Gravitational stability computed through the limit equilibrium method revisited

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SUMMARY
The stability of slopes is a problem of great relevance for geologists and geophysicists as well as for geotechnical and geoenvironmental engineers. The classical approaches are the method of limit equilibrium, and the finite-element and finite-difference analyses of deformations. Since the former is computationally simpler and less expensive, it is more widely used in common practice, though it has some weakness points from a theoretical point of view. Essential in this technique is the definition and computation of the factor of safety $F$ for the slope, a parameter indicating that the slope is stable, if it is larger than unity. The method is known to have not a unique solution, but it is common belief that the safety factors associated with all the solutions fulfilling the basic equilibrium equations do not differ more than 5–10 per cent from each other, which is a range of variability considered acceptable by most. Here the non-uniqueness of the solution is discussed, and it is shown that the magnitude range of $F$ can be so large as to undermine the meaning of the safety factor criterion. The classical limit equilibrium methods based on the assumptions of cutting the sliding body into a set of vertical slices are revised, and the new concept of minimum lithostatic deviation (MLD) is introduced as a means to mitigate the effect of non-uniqueness. The paper suggests that the proper solution to the problem is the one that satisfies the equilibrium equations and minimizes the lithostatic deviation that is defined here as the ratio of the average intensity of the interslice forces and the total weight of the body. Accordingly, the factor of safety $F$ associated with such a solution is suggested to be the value appropriate to evaluate the stability of the slope. Remarkably, the MLD principle gives us the means to introduce a completely revolutionary approach to study stability. We derive expressions that account for gravitational loading, and for additional effects such as seismic loading and the overpressure due to the overlying water mass in case of underwater slopes.

Key words: earthquake-triggered sliding, limit equilibrium method, MLD, slope stability, underwater slope.

1 INTRODUCTION
The problem of assessing the stability of a slope has a large theoretical and practical interest, and since long time has attracted and involved geologists, engineers and geophysicists, who developed several theoretical tools and processed a countless number of applications in different fields and on different scales: stability of embankments and excavations, of dams and reservoir flanks (see Duncan 1996 for a state-of-the-art review), of rocky (Ersismann & Abele 2001), clayey or sandy slopes, of unconsolidated material forming volcanic edifices (Carracedo 1999; Reid et al. 2000; Donnadieu et al. 2001) of thick and large sedimentary layers on submarine continental margins (Dawson et al. 1988; Nisbet & Piper 1998; Leynaud et al. 2004), of earthquake-induced landslides (Keefer 2002; Del Gaudio et al. 2003) are only a few of the very many illustrative examples of cases that scientists have to cope with and to analyse, often with the purpose of reducing instability conditions by means of reinforcing works or of evaluating and mitigating the hazard of catastrophic failures and collapses.

Realistic analyses of slope stability could require a detailed knowledge of soil conditions and subsurface properties that vary in time and space and that are often difficult and expensive to acquire in field or from field samples examined in the laboratory. Frequently only the most relevant parameters are available, which hampers the use of very sophisticated theoretical tools. Indeed the lack of knowledge of the subsurface media is the most relevant and recognized obstacle to make widespread use of 3-D finite-element and finite-difference analysis in order to compute strain and stress pattern and to evaluate instability conditions of slopes. Usually, this approach or its cheaper 2-D variant are
adopted in engineering applications involving investments of big economic value (dams, large industrial plants, bridges, mines, etc.), when *in situ* high-quality and high-resolution measurements can be afforded and can provide an adequate data set of the subsoil geometry and rheology (Duncan 1996; Jeremic 2000; Griffiths & Lane 2001). Similar considerations hold for the stability analysis performed by means of the distinct-element method (Stead & Eberhardt 1997; Coggan *et al.* 1998; Huang *et al.* 2003) that does not differ substantially from the finite element one: the former consists in discretizing the subsurface medium in a coarse set of different blocks recognized on physical grounds (typically, block boundaries are inner discontinuities, and blocks may be in turn subdivided in finite-element or finite-difference subgrids), rather than building a fine mesh of elements with boundaries determined predominantly by computational reasons.

The method of limit equilibrium is computationally simpler and demands the use of a more restricted set of data on the properties of the subsoil, which is the main reason for it to be so largely and conveniently employed in the common practice of slope stability analysis. The bulk of the method was chiefly developed in the 1950s and 1960s, through the master works by Bishop (1955), by Morgenstern & Price (1965), by Spencer (1967) and by Janbu (1968), who introduced different hypotheses to improve the basic Fellenius formula (1927). However, theoretical developments to enhance the reliability of the method and to widen its range of applicability have never ceased to be produced in the following years until today (Duncan & Wright 1980; Chen & Morgenstern 1983; Leshchinsky & Huang 1992; Chen *et al.* 2001; Zhu *et al.* 2001, 2003; Chugh 2003; Jiang & Yamagami 2004). This paper addresses one aspect of the method that has been surprisingly overlooked so far. The method involves more unknowns than equations and needs assumptions to be made on the unknowns to reach an acceptable balance. In the conventional versions the assumptions regard the internal forces or side forces, and is commonly believed that the final results of the analysis that is the computed value of the factor of safety for the slope and the surface of sliding, do not depend too much on the initial surmises. The paper explores this issue showing that, on the contrary, there exist exact solutions to the equilibrium equations that are associated with very different values of the safety factor $F$. The consequence is that a new tool is needed to determine the value of $F$ that has to be used to in the slope stability criterion. The paper introduces the parameter called ‘lithostatic deviation’ and suggests the principle of the minimum lithostatic deviation (MLD) as the most appropriate means to compute the factor of safety and to find the stability condition of a slope.

## 2 FORMULATION OF THE PROBLEM

The method of limit equilibrium was initially conceived 60 yr ago for 2-D problems that is to analyse the stability of slopes along cross-sections, since full 3-D computations were prohibitively expensive and complicated due to the scarce availability of computers (see Whitman & Bailey 1967). The extension to 3-D analyses is an advancement that was systematically pursued in 1980 (Chen & Chameau 1982; Hungr 1987), and that can now be performed by using economic PC, due to the dramatic progress in computing power and in software packages. Since 1967). The extension to 3-D analyses is an advancement that was systematically pursued in 1980 (Chen & Chameau 1982; Hungr 1987), and that can now be performed by using economic PC, due to the dramatic progress in computing power and in software packages. Since the emphasis is here on the principles at the basis of the method rather than on the computational aspects, the attention will be restricted to formulating the problem in a 2-D space, with the advantage that the illustration will be easier.

