we obtain the usual growth rate:

$$\omega_s \sim \pm \sqrt{\frac{-g\beta h_\infty (\rho_0 - \rho_\infty)}{\rho_0 + \rho_\infty \xi h_\infty}}.$$

which is proportional to $h_\infty^{1/2}$ for small $h_\infty$.

The last result may be relevant to the problem of boiling heat transfer. As is well known, the state of nucleate boiling will go over to stable film boiling when the temperature gradient exceeds certain critical value. For the film boiling, a fairly stable layer of vapor lies underneath the heavier liquid contrary to the criterion of Rayleigh-Taylor instability. Although, our result shows the instability will still persist, yet the growth rate can be greatly reduced. It is conceivable that when the disturbances grow slowly to the finite amplitude, the nonlinear effect will eventually stabilize the system.

For the unusual case that the latent heat is very small, if there is also no temperature gradient at equilibrium, then the dispersion relation (43) reduces to

$$P_* = 0,$$

or

$$\omega^2 = \frac{g\beta (\rho_0 - \rho_\infty) \sinh \beta h_\infty \cosh \beta h_\infty + \rho_\infty \sinh \beta h_\infty \cosh \beta h_\infty}{(\rho_0 - \rho_\infty) \cosh \beta h_\infty \cosh \beta h_\infty}.$$  \tag{52}\]

We may also remark that the effect of surface tension can be easily included. We need only to replace $g$ by $\frac{g + \frac{\sigma^2}{k}}{(\rho_0 - \rho_\infty)}$ in the dispersion relation (43).

References


**DISCUSSION**

Vijay Dhir$^2$ and John Lienhard$^6$

The author has presented a very comprehensive extension of the Taylor stability theory. It will doubtless find applications in many important problems. He is quite correct, for example, in noting its potential application to film boiling. We have been involved for some time in this area and are delighted to see someone address the combined influences of heat and mass transfer, and of finite fluid depth, on the stability problem. Indeed we are presently investigating the influence of another system variable—the liquid viscosity—upon the problem, and Prof. Hsieh's study will provide verification for some of the assumptions we wish to make.

The purpose of our discussion is twofold. First we shall cast Prof. Hsieh's very general dispersion relation, equation (43), into dimensionless form. This will make it far easier to determine which terms can be neglected in a given application. We shall then apply the equation to the process of film boiling from a cylinder—a process for which there exists plenty of experimental data.

Order of Magnitude Analysis. To facilitate the nondimensionalization we are including a Nomenclature section in which, for convenience, we combine all the nomenclature related to Prof. Hsieh's equation (43) with our additional nomenclature. In particular the reader should note the distinction between Hsieh's effective gravity, $g$ (which depends upon a variety of system variables) and the actual acceleration on the system $g_\infty$. Since, for any boiling problem, the function, $g$, will introduce the surface tension, $\sigma$, it is proper to use $\sigma$ in our nondimensionalization. We then introduce the following dimensionless variables:

$$A = \frac{D^{(0)}}{D^{(0)}}, \quad B_i = \frac{h_i}{h}, \quad B_2 = \frac{h_2}{h_\infty},
C = \frac{g_\infty}{g}, \quad K = \frac{q^{(0)}}{q^{(0)}}; \quad \Gamma = \frac{\beta^{(0)}}{\beta^{(0)}},
J = \frac{\rho^{(0)}D^{(0)}}{G/L^{(0)}}, \quad E = \frac{\varphi_T^{(0)}}{G/L},
$$

$$\Lambda = \frac{1}{\sqrt{3k}} \left[ \frac{\varphi_T^{(0)}}{\sigma} \right]^{1/2}, \quad \Omega = -\omega \left[ \frac{\varphi_T^{(0)}}{\sigma} \right]^{1/4},$$

$$N^{(0)} = \frac{D^{(0)}}{\frac{\sigma}{k} \left( \frac{\varphi_T^{(0)}}{\sigma} \right)^{1/4}}.$$  \tag{62}\]

