{v^2}{2} \right) \log \left( \frac{m}{\mu} \right)^2 \frac{4}{1 - v^2} \right].
$$

If we replace the factor $\log \left( \frac{m^2 E_1^2}{\mu^2 e_+ e_-} \right)$ in (30) by $\log \left( k'^2 m^2 E_1^2 / \mu^2 e^2 \right)$ where $k'$ is a number of order unity, (30) is the same as that of B, but the indefinite $k'$ has disappeared in our calculations. As was expected, (30) corresponds only to the process that the spin does not flip and the direction of the polarization of the virtual photon is transverse.

In the domain II, it will be necessary to note that there are two regions according to whether or not the energy transferred to the electron is almost equal to that transferred to the positron because from (26) and (29)

$$
x \gg 1 \quad \text{for} \quad v^2 \ll 1, \quad (32)
$$

$$
x \ll 1 \quad \text{for} \quad 1 - v^2 \ll 1. \quad (33)
$$

The approximate formulas in these two domains are written as

$$
\sigma_{IIa} = \frac{2}{3\pi} (Z^2)^{\frac{3}{2}} \frac{e^3 \gamma^3}{m^3} \frac{d \epsilon_+ d \epsilon_-}{e^4} \left[ L \times \left( \epsilon_+^3 + \epsilon_-^3 \right) \left( 1 + \frac{E_1^2}{E_1^2} \right) + 8 \epsilon_+ \epsilon_- E_1 \frac{\alpha E_1}{\mu} \log \left( 1 - v^2 \right)^{1/2} \left( 1 - u \right)^{1/2} \right]
$$

$$
= \frac{4}{3\pi} (Z^2)^{\frac{3}{2}} \frac{e^3 \gamma^3}{m^3} \frac{d \epsilon_+ d \epsilon_-}{e^4} \left[ \log \frac{\alpha E_1}{2m} u \left( 1 - v^2 \right) \left( 1 + \frac{v^2}{2} \right) \log \left( \frac{m}{\mu} \right)^2 \frac{4}{1 - v^2} \right],
$$

and

$$
\sigma_{IIb} = \frac{4}{\pi} (Z^2)^{\frac{3}{2}} \frac{e^3 \gamma^3}{m^3} \frac{d \epsilon_+ d \epsilon_-}{e^4} \left[ \log \frac{2 \alpha e_+ e_-}{\epsilon m} \right] \left( \epsilon_+^3 + \epsilon_-^3 + \frac{8}{3} e_+ e_- \right) \frac{m^2 E_1 E_2}{\mu^2 e_+ e_-}
$$

$$
- \frac{4}{3} \left( \epsilon_+^3 + \epsilon_-^3 \right) \left( 1 + \frac{\epsilon_+}{E_1} \right) - \frac{4}{3} \left( \frac{\epsilon_-}{E_1} \right)^2 \left( \epsilon_+^3 + \epsilon_-^3 + e_+ e_- \right)
$$

$$
= \frac{4}{3\pi} (Z^2)^{\frac{3}{2}} \frac{e^3 \gamma^3}{m^3} \frac{d \epsilon_+ d \epsilon_-}{e^4} \left[ \log \frac{\alpha E_1 u \left( 1 - v^2 \right)}{2m} - 1 \right]
$$

$$
= \frac{4}{3\pi} (Z^2)^{\frac{3}{2}} \frac{e^3 \gamma^3}{m^3} \frac{d \epsilon_+ d \epsilon_-}{e^4} \left[ \log \frac{m}{\mu} \left( \frac{2m}{\mu} \right)^{1-\frac{u}{1-v^2}} - (1+v^2) (1-v) - \frac{1}{4} (3+v^2) u^2 \right].
$$
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§ 3. Total cross section

(a) Non-screened total cross section

To get the contribution to the total cross section from the domain I, we take the integral domains of \( u \) and \( v \) as follows,

\[
0 \leq u \leq \frac{m}{\mu}, \quad -1 + \frac{m}{\epsilon} \leq v \leq 1 - \frac{m}{\epsilon}.
\]

(35)

Though the expression (31) does not hold at the two limits of integration, (35), \( u \) and \( v \) occur only in logarithmic functions after the integration of (31) so that the indefiniteness of the domains of integration of \( u \) and \( v \) does not appreciably affect the final results.

