researches of Dr. Fraser, his comments are not unexpected. It is encouraging that he has been able to extend the Ross analysis method for plane two-dimensional flows to the axisymmetric case, since this was the unachieved objective of the authors' work. The indication that $\frac{\delta}{D}\frac{U_{10}}{U_1}R$ rather than $\frac{\delta}{D}\frac{U_{10}}{U_1}\frac{R}{R_0}$ is the proper outer-flow parameter emphasizes the need for experiments in diffusers of other initial radii. Work in this direction is currently under way in England at the Mechanical Engineering Research Laboratory, as evidenced by a recent paper of Winternitz and Ramsay.[8] It is hoped that this research will soon yield us the needed experimental information.

Mr. Uram's remarks concerning the surprisingly short response time of the outer flow to changes in geometry, or pressure gradient, are concurred in. One wonders which of the two changes, geometry or pressure, induces the change. The outer-flow parameter indicates that both R and U_1 enter in, but it does not tell us why. It is hoped that Mr. Uram will continue his study of velocity profiles and attempt to tie down the factors a, b, and c. The results indicated are very suggestive but require a more detailed study for adequate comment. There may be some question as to what kind of progress we are making if we substitute a relation involving three empirically determined factors (a, b, and c) for one involving one empirically determined factor (D). Equation [2] does have some rational basis and works reasonably well. It may be that for other reasons such as consideration of the "equilibrium" boundary layers [cf. reference (5) of paper], other relations for the velocity are desirable.

The authors wish to thank Messrs. Fraser and Uram for their interesting discussions.

Improvement of Holzer Table Based on a Suggestion of Rayleigh's[1]

N. O. Myklestad.[2] In this paper a method is developed by which the time involved in finding a natural frequency of torsional vibration by means of the Holzer table can be reduced; but in general this requires judgment on the part of the computer.

Since the task of obtaining a natural torsional mode is an easy one in either case, the method of this paper becomes advantageous only when such calculations must be performed frequently. When this is the case, the Holzer calculations are now nearly always programmed for a high-speed digital computing machine so that any just comparison should be made on this basis.

The usual way of finding natural modes then is to program the calculations so that the machine automatically calculates the shaking torque for regular intervals of ω^2, and every time a reversal of sign is obtained it backtracks and proceeds with smaller intervals of ω^2 until a reversal is again obtained; and this is kept up until the shaking torque is smaller than some predetermined value.

In the method of this paper the value of ω_{Ray}^2 could be calculated every time there was a reversal of sign of the shaking torque; but whether or not this is an advantage probably can be determined only by experience.

[8] "Effects of Inlet Boundary Layer on Pressure Recovery, Energy Conversion and Losses in Conical Diffusers," by F.A.L. Winternitz and W. J. Ramsay, *Journal of the Royal Aeronautical Society*, vol. 61, Feb. 1957, pp. 116–124.

[1] By S. H. Crandall and W. G. Strang, published in the June, 1957, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, pp. 228–230.

[2] Senior Engineer, Specialist, Phoenix Division, AiResearch Manufacturing Company, Phoenix, Ariz. Mem. ASME.

E. W. Gaylord.[3] Convergence of successive trials would be assured for any mode if, as in Stodola's method for higher modes,[4] the value of the maximum deflection β, found in the Holzer table, were purified of all the lower modes including the mode for $\omega = 0$.

In terms of the orthogonal eigenfunctions X the β column of the Holzer table may be expressed as

$$V = \Sigma a_j X_j \quad [1]$$

The Rayleigh method reduces to

$$\omega_{\mathrm{Ray}}^2 = \frac{\Sigma\omega_j^2 a_j^2 X_j\cdot BX_j}{\Sigma a_j^2 X_j\cdot BX_j} \quad [2]$$

or with X normalized to

$$\omega_{\mathrm{Ray}}^2 = \frac{\Sigma\omega_j^2 a_j^2}{\Sigma a_j^2} \quad [3]$$

In terms of ω_k

$$\omega_{\mathrm{Ray}}^2 = \omega_k^2\,\frac{\Sigma\,\frac{\omega_j^2}{\omega_k^2}\,a_j^2}{\Sigma a_j^2} \quad [4]$$

Since $\frac{\omega_j}{\omega_k} > 1$ for all $j > k$

the Rayleigh method gives an upper bound for ω_k^2 if all a_j for $j < k$ are made zero in Equation [4].

This may be readily done if all the lower mode eigenfunctions are known, as would be the case when one successively used the Holzer method to evaluate the preceding modes of vibration.