Under this nondimensionalization equation (43) becomes

$$\left\{ \begin{array}{l}
\Gamma + \frac{1}{B_i + 1} \left[ \frac{3J_B B_i \Omega A}{N^{(0)}} + \frac{B_1 \sqrt{1 + \frac{3J_B A}{N^{(0)}}}}{1 + \frac{3J_B A}{N^{(0)}}} \right] \cos (B_1 \sqrt{1 + \frac{3J_B A}{N^{(0)}}}) \\
\left( \frac{B_1 \sqrt{1 + \frac{3J_B A}{N^{(0)}}}}{1 + \frac{3J_B A}{N^{(0)}}} \right) \cos (B_1 \sqrt{1 + \frac{3J_B A}{N^{(0)}}}) \end{array} \right\} = C$$

$$= \left\{ \begin{array}{l}
\sqrt{3} (\cos B_1 \sinh B_1 + \Gamma \cosh B_1 \sinh B_1) \Omega A \\
B_1 (\Gamma - 1)
\end{array} \right\}$$

$$= \left\{ \begin{array}{l}
\cos B_1 \cosh B_1 - C \\
B_1 + \Gamma
\end{array} \right\}$$

$$= \left\{ \begin{array}{l}
\cos B_1 \cosh B_1 + \Gamma \sinh B_1 \cos B_1 \\
B_1 + \Gamma
\end{array} \right\}$$

$$\times \left\{ \begin{array}{l}
1 + \sqrt{3} \Omega A E \left[ \sqrt{1 + \frac{3J_B A}{N^{(0)}}} \cos (B_1 \sqrt{1 + \frac{3J_B A}{N^{(0)}}}) \\
1 + \frac{3J_B A}{N^{(0)}} \cos (B_1 \sqrt{1 + \frac{3J_B A}{N^{(0)}}}) \right] \end{array} \right\}.$$  \tag{43a}\]

Equation (43) contains 16 variables (if we include $\sigma$) in 3 dimensions. Our reduction of this expression into an equation in 11 dimensionless groups is thus consistent with the Buckingham Pi-theorem.

Fig. 1 shows a typical liquid-vapor interface in film boiling on a horizontal heater, both in a photograph from [8]$^8$ and in a schematic idealization. The large wave amplitudes in the photographs represent growth to beyond the range in which the present linear theory can be expected to apply. However, since our concern is with the selection of the original small amplitude wave

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$^6$ Numbers in brackets designate Additional References at end of Discussion.
which eventually grow to this size, it is reasonable to compare such data with the theory. The waves in the actual case are not stable as Prof. Hsieh suggests, by the way. They are actually executing a kind of cyclic bubble escape process in which Taylor-unstable waves grow and collapse to release bubbles.

The original hydrodynamic theory describing such behavior was formulated by Zuber [9], who assumed infinite depth and used the dispersion relation:

$$ \Omega = \sqrt{\frac{1}{\sqrt{3} \lambda} \Gamma - 1 \frac{1}{C}}; \quad C = 1 - \frac{1}{3\lambda^2} $$

(53)

This was modified by Lienhard and Wong [10] for the horizontal cylindrical geometry. They retained the infinite depth assumption and obtained the same $$Q$$, but with

$$ C = 1 - \frac{1}{3\lambda^2} + \frac{1}{2R'^2}; \quad R' = R \sqrt{\frac{\beta(\rho^{(3)} - \rho^{(1)})}{\sigma}} $$

(54)

For comparison we can nondimensionalize Hsieh's equation (46)—a simplified form of his dispersion relation—which is based on all the same assumptions except that of infinite vapor depth, and write it in the same form for this problem:

$$ \Omega = \sqrt{\frac{1}{\sqrt{3} \lambda} (\Gamma - 1)} $$

(55)

For $$\Gamma \gg B_1^{-1}$$ this is the same as equation (53).