### 2.1 Equilibrium equations for a 2-D slope

The static equilibrium of a 2-D body $B$ implies that anyone of its subportions is in equilibrium as well. In principle there is no practical loss of generality if we assume that in the space $(x, z)$ (where the axis $z$ points upward) $B$ is bounded by a bottom surface $S_1$ and a top surface $S_2$ that can be given explicit analytical expressions, that is:

$$S_1 \equiv \{ x, z : x_1 \leq x \leq x_f, z = z_1(x) \},$$

$$S_2 \equiv \{ x, z : x_1 \leq x \leq x_f, z = z_2(x) \},$$

where $z_1(x)$ and $z_2(x)$ are continuous and differentiable functions, defined within the interval $I = [x_1, x_f]$, as is sketched in Fig. 1. The tractions acting on the bottom and top surfaces are denoted by $T_1$ and $T_2$ with respective components $T_{1x}(x), T_{1z}(x)$, and $T_{2x}(x), T_{2z}(x)$. Usually, in slope stability problems the top surface $S_2$ and the external traction $T_2$ are taken as known quantities, while the bottom surface $S_1$ and the corresponding traction $T_1$ are unknown. Here we consider that the body $B$ is subjected to the gravity acceleration $g$ and to an external known seismic load. If we take the subportion $\Sigma$ of $B$, delimited by the boundary $\partial \Sigma$ with external unit vector $\mathbf{m}$, then the conditions for the horizontal and vertical equilibrium of $\Sigma$ can be written in the following form:

$$\int_{\partial \Sigma} (\sigma_{1x} m_x + \sigma_{1z} m_z) d(\partial \Sigma) = -k_{H} M_{\Sigma} g,$$

$$\int_{\partial \Sigma} (\sigma_{2x} m_x + \sigma_{2z} m_z) d(\partial \Sigma) = (1 + k_{V}) M_{\Sigma} g.$$  

Here $\sigma_{1x}, \sigma_{1z},$ and $\sigma_{2z}$ are the components of the stress tensor and $M_{\Sigma}$ is the total mass of $\Sigma$. Notice that in this expression the seismic loading is accounted for in the simplified form of an additional acceleration with magnitude proportional to $g$ through the horizontal and vertical coefficients $k_{H}$ and $k_{V}$ (see Graham 1984; Loukidis *et al.* 2003). In principle, both $k_{H}$ and $k_{V}$ vary in time and space, and accordingly should be considered functions of $x, z$ and $t$. However, following the conventional approach, we will consider a time-independent load (which generally correlates with the peak seismic acceleration) that is uniform over the sliding body $B$: hence $k_{H}$ and $k_{V}$ are supposed here to be constant coefficients. Notice further that the seismic load is assumed to act downward when $k_{V} > 0$ and in the direction of increasing $x$ when $k_{H} > 0$. 

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The further condition of equilibrium is that the total torque on $\Sigma$ is null. Since the internal stresses do not contribute to the torque, the rotational balance can be expressed as follows:

$$
\int_\Sigma (\sigma_{xz} m_x + \sigma_{zz} m_z)(z-z_e) \, d\Sigma - \int_\Sigma (\sigma_{xx} m_x + \sigma_{zz} m_z)(x-x_e) \, d\Sigma + (1+k_f) \int_\Sigma \rho g(x-x_e) \, d\Sigma + k_H \int_\Sigma \rho g(z-z_e) \, d\Sigma = 0,
$$

(1c)

where $\rho$ is the density of the body, and the pole of rotation $C$ with coordinates $x_c$ and $z_c$ can be chosen arbitrarily on the plane $(x,z)$. The integral eq. (1) provide the formulation of the problem of the static equilibrium under the hypothesis that they hold for any subset $\Sigma$ of the body $B$.

The body $B$ may be partitioned into a set of subportions by cutting it by means of planar vertical surfaces. This kind of partition is useful with the Lagrangian treatment of the equation of motion (see Hungr 1995; Tinti et al. 1997, 1999; Bortolucci et al. 2001). Let us now take $\Sigma$ to be a vertical slice of $B$ that has an infinitesimal width $dx$, being delimited by the vertical planes $x' = x - dx/2$ and $x' = x + dx/2$ with $x \in I$. The three eq. (1) can be easily specialized for this slice and take a simple form. With the aid of Fig. 1, let us first define the local slope angle $\alpha(x)$ of the bottom surface $S_1$ through the relation $\tan \alpha(x) = -dz_1(x)/dx$, and analogously for the inclination angle $\beta(x)$ of $S_2$ with $\tan \beta(x) = -dz_2(x)/dx$. Then, if the midpoint of the bottom side of the slice with coordinates $x_c = x$ and $z_c = z_1(x)$ is taken as the pole of rotation, the eq. (1) become:

$$
\frac{dE(x)}{dx} + T_{1x}(x) \sec \alpha + T_{2x}(x) \sec \beta = -k_H g \int_{z_1(x)}^{z_2(x)} \rho(x,z) \, dz,
$$

(2a)

$$
\frac{dX(x)}{dx} + T_{1z}(x) \sec \alpha + T_{2z}(x) \sec \beta = (1+k_f)g \int_{z_1(x)}^{z_2(x)} \rho(x,z) \, dz,
$$

(2b)

$$
\frac{d}{dx} \left( A(x) - z_1(x) \frac{d}{dx} \right) E(x) + T_{2y}(x)[z_2(x) - z_1(x)] \sec \beta - X(x) = -k_H g \int_{z_1(x)}^{z_2(x)} \rho(x,z)(z-z_1) \, dz,
$$

(2c)

where the function $E(x)$, $X(x)$ and $A(x)$ are defined as:

$$
E(x) = \int_{z_1(x)}^{z_2(x)} \sigma_{xx}(x,z) \, dz,
$$

(3a)

$$
X(x) = \int_{z_1(x)}^{z_2(x)} \sigma_{zz}(x,z) \, dz,
$$

(3b)

$$
A(x) = \int_{z_1(x)}^{z_2(x)} z \sigma_{zx}(x,z) \, dz.
$$

(3c)

The functions $E(x)$ and $X(x)$ have the dimension of a force per unit length and are the zeroth-order moment of the distribution of the normal stresses $\sigma_{xx}(x,z)$ and of the shear stresses $\sigma_{zz}(x,z)$. On the other hand, the function $A(x)$ has the dimension of a torque per unit length and is the first-order moment of the distribution of $\sigma_{zx}(x,z)$. If we impose the above equilibrium conditions to all slices that can be defined on the
body B in the range \([x_i, x_j]\), hence the expressions (2) together with the related definitions (3) can be seen as a set of first-order differential equations in the interval \(I\). If we further assume that the bottom and top surfaces of B coincide at the end points of \(I\), that is:

\[ z_1(x_i) = z_2(x_j), \]  
\[ z_1(x_j) = z_2(x_j). \]

