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Short Note

Linearization of the eikonal equation

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INTRODUCTION

Seismic traveltime tomography is a nonlinear inverse problem wherein an unknown slowness model is inferred from the observed arrival times of seismic waves. Nonlinearity arises because the raypath connecting a given source and receiver depends on the slowness. Specifically, if $L(s)$ designates a raypath through the slowness model s between two fixed endpoints, then the path integral for traveltime

$$t(s) = \int_{L(s)} s \, d\ell$$

is a nonlinear functional of s because it does not, in general, satisfy the superposition condition (i.e., $t(s_1 + s_2) \neq t(s_1) + t(s_2)$ where s_1 and s_2 are two different slowness models). The tomographic inverse problem can be solved after linearizing the traveltime expression about a known slowness model s_0. This linearized expression is usually obtained by appealing to Fermat's principle (e.g., Nolet, 1987). Alternately, the required relation can be rigorously derived via ray-perturbation theory (Snieder and Sambridge, 1992). The purpose of this note is to present a straightforward derivation of the same result by linearizing the eikonal equation for traveltimes. Wenzel (1988) adopts this approach, but his method of proof cannot be generalized to heterogeneous 3-D media. A full 3-D treatment is given here. The proof is remarkably simple, and thus it is quite possible that others have discovered it previously.

DERIVATION

The traveltime of a seismic wave propagating through a 3-D slowness structure $s(\mathbf{r})$ satisfies the eikonal equation

$$\|\nabla t(\mathbf{r})\|^2 = s(\mathbf{r})^2. \tag{1}$$

This is a nonlinear partial differential equation for the traveltime function $t(\mathbf{r})$. If the eikonal equation is linearized about a reference slowness model $s_0(\mathbf{r})$, then an expression is obtained for the change in traveltime $\delta t(\mathbf{r})$ induced by a small slowness perturbation $\delta s(\mathbf{r})$. Hence, assume that the slowness $s(\mathbf{r})$ is given by

$$s(\mathbf{r}) = s_0(\mathbf{r}) + \delta s(\mathbf{r}). \tag{2}$$

Then, the traveltime function can also be split into two constituent parts, and is given by

$$t(\mathbf{r}) = t_0(\mathbf{r}) + \delta t(\mathbf{r}), \tag{3}$$

with $\|\nabla t_0(\mathbf{r})\|^2 = s_0(\mathbf{r})^2$. Substituting these relations into equation (1) and engaging in some simple algebraic manipulation yield

$$\nabla\delta t(\mathbf{r}) \cdot \left\{ \left[\frac{\nabla t_0(\mathbf{r})}{s_0(\mathbf{r})}\right] + \frac{1}{2}\left[\frac{\nabla\delta t(\mathbf{r})}{s_0(\mathbf{r})}\right] \right\} = \delta s(\mathbf{r}) \left\{ 1 + \frac{1}{2}\left[\frac{\delta s(\mathbf{r})}{s_0(\mathbf{r})}\right] \right\}. \tag{4}$$

The quantity $\mathbf{l}_0(\mathbf{r}) \equiv \nabla t_0(\mathbf{r})/s_0(\mathbf{r})$ is a dimensionless unit vector tangent to the raypath through the reference medium at position $\mathbf{r}$. If the two approximations

$$\left\| \frac{\nabla\delta t(\mathbf{r})}{s_0(\mathbf{r})} \right\| \ll \|\mathbf{l}_0(\mathbf{r})\| = 1, \qquad \left| \frac{\delta s(\mathbf{r})}{s_0(\mathbf{r})} \right| \ll 1,$$

are adopted, then equation (4) reduces to the linearized eikonal equation

$$\nabla\delta t(\mathbf{r}) \cdot \mathbf{l}_0(\mathbf{r}) \simeq \delta s(\mathbf{r}). \tag{5}$$

The left-hand side of this expression is recognized as the directional derivative of $\delta t(\mathbf{r})$ in the direction of the raypath tangent vector $\mathbf{l}_0(\mathbf{r})$. Hence, it is rewritten as

$$\frac{d\delta t(\mathbf{r})}{d\ell_0} \simeq \delta s(\mathbf{r}), \tag{6}$$

where $d\ell_0$ is an infinitesimal displacement along the reference raypath. Finally, integrating equation (6) yields the familiar result for the traveltime perturbation

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$$\delta t(\mathbf{r}) \simeq \int_{L_0(\mathbf{r})} \delta s \; d\ell_0, \tag{7}$$

where $L_0(\mathbf{r})$ is the raypath through the reference medium that connects the source with the position $\mathbf{r}$. The integral is readily evaluated because the locus of this raypath is known.

CONCLUSION

The approximate relation (7) is the basic theoretical expression underlying linearized traveltime tomography. It implies that a small change in traveltime is linearly related to a small change in slowness. In general, the accuracy of the expression improves as the relative slowness perturbation $|\delta s(\mathbf{r})|/s_0(\mathbf{r})$ diminishes. Although this equation is usually obtained by physical reasoning based on Fermat's principle, the present work demonstrates that it can be derived in a straightforward manner by perturbing the eikonal equation, neglecting all nonlinear terms, and solving the resulting expression.

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