The result of the integration of (31) is

\[
Q_1^o = \frac{28}{(27\pi)} \cdot (Ze^2)^2 \left( \frac{\alpha E_1}{\mu} \right)^3.
\]

(36)

We have neglected the terms of lower power of \( \log \left( \frac{E_1}{\mu} \right) \) in (36), because the errors which come from the above mentioned indefiniteness of the integration domains are of the order of \( \left( \log \frac{E_1}{\mu} \right)^2 \).

Similarly we obtain in the domain IIa and IIb, respectively,

\[
Q_1^{\text{IIa}} = \frac{4}{3\pi} \left( \frac{\alpha E_1}{\mu} \right)^2 \left( \frac{e^2}{m} \right)^2 \left( \log 2 + \frac{1}{2} \right) \log \frac{E_1}{\mu},
\]

(37a)

\[
Q_1^{\text{IIb}} = \frac{4}{3\pi} \left( \frac{\alpha E_1}{\mu} \right)^2 \left( \frac{3}{2} \log 2 - \frac{1}{2} \right) \log \frac{E_1}{\mu},
\]

(37b)

where \( \beta \) is the number which occurs from the limit of the integration over \( \nu \), i.e., \( \beta \left( \frac{m}{\mu} \right) \leq u \leq 1 \). The sum of the above two values (37a) and (37b) gives the contribution from the domain II to the non-screened total cross section, i.e.,

\[
Q_1^o = Q_1^{\text{IIa}} + Q_1^{\text{IIb}} = \frac{4}{3\pi} \left( \frac{\alpha E_1}{\mu} \right)^2 \left( \frac{e^2}{m} \right)^2 \left( \frac{5}{2} \log 2 + \frac{3}{2} \right) \log \frac{E_1}{\mu}. \]

(38)

It must be noted that \( Q_1^o \) depends on the inverse square of \( \beta \), hence the indefiniteness of \( \beta \) seriously affects the final result. This point will be discussed again in the last section.

(b) Completely screened total cross section

In this case each domain consists of two parts, one of which satisfies (22) and the other does not. The contribution to the cross section from each domain is given by the sum of the contributions from the two parts. The results are as follows:

* Including the terms of lower power of \( \log \left( \frac{E_1}{\mu} \right) \), the full expression of (36) is written as

\[
Q_1^o = \frac{4}{3\pi} (Ze^2)^2 \left( \frac{e^2}{m} \right)^2 \left[ \frac{7}{9} \left( \log \frac{E_1}{\mu} \right)^3 + C_1 \left( \log \frac{E_1}{\mu} \right)^2 + C_2 \log \frac{E_1}{\mu} \right],
\]

where \( C_1 = (4/3) \log \alpha - 0.85, C_2 = 20.81 - \left( 14/3 \right) (\log \beta_1)^2 + \left( 62/9 \right) \log \alpha + \left( 28/3 \right) \log \beta_2 \) and \( \beta_1 \) and \( \beta_2 \) are the numbers (\( \geq 1 \)) which occur from the limits of the integrations.
\[ Q_i^s = \frac{28}{27\pi} (Ze)^3 \left( \frac{e^3}{m} \right)^2 \log(137 Z^{-1/3}) \left[ 3 \log \frac{\alpha E}{\mu} \log \frac{E}{137 Z^{-1/3}} + (\log 137 Z^{-1/3})^2 \right], \] (39)

\[ Q_{ii}^s = Q_{ia}^s + Q_{ii}^s = \frac{4}{3\pi} \left( \frac{e^3}{m} \right)^2 \left( \frac{5}{2} \log 2 + \frac{3}{2} \right) \log \left( \alpha \cdot 137 Z^{-1/3} \right), \] (40)

where

\[ Q_{ia}^s = \frac{4}{3\pi} \left( \frac{e^3}{m} \right)^2 \left( \frac{e^3}{m} \right) \left( \frac{3}{2} \log 2 - \frac{1}{2} \right) \log \left( \alpha \cdot 137 Z^{-1/3} \right). \] (41b)