It follows immediately from the definitions (3) that the functions \(E(x), X(x)\) and \(A(x)\) have to satisfy the boundary conditions:

\[ E(x_i) = E(x_j) = 0, \quad (4a) \]
\[ X(x_i) = X(x_j) = 0, \quad (4b) \]
\[ A(x_i) = A(x_j) = 0. \quad (4c) \]

Hence, by introducing the vertical slices the original formulation of stability has been reduced from a 2-D to a 1-D problem where the unknowns are the functions \(E(x), X(x)\) and \(A(x)\) as well as the bottom traction \(T_i(x)\) and the bottom surface \(z_i(x)\). This is a notable simplification, but it is worth outlining that the problem we get is not logically equivalent to the original one, which is expressed by the integral eq. (1) applied to all possible subsets of the body. Indeed, the equilibrium of all slices (that are special subsets of B) is a necessary condition for the equilibrium of the entire body, but it is not sufficient. Despite this theoretical deficiency, the plainness of this formulation is so appealing that it can be taken as an alternative, though weak form of stability for a slope.

### 2.1.1 The method of slices

The idea of partitioning the body B into vertical slices we have adopted in the previous section is not new, for it was introduced more than 60 yr ago in the pioneer works on the limit equilibrium. In traditional analyses the slices, either of finite (e.g. Bishop 1955) or infinitesimal thickness (e.g. Chen & Morgenstern 1983), are considered as rigid bodies, and consequently their equilibrium is studied in terms of the balance between their weight, the possible loads (in our case, the seismic load already accounted for and the overlying water load that will be introduced later in Section 3) and the forces of mutual interaction between contiguous slices. This slice–slice interaction is analysed by introducing the interslice force \(R(x)\) and the forces of mutual interaction \(A(x)\) (as well as the bottom traction \(T_i(x)\) and the bottom surface \(z_i(x)\)).

Therefore, for the set of unknowns of the traditional approaches results to be composed by the variables \(R_i(x), R_f(x)\) and \(z_i(x)\), in addition to the bottom tractions and the bottom surface (i.e. the quantities \(T_i(x)\) and \(z_i(x)\) in our notation). In our model, the internal forces are represented via the distributions \(dJ(x, z) = \sigma(x, z) dz\) and \(dK(x, z) = \sigma(x, z) dz\) associated with the normal and shear stresses, respectively. The zeroth-order moments \(E(x)\) and \(X(x)\) can be easily identified with the horizontal and vertical components of the total force \(R(x)\) and \(T(x)\). However, there is no equivalent for \(z(x)\). Indeed, in terms of force distributions, the ratio of the first-order moment to the zeroth-order moment can be taken as the position of the point of application of the total force. This is a notable simplification, but we could define a similar variable \(z(x)\) for the total vertical force \(X(x)\).

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However, because no constraint imposes that \(z(x)\) since the two distributions of forces are independent, the above variable \(z(x)\) has no meaning in our approach. Our derivation of the weak form of the equilibrium in 1-D space shows that \(E(x), X(x)\) and \(A(x)\), or equivalently \(E(x), X(x)\) and \(z(x)\), are the only variables associated with the internal forces that have relevance for the equilibrium conditions, and that neither \(z(x)\) nor any of the higher-order moments have any influence.

### 2.2 The limit equilibrium method

The analysis of equilibrium depicted so far is incomplete since it poses no limit on the stresses that the body is capable to sustain without changing its state, that is, without cracking, rupturing, fracturing, etc. This issue can be addressed only if the properties of the material composing the body B are taken into account. The simplest and most traditional way to evaluate the stability of B is to compare the shear stress \(S\) on the bottom surface with the shear strength of the material \(S_{\text{max}}\), and to require that \(S_{\text{max}}\) cannot be exceeded by the modulus of \(S\), since it would produce a material failure. In 1-D this condition assumes the form:

\[ |S(x)| \leq S_{\text{max}}(x), \quad x \in [x_i, x_j], \quad (5) \]

where \(S(x)\) can be expressed by means of the bottom traction \(T_i(x)\), that is:

\[ S(x) = -T_i(x) \cos \alpha + T_i(x) \sin \alpha. \quad (6a) \]

In a similar fashion, also the normal stress in the inward direction, denoted here by \(P(x)\), may be written as:

\[ P(x) = T_i(x) \sin \alpha + T_i(x) \cos \alpha. \quad (6b) \]

The usual way in which the shear strength is related to the property of the materials in slope stability analysis is by means of the linear empirical law (Graham 1984):

\[ S_{\text{max}}(x) = c'(x) + \sigma'(x) \tan \phi'(x), \quad (7a) \]
\[ \sigma'(x) = P(x) - u(x), \]  
\[ \text{including the cohesive and the frictional terms } c'(x) \text{ and } \tan \phi'(x), \text{ respectively, as well as the effective normal stress } \sigma'(x) \text{ that in turn is} \]
\[ \text{defined as the difference between the normal stress } P(x) \text{ and the pore pressure } u(x). \]

Since what matters most in the mathematical formulation of stability is that \( S_{\text{max}} \) is a linear function of \( P \), the eq. (7a) may be more conveniently written as:
\[ S_{\text{max}}(x) = c'(x) + P(x) \tan \phi'(x), \]  
with:
\[ c'(x) = c(x) - u(x) \tan \phi'(x). \]
The factor of safety \( F \) may be seen as the ratio between the shear strength of the involved material and the shear stress needed for the equilibrium:
\[ F(x) = \frac{S_{\text{max}}(x)}{|S(x)|} \geq 1 \quad x \in [x_i, x_f]. \]  
The inequality in eq. (9) completes the set of conditions that have to be satisfied for the equilibrium of the body \( B \). In view of the previous analysis, a state of equilibrium for the body \( B \) can be defined as a state where the functions \( E(x), X(x) \) and \( A(x) \) depending on the internal stresses satisfy the eqs (2) and (3) and where the stresses \( S(x) \) and \( P(x) \) on the bottom surface \( S_1 \) satisfy the above inequality (eq. 9).

If \( F(x) \) equals unity in a subset \( I_f \) of the interval \( I \), then the corresponding slices are subjected to the maximum shear bottom stresses that they can sustain and are very close to instability. If the bottom stresses are increased here, then the body could react and still maintain the equilibrium by redistributing the extra stresses over the neighbouring slices, with the consequence, however, of reducing the factor of safety outside \( I_f \) and of enlarging the interval \( I_f \), since other slices could reach the extreme condition of shear stresses equalling the shear strength.