Table 1 lists, in the second column, the relative order of magnitude of the dimensionless variables in equation (43a). Using the fact that $$B_1 \ll 1$$, $$B_2 \gg 1$$ and $$B' \ll 1$$ we can reduce equation (43a) to:

$$ \frac{\Gamma}{B_1 + \Gamma} \left[ \frac{3JB_i(3\lambda^2)}{N^{(1)}} + B_1 \left( 1 + \frac{3\lambda^2}{N^{(0)}} \right) \right]^{1/2} $$

$$ \times \cosh \left( B_1 \sqrt{1 + \frac{3\lambda^2}{N^{(0)}}} + B_1 \left( 1 + \frac{3\lambda^2}{N^{(0)}} \right) \right) $$

$$ \times \left\{ \left( 1 + B_1 \Gamma \sqrt{3\lambda^2} \right) - \left( 1 - \frac{1}{3\lambda^2} + \frac{1}{2R'^2} \right) \right\} $$

$$ = \left( \frac{\Gamma}{B_1 + \Gamma} \right) \left( \frac{3JB_i(3\lambda^2)}{N^{(1)}} \right) \left( 1 - \frac{1}{3\lambda^2} + \frac{1}{2R'^2} \right) $$

(56)

Equation (56) is the equivalent expression to Prof. Hsieh's equation (49). However he has carried the deletion of terms farther than would be admissible for the film boiling case, by taking $$B_1$$ (or $$B_2$$ in his configuration) to be very very small.

Table 1 Magnitude of dimensionless variables for the film boiling configuration

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Order of magnitude</th>
<th>Situations treated in text</th>
</tr>
</thead>
<tbody>
<tr>
<td>$$\Gamma$$</td>
<td>$$\infty$$ to 2</td>
<td>$$\infty$$</td>
</tr>
<tr>
<td>$$B_1$$</td>
<td>0.1</td>
<td>0.157</td>
</tr>
<tr>
<td>$$B_2$$</td>
<td>$$\gg 1$$</td>
<td>0.180 0.249 0.508 0.817</td>
</tr>
<tr>
<td>$$A$$</td>
<td>0 to 1</td>
<td>1.1 1.0 0.9 1.0</td>
</tr>
<tr>
<td>$$N^{(1)}$$</td>
<td>0.01 to 0.001</td>
<td>0.018 0.03 0.01</td>
</tr>
<tr>
<td>$$J$$</td>
<td>1 to 0.01</td>
<td>0.054 0.775</td>
</tr>
<tr>
<td>$$R'$$</td>
<td>$$\ll 1$$</td>
<td>3.4</td>
</tr>
<tr>
<td>$$R$$</td>
<td>0.1 to 1</td>
<td>0.041</td>
</tr>
<tr>
<td>$$A$$</td>
<td>Variable</td>
<td>0.044 0.004</td>
</tr>
<tr>
<td>$$C$$</td>
<td>$$\gg 1$$</td>
<td>Equations (58) and (54)</td>
</tr>
<tr>
<td>$$R'$$</td>
<td>$$\gg 0$$</td>
<td>0.06</td>
</tr>
</tbody>
</table>

(a) Evaluated using a value of $$\kappa$$ which corresponds with $$\Omega_{max}$$. (b) $$CO_2$$ and Acetone columns give coordinates for maximum $$\Omega$$. The left-hand entry in each column is for $$R' = \infty$$ and the right-hand entry is for $$R' = 1.02$$. (c) Reference [8] shows Taylor waves cannot exist for $$R' \leq 0.06$$.