### § 4. The case in which the incident particle is an electron

Now the cross section consists of three parts proportional respectively to \(|M_\mu|^2\), \(|M_{\mu'}|^2\) and \(|M_\mu M_{\mu'} + M_{\mu'} M_\mu|^2\). Until now, we have neglected the latter two parts. This is permitted if the rest mass, \(\mu\), of the incident particle is much larger than that of an electron, because the part proportional to \(|M_{\mu'}|^2\) contains the factor \((\mu^2/\mu)^2\) instead of \((e^3/\mu)^2\) as in (23) and the interference part proportional to \(|M_\mu M_{\mu'} + M_{\mu'} M_\mu|^2\) contains the factor \((e^3/\mu) (e^3/\mu)\).

However, when the incident particle is an electron, the above discussion is no longer valid and moreover the exchange effect must be considered. The part proportional to \(|M_{\mu'}|^2\) can be evaluated in a manner similar to that used for the part proportional to \(|M_\mu|^2\) and is given by

\[ \sigma' = \frac{2}{\pi} (Ze)^3 \left( \frac{e^3}{\mu} \right)^2 \int d\epsilon_+ d\epsilon_\mu \frac{E^2}{E_\mu^2} \left[ \left( 1 + \frac{4}{3} x' \right) \log \left( 1 + \frac{1}{x' \cdot 3} \right) - \frac{4}{3} - \frac{2}{3} \frac{E_\mu E}{\epsilon^4} \left( 1 + 2x' \right) \log \left( 1 + \frac{1}{x' \cdot 3} \right) - 2 \right], \] (42)

where

\[ x' = 1/x, \]

\[ L' = \begin{cases} \log (2\alpha E_\mu E \epsilon_\mu (1 + x')^{1/2}) - 1 & \text{for non-screening,} \\ \log \alpha \cdot 137 Z^{-1/3} (1 + x')^{1/2} & \text{for complete screening.} \end{cases} \]

As is expected, (42) is a type of Bremsstrahlung.

Since \(\mu = m\), the domain II disappears and the only remaining domain is I. The contribution to the non-screened total cross section is evaluated approximately.

\[ Q_{\mu} \sim (Ze)^3 (e^3/m)^2 \log (E_\mu/m). \] (43)

The direct evaluation of the interference parts is rather complicated, but its magnitude can easily be estimated as follows:
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\[ d\sigma_{DD'} = \frac{1}{v} |M_D M_{D'}^* + M_{D'}^* M_D| \rho_\Psi \]

\[ \leq \frac{2}{v} |M_D| \cdot |M_{D'}| \rho_\Psi = 2 \sqrt{\frac{1}{v} |M_D|^2 \rho_\Psi \cdot \frac{1}{v} |M_{D'}|^2 \rho_\Psi} \]

\[ = 2 \sqrt{d\sigma_D d\sigma_{D'}} , \quad (44) \]

where \( v \) is the velocity of the incident electron. After integration of (44)

\[ Q_{DD'} \leq 2 \int \sqrt{d\sigma_D d\sigma_{D'}} \leq 2 \sqrt{\int d\sigma_D \cdot \int d\sigma_{D'}} = 2 \sqrt{Q_D Q_{D'}} . \quad (45) \]

The right hand side is then proportional to \( (Ze')^2 (e'/m)^2 (\log E_1/m)^2 \) which is of the same order as that of the terms neglected in the evaluation of (36).

Next we shall estimate the exchange effect. We represent by \( M_{D'}^* \) the matrix element in which \( P_1 \) and \( P_- \) are interchanged in \( M_D \) and put

\[ M_D = \bar{M}_D/k^2 , \quad M_{D'} = \bar{M}_{D'}/k'^2 , \quad k' = P_1 - P_- . \quad (46) \]

Then the cross section is given by

\[ \sigma \sim \frac{1}{2} |M_D - M_{D'}| = \frac{1}{2} \frac{1}{k} |\bar{M}_D|^2 + \frac{1}{2} \frac{1}{k'} |\bar{M}_{D'}|^2 - \frac{1}{k' k'^2} \text{Re}(\bar{M}_D \bar{M}_{D'}). \quad (47) \]

The contribution of the second term to the total cross section is the same as that of the first, and the third term can be estimated by comparing \( 1/k'^2 \) with \( 1/k^4 \). Thus, putting

\[ I_D = \int \frac{1}{k'} d\Omega_d d\Omega_- , \quad I_{DD'} = \int \frac{1}{k k'^2} d\Omega_d d\Omega_- , \quad (48) \]

we get

\[ I_{DD'}/I_D < (m/E_1)^2 (\log 2E_1/m)^2 , \quad (49) \]

therefore the exchange effect can be neglected as long as \( E_1 \gg m \).