This process of stress redistribution can continue until the interval \( I_f \) covers the entire interval \( I \). This is called the limit equilibrium case, and any increment, though infinitesimally small, of \( |S| \) will lead irreversibly the body to the instability. The process of stress redistribution tends to produce uniform distributions of the factor of safety \( F \) over \( I \), smoothing progressively out the differences from slice to slice. The method of limit equilibrium assumes that \( F(x) = F = \text{constant} \) and searches for the equilibrium state of the body \( B \) by replacing the inequality (eq. 9) with the conditions:
\[ FS(x) = S_{\text{max}}(x) \quad x \in [x_i, x_f], \quad F \geq 1. \]  

### 3 The Basic Set of Equations

The 1-D equilibrium equations derived in the previous section can be further manipulated to account explicitly for the effect of external loads on the stability of the body \( B \). In this section, besides the seismic load, we will account also for the additional contribution due to the pressure of the water mass overlying the body, which allows us to deal with slopes that are partially or totally underwater, as occurs for submarine sedimentary bodies or for mountain flanks watered by natural lakes or reservoirs. We assume, therefore, that an external load is imposed on the upper surface \( S_2 \) of the body \( B \) under the form of a water pressure \( D(x) \), which acts normal to \( S_2 \), and that has the expression:
\[ D(x) = \rho_w g(z_w - z_2(x)), \quad z_2(x) < z_w, \]  
\[ D(x) = 0 \quad z_2(x) \geq z_w, \]  
where \( z_w \) is the still water level and \( \rho_w \) is the water density. Hence, after expressing the upper traction \( T_2(x) \) in terms of \( D(x) \) and the bottom traction \( T_1(x) \) in terms of the stresses \( P(x) \) and \( S(x) \), it is straightforward to write the set of the slice stability conditions in the following form:
\[ \frac{d}{dx} E + P \tan \alpha - S - D \tan \beta = -k_H w, \]  
\[ \frac{d}{dx} X + P + S \tan \alpha - D = (1 + k_F)w, \]  
\[ \frac{d}{dx} A - z_1 \frac{d}{dx} E - X - D \tan \beta(z_2 - z_1) = -k_H w \frac{z_2 - z_1}{2}, \]  
\[ FS = c^* + P \tan \phi' \]  
where \( w \) is the weight of the slice per unit horizontal length, that is:
\[ w(x) = \rho(x) g(z_2(x) - z_1(x)), \]  
and where the assumption is made that the density of the body is vertically uniform, that is, \( \rho = \rho(x) \).

The set of differential eq. (12) can be taken as the basic differential system that allows one to analyse the stability of the body \( B \) on the slip surface \( S_1 \). They have to be complemented by the boundary up-slope conditions (see eq. 4) and by further conditions that one can select.

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Figure 2. Slide (light grey) that is used to illustrate all the methods applied in the paper to evaluate the stability. The material parameters $c', \phi'$ and the specific weight $\rho g$ are taken to be constant with respective values: 6 kPa, 25° and 25 kN m$^{-3}$. The variation of $c'(x)$ with $x$ are due to the varying piezometric level (thick dashed line). The bottom surface is taken to be an arc of circumference centred in $C$, which permits the application of methods based on the constraint (eq. A7), such as Bishop's. The body is partially submerged on the right-hand side under a 5-m-deep water basin.

Among different options: one can use the down-slope conditions (eq. 4), or equivalently the integral conditions (A1), (A2) and (A3) derived in the Appendix, with the last one replaceable either by the integral balance (eq. A4) or by the expression (A7) in case of arc-like slip surface. In conclusion, it is a system of three first-order ordinary differential equations plus a further functional relationship (eq. 12d), contemplating five unknown functions defined over the finite domain $[x_i, x_f]$, namely the bottom stresses $P(x)$ and $S(x)$, and the variables associated with the internal stresses $E(x), X(x)$ and $A(x)$. In the basic formulation, the factor of safety $F$ is considered as a further unknown variable that is, however, a parameter and not a function. The model makes use of a very limited number of properties of the constituent material of B, accounting for lateral heterogeneity, since all involved functions are allowed to take different values over the interval $[x_i, x_f]$; these are the specific weight $\rho g(x)$, the effective cohesion $c'(x)$ and the friction angle $\phi'(x)$, as well as the pore pressure $u(x)$ that can be calculated from the local piezometric level. A further unknown of the problem would be the profile of the failure surface $z_1(x)$, but in this paper, for the sake of simplicity, it will be supposed to be given a priori. This is not a critical limitation, since the most common methods examine the stability on a specified family of possible curves, and take as the slide bottom profile the one with the lowest value of the safety factor $F$ (Duncan 1996). Among the given data to the problem are, as usual, the seismic and the water pressure loads, namely the coefficients $k_H$ and $k_V$, and the basin water level $z_w$. Fig. 2 shows a theoretical slide that will be considered throughout the rest of the paper as a case to illustrate methods and concepts of the analysis of stability. On purpose we did not select a special case, and the body has a rather ordinary geometry with an average slope of about 30° and an arc-like sliding surface. The body is homogeneous, is delimited by a piecewise upper profile, and is partially submerged. A piecewise line also describes the piezometric level.

3.1 Non-uniqueness of the solution

Since the problem includes four equations and the unknowns are five functions and the parameter $F$, it is intrinsically underdetermined and the solution cannot be unique. This observation is trivial and was dealt with since the first applications of the limit equilibrium method, but the problem of non-uniqueness and of its consequences on the analysis of slope stability has been explored only partially. The point of view that is most widely accepted is that the lack of uniqueness is not crucial, since it has only little influence on the value of the safety factor, which is practically the only unknown relevant for making some sensible statements on slope stability. This paper will show that this is not true, and will propose some further conditions to limit the arbitrariness of the solution within a reasonably acceptable range of possibilities. In the course of the discussion it will be convenient to make reference to some of the most conventional methods that were proposed in the literature to evaluate $F$.

3.1.1 The Fellenius method in a new view-point

The Fellenius or ordinary method (Fellenius 1927) was the first attempt to evaluate the safety factor. In spite of the theoretical existence of an infinite set of solutions because the problem is underdetermined, this method does not provide an exact solution, but only a crude
approximation. Its importance resides in the fact that it was used as a reference by all later approaches. It supposes that the stresses in the slide are essentially lithostatic, which in terms of the stress tensor components of the 2-D formulation entails that \( \sigma_{xx} = \sigma_{zz} = 0 \) and that \( \sigma_{xz}(x, z) = \rho(x, z) g(z/(z(x) - z)) \). In terms of the 1-D space this is equivalent to set \( E(x) = 0, X(x) = 0, A(x) = 0 \) for any points of the slope, that is, for \( \forall x \in [x_1, x_2] \).