The purified shape will be $V' = V - \Sigma a_j X_j;\ j < k$

or

$$\beta' = \beta - \Sigma a_j \beta_j \quad [5]$$

To evaluate the a_s form the orthogonal product

$$\sum_{n\neq j} a_j X_j\cdot BX_n = V\cdot BX_n - a_n X_n B\cdot X_n = 0 \quad [6]$$

giving

$$a_n = \frac{V\cdot BX_n}{X_n\cdot BX_n} \quad [7]$$

For the zeroth mode $X_0 = 1, 1, 1 \ldots 1$

$$a_0 = \frac{\Sigma I\beta}{\Sigma I} \quad [8]$$

For purification of the first mode it would be

$$a_1 = \frac{\Sigma I\beta\beta_1}{\Sigma I\beta_1^2}$$

where β_1 is found from the Holzer table for the correct value of ω_1.

Purification of the zeroth and first mode from column 3 of den Hartog's example gave purified values of β as follows

β	$a_0\beta_0$	$a_1\beta_1$	$\beta' = \beta - a_0\beta_0 - a_1\beta_1$
1.00	−2.20	2.42	0.78
−26.78	−2.20	−0.64	−23.94
19.16	−2.20	−0.65	22.01
63.97	−2.20	−0.66	66.83
104.99	−2.20	−0.67	107.86
139.78	−2.20	−0.67	142.65
166.29	−2.20	−0.67	169.16
182.95	−2.20	−0.68	185.83

[3] Assistant Professor of Mechanical Engineering, Carnegie Institute of Technology, Pittsburgh, Pa.

[4] "Mechanical Vibrations," by J. P. den Hartog, McGraw-Hill Book Company, Inc., New York, N. Y., fourth edition, 1956, p. 162.

The purified values of β resulted in $\omega_{Ray}^2 = 18{,}332$ which, being an upper bound, is somewhat higher than the value $\omega^2 = 18{,}000$, given by den Hartog.

The authors are fortunate in getting so close to the correct frequency. This is due to the corrections for the first and zeroth mode having a small net effect on the value of β.

F. Söchting.[5] In this valuable paper the authors derive a method to shorten the necessary time to calculate the torsional natural frequencies by using a suggestion of Rayleigh. The authors are inclined to assume that the suggestion of Rayleigh was never used before. The writer thinks it may be of interest to say that this method was used two times before, to calculate the torsional frequency based on Holzer's equation[6] and to determine the frequencies of strings and beams.[7]

Authors' Closure

The suggestion is made by Professor Gaylord that the approximate mode shape obtained in a Holzer table should be orthogonalized with respect to all known modes before computing Rayleigh's quotient. This would insure that ω_{Ray} would be an upper bound for the next natural frequency. Furthermore, the quadratic convergence in the neighborhood of the true mode would still be retained. The authors are in complete agreement so far as the mathematical principles involved are concerned. They are somewhat in doubt, however, as to the practical wisdom of such a procedure. One of the advantages cited[8] for the method as given in the paper was that orthogonalization was *not* required. Systematic orthogonalization of every trial mode entails considerable computational labor, more than the Holzer table itself, for the higher modes.

Dr. Myklestad suggests that a procedure requiring judgment from the computer may not be desirable when a program for a high-speed digital computer is being set up. There is considerable validity in this viewpoint. Unsubtle, overly repetitious procedures are often the easiest to program and trouble shoot, and give the most reliable operation. Furthermore, the present high computing speeds in comparison with set-up and input-output times means that often there is no appreciable penalty involved in using nonoptimum procedures. As for the particular procedure described in the paper, it must be admitted that it was originally designed for hand computation using a desk calculator. The sign pattern criteria which can be evaluated almost instantaneously by eye would require fairly elaborate programming to insure foolproof operation in a digital computer.

It is a pleasure to acknowledge the prior claim of F. Söchting to the process we have called "the Rayleigh-Kohn-Holzer Method." After the publication of our paper we were informed[9] that this method was known in the German literature[10] as Söchting's method. There is a remark in Klotter's paper[10] that the convergence of Söchting's method may be doubtful. An attempt to clarify this was made by Schaefer.[11] In view of this, perhaps the principal contribution of our paper has been in supplying a more complete picture of the convergence of the method. The main features are illustrated[12] in Fig. 1 of this discussion. The Rayleigh quotients of the Söchting process are plotted against the trial values of ω^2. The properties of the curve shown may be deduced from Equation [9] of the paper. The curve crosses the 45-deg line at the Ω_j^2 and the μ_j^2. Strict iteration with the Söchting process can be represented by paths such as ABCDE. The quadratic convergence in the neighborhood of an Ω_j^2 is indicated by the horizontal tangent at $\omega^2 = \Omega_j^2$. The μ_j^2 are unstable foci for the process. The curve crosses the 45-deg line with a slope of 2 at each μ_j^2. At the eigenvalue Ω_p^2 the second derivative of the curve is