\section{5. Summary and discussions}

First we shall confine our discussions to the case in which the incident particle is an electron. Since the domain II disappears for \( \mu = m \), the total cross section is given by (39) or (36) according to whether the screening is effective or not. In (39) and (36) the terms of lower power of \( \log(E_1/m) \) are neglected. This is justified if the energy of the primary electron is larger than about 10 Bev. For example the full expression for (36) is given by

\[ Q_{\mu}^1 = \frac{4}{3\pi} (Ze')^2 \left( \frac{e'}{m} \right)^6 \left[ \frac{7}{9} (\log E_1/m)^3 - C_1 (\log E_1/m)^3 + C_2 (\log E_1/m) \right] , \]

where \( C_1 < 0.85 \) and \( C_2 < 20.81 \).* Thus the second and the third terms can be neglected

* See the footnote on page 491.
if the incident electron has an energy larger than the above mentioned value. The error
due to this neglect is at most 20%. For such high energies the contribution from the
diagrams D' may also be neglected as estimated in § 4. If the energy of the incident
electron is smaller than 10 Bev, we can not disregard the contribution of the diagrams
D' and the terms of low power.

As is well known, the cross section derived with the Born approximation is inclined
to become larger than the actual value. This error is estimated by Bethe et al.\(^6\) in the
case of the pair creation by \(\gamma\)-rays using the distorted electron wave functions. They
showed that the value derived with the Born approximation is over-estimated by a factor
of 20%. Since (36) is essentially the same as that derived by the Williams-Weizsäcker
method\(^7\), the same situation will hold in our case. Another effect of suppression has
been pointed out by Landau and Pomeranchuk\(^8\)\(^*,\) in the cases of Bremsstrahlung and
pair creation by \(\gamma\)-rays. This effect arises from the fact that the incident particle collides
with a "medium", not with one isolated atom. This suppression becomes important at
energies higher than the critical value—\(10^{15}\) eV for lead. Though the present processes
are not the ones discussed by them, it is certain that this effect also suppresses our values
of the cross sections, because this suppression effect is essentially caused by the interference
of many waves with different phases. It is noted that for the above two reasons our
results (36) and (39) may be considered slightly larger than the actual one.

The experimental values measured by Koshiba and Kaplon may be compared with
our results since the energies of primary electrons are larger than 10 Bev. Our cross
sections give about one third of their values. This discrepancy may not be considered
conclusive, because the experimental errors due to the measurement of the energies of
the primary particles are suspected to be large in such a high energy region. Block et
al.\(^9\) showed that their experimental results are consistent with B modified so as to include
the terms of lower power of \(\log (E_1/m)\). However, since in their experiment the primary
energies lie between 0.1 and 10 Bev having the average value 400 Mev, one has to take
into account the contributions not only from the terms of lower power but also from
the diagrams D'. Otherwise, the comparison with theory is of little meaning. It is
unfortunate that a decisive conclusion can not be drawn from the above mentioned ex-
periments. Since we have derived the cross sections for this process using the current
theory and taking into account the various effects for high energy, it is desired that the
tritons process, in which the primary electron has an energy higher than 10 Bev, will
be studied more extensively.

In the case that the mass of the incident particle is heavy compared with that of
an electron, the differential cross section is given by (23), which is valid as long as the
participant particles have relativistic energies and is more correct than those considered
heretofore. As to small transferred energies the approximation formula (30) to (23) is
essentially the same as B. However for large transferred energies there are two differences

\* The authors thank sincerely to Prof. Z. Koba who kindly brought Landau and Pomeranchuk's work
to their notice.
between our calculation and that of B, one of which is the crudeness in the estimation of the domain IIa in B*; the other being the fact that the domain IIb is not discussed in B.