These assumptions satisfy the boundary conditions (eq. 4), but lead to some inconsistencies. In the first place, if we introduce them into eq. (12c), we see that the equation cannot be fulfilled for arbitrary water and horizontal seismic loads, but only if these loads respect the special almost unrealistic balance \( 2D(x) \tan \beta(x) = k_H w(x) \) for \( \forall x \in [x_1, x_2] \).

Furthermore, if we apply the lithostatic positions coherently also to the eqs (12a) and (12b) and if we solve for the remaining unknowns \( P(x) \) and \( S(x) \), we find easily that:

\[
P(x) = \left[ (1 + k_V)w(x) + D(x) \right] \cos \alpha(x) + (D(x) \tan \beta(x) - k_H w(x)) \sin \alpha(x) \cos \alpha(x),
\]

\[
S(x) = \left[ (1 + k_V)w(x) + D(x) \right] \sin \alpha(x) - (D(x) \tan \beta(x) - k_H w(x)) \cos \alpha(x) \cos \alpha(x).
\]

Eventually, with the aid of the limit equilibrium relation (12d), the value of \( F \) can be deduced as:

\[
F_O = \frac{c'(x) + \left[ (1 + k_V)w(x) + D(x) \right] \cos \alpha(x) + (D(x) \tan \beta(x) - k_H w(x)) \sin \alpha(x) \tan \phi'(x)}{\left[ (1 + k_V)w(x) + D(x) \right] \sin \alpha(x) - (D(x) \tan \beta(x) + k_H w(x)) \cos \alpha(x) \cos \alpha(x)},
\]

where the subscript \( O \) is used here to denote that the safety factor is the result of the ordinary method. Since \( F \) has to be constant over the interval \([x_1, x_2]\), the above expression (14c) can be a solution to the problem only

1. if the material properties of the slide \( c'(x) \) and \( \phi'(x) \) do not change with \( x \),
2. if the weight \( w(x) \) does not depend on \( x \), which could occur for a uniform-thickness slide,
3. if \( D(x) \) vanishes everywhere and
4. if the angle \( \alpha(x) \) does not change with \( x \) as well. The case corresponding to all these conditions is treated in the literature as the case of a uniform homogeneous layer over an incline of infinite length.

In order to find a constant value of the safety factor also when the bottom profile is circular with radius \( R \) and centre in the point \((x_0, z_0)\), the ordinary method abandons the idea of inner consistency: it accepts the normal stress as given in eq. (A3). Bishop (1955) supposed that the vertical component of the interslice force be zero \( X(x) = 0 \) and developed a simplified method by making use of the limit equilibrium condition (eq. 12d), of the vertical equilibrium eq. (12b) and of the integral balance of the torque (eq. A7). Janbu (1968) introduced a simplified method following the Bishop scheme, but he derived the expression for \( F \) no more from the eq. (A7), but from the integral form of horizontal equilibrium (eq. A1). We applied both methods in the stability analysis of the slope of Fig. 2, which is partitioned in 50 slices. From our analysis it is clear that neither the simplified Bishop’s nor the simplified Janbu’s approach can provide an exact solution to the problem. Both methods will satisfy the boundary conditions (eq. 4) for \( X \), as well as the up-slope boundary conditions for \( E \) and \( A \). As to the rest, it has to be noticed that Bishop’s results can fulfil neither \( E(x_0) = 0, A(x_0) = 0 \) because it is based only on the expression (A7) and, in particular, not on eqs (A1) and (A3). On the other hand, Janbu’s solution satisfies \( E(x_0) = 0 \), due to the application of the constraint (eq. A1), but cannot match the further requirement \( A(x_0) = 0 \) related to eq. (A3). This is shown in Fig. 3 where the curves \( E(x) \) and \( A(x) \) resulting from Bishop’s and Janbu’s simplified methods are plotted together to favour the comparison.

3.1.2 Other traditional methods

All the traditional methods used in the limit-equilibrium approach consider the safety factor \( F \) as an unknown parameter, and make the attempt to overcome the non-uniqueness of the solution by making assumptions about the interslice forces. In the Fellenius method the interslice forces are taken to be zero, although it is found that it does not lead to an exact solution to the problem. Bishop (1955) supposed that the vertical component of the interslice force be zero \( X(x) = 0 \) and developed a simplified method by making use of the limit equilibrium condition (eq. 12d), of the vertical equilibrium eq. (12b) and of the integral balance of the torque (eq. A7). Janbu (1968) introduced a simplified method following the Bishop scheme, but he derived the expression for \( F \) no more from the eq. (A7), but from the integral form of horizontal equilibrium (eq. A1). We applied both methods in the stability analysis of the slope of Fig. 2, which is partitioned in 50 slices. From our analysis it is clear that neither the simplified Bishop’s nor the simplified Janbu’s approach can provide an exact solution to the problem. Both methods will satisfy the boundary conditions (eq. 4) for \( X \), as well as the up-slope boundary conditions for \( E \) and \( A \). As to the rest, it has to be noticed that Bishop’s results can fulfil neither \( E(x_0) = 0 \) nor \( A(x_0) = 0 \) because it is based only on the expression (A7) and, in particular, not on eqs (A1) and (A3). On the other hand, Janbu’s solution satisfies \( E(x_0) = 0 \), due to the application of the constraint (eq. A1), but cannot match the further requirement \( A(x_0) = 0 \) related to eq. (A3). This is shown in Fig. 3 where the curves \( E(x) \) and \( A(x) \) resulting from Bishop’s and Janbu’s simplified methods are plotted together to favour the comparison.
Figure 3. Simplified Bishop’s and Janbu’s methods applied to the slide of Fig. 2 with no seismic and no water loads. Notice that all curves are zero at the up-slope extreme $x = x_i$, but at the down-slope body end $x = x_f$, only $E$ corresponding to the Janbu’s solution vanishes, as explained in the text.

Figure 4. Curves of the interslice forces $E$ and $X$ and of the torque $A$ for selected computation cycles indicated by the index $m$. After $m = 10$ cycles the calculations were found to converge and the corresponding curves are, therefore, to be considered as the result of the generalized Janbu’s approach.