$$\frac{d^2\omega_{ray}^2}{d(\omega^2)^2} = 2 \sum_j{}' \frac{h_j^2/h_p^2}{\Omega_j^2 - \Omega_p^2}$$

in the notation of the paper, which implies that the general tendency is for the curve to be concave down at the large eigenvalues and concave up at the small eigenvalues. This general tendency may be ineffective, however, in regions where two or more eigenvalues are very close together. We have been unable to obtain a criterion for predicting the sign of the local curvature. In our experience with practical torsional systems, the curve has been concave down at all natural frequencies except the trivial zero frequency. When the curve is concave down in the neighborhood of the eigenvalue Ω_j^2, as shown in Fig. 1, the Söchting process is sure to converge to Ω_j^2 when $\mu_j^2 < \omega^2 < \nu_j^2$, where ν_j^2 is the smallest value of $\omega^2 > \Omega_j^2$ for which $\omega_{Ray}^2(\nu_j^2) = \mu_j^2$. However, only when $\mu_j^2 < \omega^2 < \Omega_j^2$ is it readily apparent from the sign pattern in the Holzer table that the trial ω^2 is actually in this range. This was the basis for our recommendation to compute ω^2_{Ray} only when it is indicated that ω_{Ray}^2 will be higher than the trial ω^2.

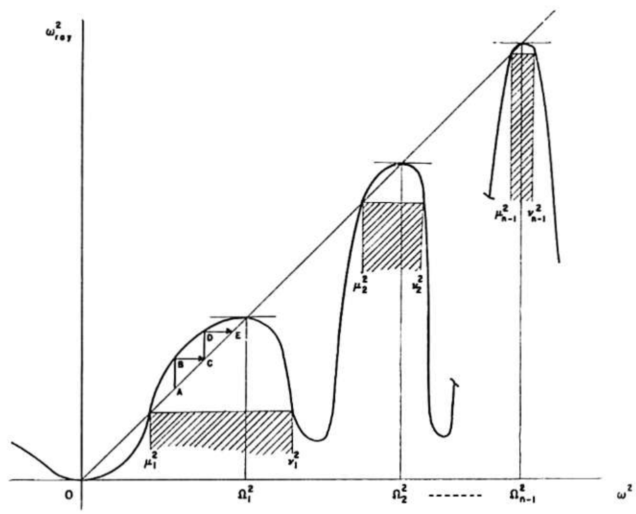

Fig. 1 The curve gives the Rayleigh quotient corresponding to the trial value of ω^2 used in the Holzer table

[5] Curtiss-Wright Corporation, Wright Aeronautical Division, Wood-Ridge, N. J.

[6] "Zur Berechnung der Eigenschwingungszahlen von Wellenleitungen," by F. Söchting, ATZ, vol. 40, 1937, p. 259; also Berechnungen Mechanischer Schwingungen, 1951, p. 234.

[7] "Zur Berechnung von Eigenschwingungszahlen," by F. Söchting, Federhofer-Girkmann Festschrift, 952, p. 365.

[8] "Iterative Procedures Related to Relaxation Methods for Eigenvalue Problems," by S. H. Crandall, Proceedings of the Royal Society of London, series A, vol. 207, 1951, pp. 416–423.

[9] Private communication from Professor H. Schaefer, Technischen Hochschule, Braunschweig.

[10] See, for example, "Analyze der verscheidenen Verfahren zur Berechnung der Torsionseigenschwingungen von Maschinenwellen," by K. Klotter, *Ingenieur Archiv*, vol. 17, 1949, pp. 1–60.

[11] "Das Restmoment der Torsions schwingungen von Maschinenwellen," by H. Schaefer, Abh. Braunschweigischen Wiss. Ges., 6, 1954, pp. 243–254.

[12] This figure was used in the oral presentation of the paper, Nov. 27, 1956. Similar figures appear in the reference of footnote (11) and in the reference of footnote (8). The fallacy in Schaefer's work results from the assumption that the curve is monotonically increasing between $\omega^2 = 0$ and $\omega^2 = \Omega_{n-1}^2$.