Equation (56) can also be summarized as

$$ [\ldots] \cdot F_1 = [1] \cdot F_2 $$

(56a)

but inspection of equation (56) shows that at low pressure

$$ F_1 \simeq F_2 = \sqrt{3\lambda^2} \left( 1 - \frac{1}{3\lambda^2} - \frac{1}{2R'^2} \right) $$

(57)

Equation (56a) can be true for $$F_1 \simeq F_2$$ only if $$\{ \ldots \} = 1$$ or if $$F_1 = F_2 = 0$$. The former is out of the question since it requires the unreasonable result: $$\Omega \sim 1/\lambda^4$$. It follows that Hsieh is correct in writing $$F_1 = 0$$ at low pressure. Thus equation (57) gives

$$ \Omega' = \frac{1}{\sqrt{3\lambda^2}} - \frac{1}{(\sqrt{3\lambda^2})^3} + \frac{1}{2\sqrt{3\lambda^2R'^2}} $$

(58)

which is exactly the dispersion relation given in [8] for low pressure.

This leads to an extremely interesting consequence of Prof. Hsieh's analysis. Equation (58) includes none of the several different parameterizations that have been suggested for film boiling. It is a unique expression and is independent of all empirical correlations.
mensionless groups related to heat and mass diffusion. We can conclude that at low pressure (or more precisely for \( \Gamma \gg \rho_{\text{c}}^{-1} \)) such processes do not influence the Taylor stability. It also shows that the widely used infinite vapor depth assumption, while wrong in concept, does not lead to error in most cases.

At higher pressures, \( F_1 \) and \( F_2 \) begin to diverge from each other and the full equation (56) must be employed. The effects of heat and mass transfer will then be introduced into the problem.

**Numerical Results and Experimental Data.** Abadzic and Goldstein [11] provide wavelength observations for the film boiling of \( \text{CO}_2 \) near the critical point, on a 0.19 mm radius wire. Their observations, like the earlier observations of Grigull and Abadzic [12] show that the Taylor waves do not remain intact to too close to the critical point. Therefore, for all cases of practical interest, \( \Gamma \) is substantially greater than unity. We shall consider their results for 29.5 deg C. Here \( \Gamma = 1.915 \) and \( R' = 1.02 \).

Accordingly, we have calculated equation (56) numerically for \( R' = 1 \), and for the corresponding flat plate case (\( R' = \infty \)). We have calculated each case both for saturated \( \text{CO}_2 \) at 29.5 deg C and for saturated acetone at atmospheric pressure (since we have data for this condition). The results of this calculation are plotted in Fig. 2. Along with these curves we include the low pressure limit given by equations (53) and (54). The values of all dimensionless parameters corresponding with these cases are given in the two left-hand columns of Table 1. The vapor blanket thickness, which cannot be measured with any accuracy, was obtained from an approximate prediction made by Baumeister and Hamill [13]. This prediction is also reproduced and discussed critically in [8].

Fig. 2 shows that very near the critical point, Hsieh predicts a strong attenuation of the wave frequency. However, the wavelength deviates very little from the low pressure values at any pressure, and equation (58) can be used for computations.

Experimental data for acetone [8] and \( \text{CO}_2 \) [11], on horizontal cylindrical heaters, are included in Fig. 2. The measured wavelength for acetone nicely embraces the fairly flat range of maximum frequency as we would expect. The measured wavelength for \( \text{CO}_2 \) however, matches the "critical" (or minimum unstable) wavelength instead. We find this result surprising and wonder what its full significance might be. It suggests that for the slow-moving waves near the critical point much less inertia is involved and the wave with the fastest natural frequency might have far less advantage. The shorter wavelength might then be favored because it provides more vapor removal locations. In this case a precise explanation would require a proper formulation of the appropriate minimum principle.

**Conclusions**

1. Prof. Hsieh’s dispersion relation, equation (43), has the potential for broad application in a host of problems. We have sought to pursue its application to film boiling.
2. Equation (43a) is an appropriate nondimensionalization of equation (43) which facilitates order of magnitude simplifications.
3. Equation (56) is the general dispersion relation applicable to film boiling. Except near the critical point, it can be replaced by equation (58).
4. The use of equation (58) is not legitimate near the critical point despite the implication by previous authors [8, 9, 10, 11] that it might be.
5. That the critical wavelength has a natural frequency equal to zero is no problem. All waves in film boiling are driven beyond their natural frequency.