Generalization of Bishop’s and Janbu’s methods were developed by the introduction of a law linking the vertical and horizontal components of the interslice forces, such as:

$$X(x) = -\tan \theta E(x), \quad (16)$$

where $\theta$ is a given parameter. These methods are more complicated compared to the simplified forms, since they need an iterative computing scheme: initially the vertical component of interslice force $X(x)$ is set to be equal to zero, and this is used to compute $F$ and all the other unknowns of the problems by adopting either the Bishop’s or the Janbu’s scheme described above. Then a new shape for $X(x)$ is obtained from eq. (16), and this allows one to iterate the procedure which is usually found to converge in a few cycles. Figs 4 and 5 show curves for the interslice forces $E(x)$ and $X(x)$ and for the torque $A(x)$, respectively, obtained after successive iterations of the Janbu’s and Bishop’s approaches. It is shown clearly that the curves satisfy the boundary conditions at $x = x_i$ and that at any iteration step they come closer and closer to respecting the conditions at $x = x_f$. It is stressed that these methods provide only approximations of the exact solution to the problem. In order to obtain exact solutions, Spencer (1967) and Morgenstern & Price (1965) assumed a relationship between the internal forces that is a generalization of the expression (16). Spencer takes the same expression (16), but considers $\theta$ as an unknown, while Morgenstern & Price assume the law:

$$X(x) = -\tan \theta f(x) E(x), \quad (17a)$$

where $\theta$ is unknown and where $f(x)$ is a given function such that

$$f(x_i) = f(x_f) = 0. \quad (17b)$$

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and has a trapezoidal or of a half-sinus shape. They considered as a good solution the one that satisfies both Bishop’s and Janbu’s generalized methods. Both Spencer’s and Morgenstern & Price’s approaches enable one to find an exact solution to the problem, which satisfies all the boundary conditions, but this does not imply that the problem has a unique solution nor that the solution found is the best one.

4 THE PRINCIPLE OF MINIMUM LITHOSTATIC DEVIATION

Since we expect that the stability problem has no unique solution, we also expect that there are different exact solutions corresponding to different values of the factor of safety $F$. If, irrespective of the solving method adopted, all the safety factors were found to range within a restricted interval of values, say $F_1$ and $F_2$ (with $\Delta F = F_2 - F_1$, and $\Delta F/F_1 \ll 1$), then the non-uniqueness could be accepted as a marginal problem and any value $F$ between the lower and upper limits, for instance the mean value $(F_1 + F_2)/2$ could be taken as a representative value for $F$ and used to make judgements on the stability conditions of the slope. Indeed, under these circumstances in most cases one could have a sensible criterion to determine the stability of the slope. If the lower limit $F_1$ falls above 1, the slope is stable. Conversely, if the upper limit $F_2$ turns out to be smaller than 1, the slope is unstable. The third possible case where $F_1$ and $F_2$ falls below and above 1, respectively, leads to the impossibility of expressing any judgements, but could be a rare and tolerable occurrence. Therefore, the spread in the resulting values of $F$ could be taken as a sort of uncertainty in the knowledge of $F$ and accepted with the same spirit as one accepts uncertainties in the knowledge of all the physical and geometrical parameters of the slope under study. Unfortunately, in the next section we will show that exact solutions usually exist corresponding to very different values of $F$. We cannot decide on the stability of the slope. This point of view is a basic issue for all the traditional methods introduced in the literature and is

5 USING THE FACTOR OF SAFETY AS A FREE PARAMETER

In the conventional approaches of the limit equilibrium analysis the factor of safety is the most important unknown since its value provides the means to decide on the stability of the slope. This point of view is a basic issue for all the traditional methods introduced in the literature and is
widely used in practice. The MLD criterion introduced in this paper poses less emphasis on $F$ and more emphasis on the lithostatic deviation $\delta$, and this can suggest exploring a totally different strategy to tackle the problem, namely (1) assuming $F$ as a free parameter ranging within two given limits $F_1$ and $F_2$; and then (2) computing $\delta$ as a function of $F$. The application of the MLD criterion tells us that the value of $F$ that has to be used in expressing the judgement on the stability of the body is the one that is found in association with the minimum value of $\delta$. This strategy is shown here by means of an illustrative example that has also the purpose to show that $F$ can vary over a very large range of values. Let us pose that $X$ can be expressed as:

$$X(x, \lambda_2, \lambda_3; \lambda_1) = \sum_{k=1}^{3} \lambda_k \sin \left[ kL^{-1} \pi (x - x_i) \right],$$

(19)

where $\lambda_2$ and $\lambda_3$ are unknown parameters, while $\lambda_1$ is a free parameter, and $L = x_f - x_i$ is the horizontal length of the slide. It may be observed that $X$ is a Fourier sine expansion truncated to the third term with the last two coefficients that are unknown. Observe further that the boundary conditions for $X(x)$ are automatically satisfied since all functions vanish at $x = x_i$ and $x = x_f$. The unknown parameters are determined by imposing the down-slope conditions $E(x_f) = A(x_f) = 0$. The expression for $P$ can be derived on combining the vertical equilibrium eq. (12b) with the limit equilibrium eq. (12d), that is:

$$P(x, \lambda_2, \lambda_3; F, \lambda_1) = \left[ (1 + k_f)u(x) - \frac{d}{dx}X(x, \lambda_2, \lambda_3; \lambda_1) \right] - c^*(x) \tan \alpha(x)/F + D(x),$$

where both $F$ and $\lambda_1$ are assumed to be known parameters given a priori. The derivative of $X$ to be used in this formula can be easily obtained by differentiating the definition (19). Furthermore, by making use of the eqs (12a) and (12d), after some manipulations the expression for the horizontal forces can be derived:

$$E(x, \lambda_2, \lambda_3; F, \lambda_1) = \frac{1}{F}G(x; F) + \pi L^{-1} \sum_{k=1}^{3} \lambda_k \int_{x_i}^{x} H(x'; F) \cos \left[ \frac{k\pi (x' - x)}{L} \right] dx',$n

(21)

where:

$$G(x; F) = \int_{x_i}^{x} g(x'; F) dx',$$

(22a)

$$g(x; F) = c^*(x)[1 + H(x; F) \tan \alpha] - Fw(x)(1 + k_f)H(x; F) + k_d + FD(x)[\tan \beta - H(x; F)]$$

(22b)

and

$$H(x; F) = \frac{\tan \alpha(x) - F^{-1} \tan \phi(x)}{1 + F^{-1} \tan \phi(x) \tan \alpha(x)}.$$  

(22c)

Observe that $H(x; F)$ is a known function of $x$, since $F$ is prescribed a priori, which implies that also $g(x; F)$, $G(x; F)$ and all the integrals in the formula (21) can be computed numerically a priori. Therefore, as expected, everything is known in the expression for $E$, except the numerical coefficients $\lambda_2$ and $\lambda_3$. The expression for $A$ can be derived with the aid of the eq. (4) and results to be:

$$A(x, \lambda_2, \lambda_3; F, \lambda_1) = \pi L^{-1} \left\{ \sum_{k=1}^{3} k^{-1} \lambda_k \left[ 1 - \cos (k\pi L^{-1}(x - x_i)) \right] \right\} +$$

$$+ \frac{1}{F} \int_{x_i}^{x} z_1(x')g(x'; F) dx' + \int_{x_i}^{x} z_1(x')H(x'; F) \frac{d}{dx'}X(x', \lambda_2, \lambda_3; \lambda_1) dx' +$$

$$+ \int_{x_i}^{x} D(x') \tan \beta(x') \left( z_2(x') - z_1(x') \right) dx' - k_d \int_{x_i}^{x} w(x) \frac{z_1(x') - z_2(x')}{2} dx',$$  