Now the total cross section is the sum of the contributions from the domain I and II, i.e., $Q^T_1 + Q^T_{II}$ for non-screening and $Q^T_{II} + Q^T_{II}^t$ for complete screening. $Q^T_{II}$ (or $Q^T_{II}^t$) depends on the indefinite number $\beta$; the maximum estimate of $Q^T_{II}$ (or $Q^T_{II}^t$) is given by putting $\beta = 1$. But in any case we may neglect $Q^T_{II}$ (or $Q^T_{II}^t$) compared with $Q^T_1$ (or $Q^T_1^t$). For the energy loss of the incident particle we cannot neglect the contribution from domain II, which will be of the same order as that from domain I. Thus we see that the final results seriously depend on the value of $\beta$. Therefore the value of the energy loss using B may have an error of a factor about two.***

Finally, we wish to make one more remark. It was shown that the main contribution to the process comes from the transition in which the spin of the incident particle does not flip and the virtual photon polarizes transversely. This fact seems to show that the Williams–Weizsäcker method should be a good approximation. We shall discuss this point in a forthcoming paper.

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Appendix

We shall call a coordinate system, in which the $z'$-axis is parallel to $p_1$, the $S'(x', y', z')$-system, and a coordinate system having its $z$-axis parallel to $k = p_1 - p_2$, the $S(x, y, z)$-system. We take the $x$- and $y$-axes of the $S$-system as shown in fig. A1.

The density of the final state in the $S'$-system is given by

$$\rho_{s} = dp_1'/ (2\pi)^3 \cdot (m^2/\epsilon) \cdot dp_1 dp_2/ (2\pi)^4,$$

* In order to derive (34a), we used an approximation: $\log(1 + 1/x) = -1/x - 1/(2x^2)$, but if we use such a rough approximation as $\log(1 + 1/x) = -1/x$, we obtain the same expression as derived in B (see eq. (36) in B).

** In B, $\beta$ is taken as $3 \approx 4$. Putting this value into our expressions, our results become the same as B.

*** For large transferred energies there may arise the effect of virtual meson cloud of the target nucleus. Considering the indefiniteness of $\beta$, however, our result will not be changed largely by this effect.
where

\[ dp' = dp_1 dp_2 dp_3 dp_4 = p_2 dp_2 \sin \theta_2 d\theta_2 dp_4, \]
\[ p_2 = |p_2'| = |p_2|. \]

We shall express it by the variables in the S-system. The form of the density with respect to the electron pair is not changed and expressed as \((m^2/\varepsilon_+ \varepsilon_-) \cdot (dp_1 dp_2/(2\pi)^6)\), because the transformation between the two systems is orthogonal. With respect to the incident charged particle we obtain from figs. A1 and A2,

\[ \varphi_3' = \varphi_a, \]
\[ p_1 \sin \theta_2 = |k| \sin \theta_2 \]

or, using the small angle approximation,

\[ \theta_2' = \varepsilon/E_1 \cdot \theta_2. \]

Therefore

\[ dp_2' = p_2^2 dp_2 \sin \theta_2' d\theta_2' dp_4' \sim (\varepsilon/E_1)^2 p_2^2 dp_2 d\theta_2 dp_2 = (\varepsilon/E_1)^2 dp_2 dp_2 \]

and the density of the final state in the S-system is expressed as

\[ \rho' = \frac{1}{(2\pi)^6} \left( \frac{\varepsilon}{E_1} \right)^2 \frac{m^2}{\varepsilon_+ \varepsilon_-} \frac{dp_1 dp_2 dp_3 dp_4}{(2\pi)^6} \]
\[ \sim \frac{m^2}{\varepsilon_+ \varepsilon_-} \frac{1}{(2\pi)^6} \left( \frac{\varepsilon}{E_1} \right)^2 dp_1 dp_2 dp_3 dp_4 dp_5 dp_6. \]

It must be noted that \( B_a \) does not depend on \( \varphi_a \), because \( B_a \) depends on \( p_2' \) only through the momentum \( k = (p_1' - p_2') = p_2 - p_2 \) which is independent of \( \varphi_a \). Therefore we may take into consideration \( A_a \) only, when we integrate or average over \( \varphi_a \).

References

7) E. J. Williams, loc. cit.