(23)

where the integration constants are adjusted in such a way to match the up-slope vanishing condition for $A$. Imposing the down-slope boundary conditions for $E(x)$ and for $A(x)$ leads us to an algebraic system of two equations in the two unknowns $\lambda_2$ and $\lambda_3$ that can be easily solved, with solutions depending both on $\lambda_1$ and $F$, namely $\lambda_2^*(F, \lambda_1)$ and $\lambda_3^*(F, \lambda_1)$. By making use of these values, the functions $E$ and $A$ can be computed through the above expressions (21) and (23), the function $X$ can be calculated from its definition (19), which further allows the calculation of the bottom stresses $P$ with the aid of (20) and $S$ via the limit equilibrium eq. (12d).

This approach has been used to run experiments for three cases all concerning the body depicted in Fig. 2, but with no piezometric line (i.e. we assume $u(x) = 0$ in the eq. (8b) or equivalently $c^*(x) = c(x)$); in case 1 no seismic and water loads are applied, in case 2 a modest water load is applied down-slope (see Fig. 2) and in case 3 a rather strong seismic load is considered ($k_d = 0.27, k_f = 0.25$). The free parameter $\lambda_1$ has been assumed to range in a symmetric interval $I_1$ around zero, and the safety factor $F$ has been varied in the interval $I_F = [0.7, 1.6]$, which is rather large and encompasses the critical value 1. The lithostatic deviation $\delta$ associated with each solution depends on $\lambda_1$ and $F$. Since here most interest focuses on the dependence on $F$, for any given value of $F$ we have computed the minimum value of $\delta$ within $I_1$ that may be written as $\delta^*(F) = \min[\delta(F, \lambda_1); \lambda_1 \in I_1]$, and have plotted it in the first graph of Fig. 6.

The first consideration is that a solution to the basic set of limit equilibrium equations can be computed for all the values of $F$ specified within the selected interval $I_F$, that is even for values of the factor of safety that are very small (0.7) or rather high (1.6). This outcome is not exceptional since we have found it in many other runs handling with conventional slopes. On the basis of our experience we can state that all
slopes admit solutions both with $F$ much smaller and much larger than 1. This is dramatic for the traditional approach of the limit equilibrium method, since it has to face a basic theoretical ambiguity with no means to decide on the stability of slopes, even though reaching such a decision is precisely the goal for which it was conceived. And this is the main reason that favours the introduction of an additional principle such as the MLD criterion proposed here. The second observation is that $\delta^*$ is rather sensitive to changes in $F$ and that all the plotted curves exhibit well-defined minima. The values of $F$ corresponding to these minima are taken as the safety factors resulting from the analysis. They are $F_1 = 1.409$, $F_2 = 1.510$ and $F_3 = 0.925$ for the respective cases 1, 2 and 3. Observe that case 2 is slightly more stable than case 1, and that case 3 is undoubtedly unstable with $F$ well below 1. Table 1 summarizes the results for all cases. In the Table, also the results of the traditional methods illustrated in Section 3 are shown. Comparing the safety factors obtained by applying the MLD principle with the ones

Table 1. Summary of the results.

<table>
<thead>
<tr>
<th>Method</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F$</td>
<td>$\delta$</td>
<td>$F$</td>
</tr>
<tr>
<td>Ordinary(1)</td>
<td>1.276</td>
<td>—</td>
<td>1.356</td>
</tr>
<tr>
<td>Simplified Bishop(1)</td>
<td>1.469</td>
<td>0.1043</td>
<td>1.579</td>
</tr>
<tr>
<td>Simplified Janbu(1)</td>
<td>1.276</td>
<td>0.0983</td>
<td>1.363</td>
</tr>
<tr>
<td>Spencer(3)</td>
<td>1.468(2)</td>
<td>0.1052</td>
<td>1.579(3)</td>
</tr>
<tr>
<td>Morgenstern &amp; Price</td>
<td>1.470(3)</td>
<td>0.1082</td>
<td>1.579(3)</td>
</tr>
<tr>
<td>Section 5(4)</td>
<td>1.409(5)</td>
<td>0.0772</td>
<td>1.510(5)</td>
</tr>
</tbody>
</table>

(1) These methods do not provide exact solutions to the limit equilibrium problem.
(2) The resulting value of the unknown parameter $\theta$ are: $\theta_1 = 18.18^\circ$, $\theta_2 = 16.49^\circ$, $\theta_3 = 27.66^\circ$. Subscripts 1, 2 and 3 refer to cases 1, 2 and 3, respectively.
(3) The resulting value of the unknown parameter $\theta$ are: $\theta_1 = 23.00^\circ$, $\theta_2 = 21.21^\circ$, $\theta_3 = 34.94^\circ$.
(4) This method applies the MLD criterion.
(5) The unknown parameters are $\lambda_1(\lambda_{1min})$ and $\lambda_2(\lambda_{1min})$. The results are:
for case 1: $\lambda_{1min} = 0.379$, $\lambda_2(\lambda_{1min}) = 2.726$, $\lambda_3(\lambda_{1min}) = 0.929$.
for case 2: $\lambda_{1min} = 0.320$, $\lambda_2(\lambda_{1min}) = 3.077$, $\lambda_3(\lambda_{1min}) = 0.876$.
for case 3: $\lambda_{1min} = 2.135$, $\lambda_2(\lambda_{1min}) = 4.329$, $\lambda_3(\lambda_{1min}) = 2.272$.
All these variables have the dimension of a force per unit length and are given in units of MNm$^{-1}$. The resulting $F(\lambda_{1min})$ are shown in the Table.

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computed through Spencer’s and Morgenstern & Price’s methods (namely with the other methods providing exact solutions), one sees that safety factors we found are all between 4–6 per cent smaller.

The examples treated here have been used only with the goal to prove that the space of the solutions to our problem comprehends solutions with very different values of \( F \), which implies the need to use an additional law to identify the most appropriate factor of safety for the body under analysis. It is not our purpose to state that the position (eq. 19) is more physically sound than other positions introduced in the previous sections. Consider that the position (eq. 19) itself could be easily generalized by truncating the Fourier sine series to an order higher than 3, which would entail the introduction of a free parameter vector instead of a single parameter, and hence to explore a larger set of solutions. However, we limited us to the simple form (eq. 19) since it was sufficient to point out clearly the large variability of the safety factor \( F \).

6 CONCLUSIONS

The limit equilibrium method has a long tradition of application to the analysis of the stability of slopes that is a problem, which has a great relevance in various fields of geosciences from investigations on larger scale (unstable submarine shelf slopes or volcanic lateral collapses) down to smaller scale geoenvironmental studies. Starting from the general static equilibrium equations of a body we have first restricted the attention to a problem in a 2-D space \((x, z)\) considering bodies that are uniform along the transversal direction \(y\), and then after cutting the body into vertical slices, we have derived the basic set of the equilibrium equations with the addition of the limit equilibrium condition. The requirements one has to pose at the boundaries \(x = x_l\) and \(x = x_f\) have been given in the elementary form (eq. 4) or in the equivalent integral form (eqs A1–A3) that have general validity, or in the special form (eq. A7), holding for arc-like sliding surfaces. This formulation of the problem, consisting of three first-order ordinary differential equations, one functional relationship and six boundary conditions, contains five unknown functions of the space variable \(x\) (namely \(E(x)\), \(X(x)\), \(A(x)\) and the bottom stress functions \(P(x)\) and \(S(x)\)) and the unknown parameter \(F\). Therefore, the problem is essentially underdetermined and admits an infinite number of solutions. All the traditional methods of the limit equilibrium theory consider that the safety factor \(F\) is an unknown parameter and that the main goal of any solving procedure is to determine a proper value for \(F\). This emphasis on the search for \(F\) derives from the earlier approaches of stability studies. As a theoretical remedy, the MLD principle was introduced to contrast the large variability of \(F\) that was shown to be inherent in the limit equilibrium model. We proved that the MLD principle can constitute a pivot by means of which the approach to the stability problem can be revolutionized. Instead of treating \(F\) as an unknown parameter, we can use it as a free parameter.

Our application to different cases lead us to the same conclusion that can be summarized as follows: if use is made of a law containing a convenient number of unknown parameters, a solution can be found for any value of \(F\) given within very broad limits that are generally well below and well above 1. This statement has a strong value: the concept of the factor of safety alone cannot be used to establish the stability of a body, but some new concept such as the MLD criterion must be incorporated in the analysis. We suggest a new method to compute the safety factor for a slope consisting of the following steps:

1. treating \(F\) as a free parameter;
2. assuming a law such as eq. (19) with one or more additional free parameters;
3. solving the system of the equilibrium equations;
4. finding the minimum of the lithostatic deviation \(\delta\) and
5. finding the value of \(F\) corresponding to such a minimum.

This method is free from recursive schemes: this is an advantage since sometimes they do not converge, and next, the computation is less complex and needs shorter computation time. We notice further that there are no simple means to find the absolute minimum of \(\delta\), and that the search is restricted within the class of the solutions that we can compute, once a given law such as eq. (19) has been initially assumed. This means that changing the initial assumptions may lead to different values for \(\delta\), and consequently for the corresponding \(F\). There is no general antidote to this drawback, except the one consisting in performing a large number of experiments with various initial assumptions in order to test the largest possible class of solutions. For example, one truncates the series expansion like eq. (19) at various orders and finds the MLD corresponding to all the set of experiments. The larger is the number of tests, the more reliable the found minimum value of \(\delta\) and the computed value of the safety factor will be. Notice that, from a practical point of view, running many experiments is not at all a problem, since the procedure can be easily implemented on a common PC and the required memory allocation and computing time are both very low.

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A more elaborated further condition can be also derived. Let us first multiply both members of the equilibrium eqs (12a) and (12b) by \((z_1(x) - z_0)\) and \((x_0 - x)\), respectively, where \(x_0\) and \(z_0\) are constants. Let us then apply the integration over \([x_i, x_f]\) to the so-transformed...
horizontal and vertical equilibrium equations and further to the eq. (12c), and then sum up together. The result is:

\[ \int_{x_i}^{x_f} \left\{ \left( \frac{d}{dx} E \right) + [P \tan \alpha - S - D \tan \beta] + kHw \right\} (z_1 - z_0) \, dx - \int_{x_i}^{x_f} \left\{ \left( \frac{d}{dx} x \right) + [P + S \tan \alpha - D] \right\} (x - x_0) \, dx + \]

\[ + \int_{x_i}^{x_f} \left[ \frac{d}{dx} A - z_1 \frac{d}{dx} E - X - Y D \tan \beta (z_2 - z_1) + kHw \frac{z_2 - z_1}{2} \right] dx \]

\[ = (1 + k_V) \int_{x_i}^{x_f} w(x_0 - x) \, dx \]

that after some calculations and using the boundary conditions (eq. 4) can be written as:

\[ \int_{x_i}^{x_f} [(P \tan \alpha - S - D \tan \beta)(z_1 - z_0) - [P + S \tan \alpha - D](x - x_0) - D \tan \beta (z_2 - z_1)] \, dx + \]

\[ + \int_{x_i}^{x_f} kHw \frac{z_1 + z_2 - 2z_0}{2} \, dx = (1 + k_V)W(x_0 - x_B) \quad (A4) \]

Here \( x_B \) is the horizontal coordinate of the centre of mass of the body. Now, the eqs (A1) and (A2) can be interpreted as the balance of the horizontal and vertical momentum for the whole body B. In a similar fashion, the eq. (A4) can be viewed as the balance among the torques exerted on B by all the active forces, that is, the bottom shear and normal forces, the gravity force, the external water load and the seismic load, the torques being computed with respect to the pole of rotation \((x_0, z_0)\). In the special case where the profile \( z_1(x) \) of the bottom surface is a circumference arc of radius \( R \), then a simplified version of the eq. (A4) can be obtained. In fact, on making use of the equalities:

\[ z_0 - z_1 = R \cos \alpha \quad (A5) \]

\[ x_0 - x = R \sin \alpha \quad (A6) \]

we get:

\[ \int_{x_i}^{x_f} S \sec \alpha \, dx = \int_{x_i}^{x_f} [(1 + k_V)w + D] \sin \alpha \, dx + \frac{1}{R} \int_{x_i}^{x_f} D \tan \beta (z_2 - z_0) \, dx - \frac{kH}{R} \int_{x_i}^{x_f} w \frac{z_2 - z_0}{2} \, dx + \]

\[ + \frac{kH}{2} \int_{x_i}^{x_f} w \cos \alpha \, dx \quad (A7) \]

This form of the torque balance has the advantage of including only the shear stresses \( S(x) \) as unknown function, which can be exploited in some solving procedures.