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Dispersion of Pollutants in Semi-Infinite Porous Media with Unsteady Velocity Distribution

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Analytical solutions are developed for the dispersion problem in nonadsorbing and adsorbing, semi-infinite porous media in which the flow is one dimensional and the average flow velocity is unsteady. The expression that states the direct relationship of the dispersion coefficient with seepage velocity is used to solve the unsteady dispersion flow problem by introducing a new time variable. The source concentration of the pollutant varies exponentially with time. The variation in seepage velocity with time is considered because of resistance in the flow. The graphical solutions are also obtained for a set of data assumed.

Introduction

Generally the mizing of miscible fluids as they flow through porous media is referred to as hydrodynamical dispersion. With respect to the importance of dispersion process studies in water quality management and pollution control, the dispersion has been referred to as a hydraulic mixing process by which the waste concentrations are attenuated while the waste pollutants are being transported downstream. There are large occasions when waste pollutants from industrial plants, urban areas and other operations reach a natural water course. An ever increasing pressure on the waste assimilating capacity of our water resources does make it more difficult to ensure that concentrations of various contaminants will remain below the limits dictated by a variety of competing uses. Interest in dispersion in porous media has also resulted from seawater intrusions into coastal acquifers seepage from canals and streams into and through acquifers, the deliberate release of herbicides into canals to kill weeds.

These problems of dispersion have been receiving considerable attention from chemical, environmental, petroleum engineers, hydrologists, mathematicians and soil scientists. Most of the works reveal common assumptions of homogeneous porous media with constant porosity, steady seepage flow velocity and constant dispersion coefficient. For such assumptions Ebach and White (1958) studied the longitudinal dispersion problem for an input concentration that varies periodically with time and Ogata Banks (1961) for a constant input concentration. Hoopes and Harleman (1965) studied the problem of dispersion in radial flow from a well fully penetrating, homogeneous, isotropic non-adsorbing confined acquifer. Bruch and Street (1966) considered both longitudinal and lateral dispersion with in semi-infinite nonadsorbing porous media in a steady unidirectional flow field for a constant input concentration. Marino (1974) considered the input concentration varying exponentially with time. Al-Niami and Rushton (1977) and Marino (1978) studied the analysis of flow against dispersion in porous media.

The practical problems involving dispersion of solutes are that the media are seldom homogeneous and seepage flow velocity is unsteady. This fact led Shamir and Harleman (1967) to consider the solute dispersion in layered porous media. Their work appears to be the first investigation taking into account the nonhomogeneity of the medium porosity. Lin (1977) proposed several porosity equations with decreasing or increasing porosity. He developed analytical and numerical procedures for predicting the solute dispersion in a porous medium with porosity varying with distance and constant volumetric seepage flow rate. Basak (1978) presented an analytical solution to the problem of evaporation from a horizontal soil column in which diffusivity increases linearly with moisture content and also to a problem of concentration dependent diffusion with decreasing concentration at the source. Banks and Jerasate (1962) presented his paper for dispersion phenomena for unsteady porous media flow, the porous media being homogeneous for a constant input concentration introduced at any position. After the general attack upon dispersion problems with perturbation methods for the first time by G. Dagan (1971) limited to steady flow inhomogeneous acquifers, and some other investigations, Hurt (1978) applied the perturbation methods to longitudinal and lateral dispersion in nonuniform seepage flow through heterogeneous acquifers. Wang (1978) discussed the concentration distribution of a pollutant arising from a instantaneous point source in a two dimensional water channel with non-uniform velocity distribution. He employed Gill's method to solve the convective – diffusion equation. Kumar (1983) considered the unsteady flow against dispersion in porous media.

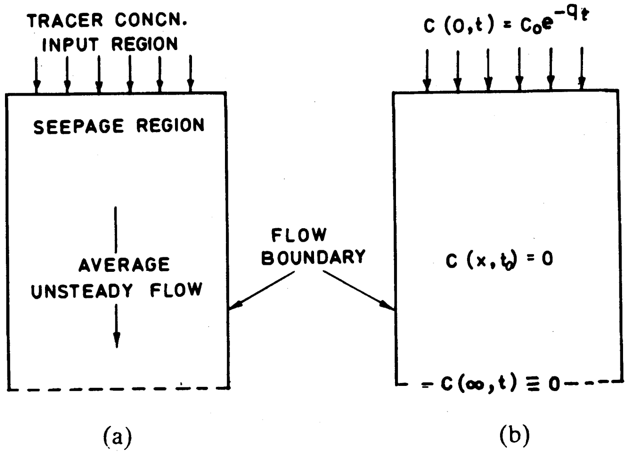

Fig. 1. Schematic illustration of the dispersion problem. (a) flow condition (b) boundary and initial concentration conditions.

The present paper presents an analytical solution to a simplified dispersion problem in nonadsorbing and adsorbing, homogeneous porous media in which the average flow velocity is unsteady for the input concentration of contaminants varying exponentially with time, Fig. 1.

Mathematical Formulation and Solution

The theory that follows is confined to dispersion in unidirectional seepage flow through semi-infinite homogeneous porous media. The seepage flow velocity is assumed unsteady. The dispersion systems to be considered are subject to an input concentration of contaminants that vary exponentially with time. The porous medium is first considered as nonadsorbing and then as adsorbing. To obtain the solutions to a class of unsteady flow problems, the direct relationship between the longitudinal dispersion coefficient and fluid velocity is used. A new time scale is further introduced.

PART I: Non-adsorbing Porous Medium

Case 1 – Consider the input concentration is C_o exp ($-qt$), C_o is a reference concentration and q is a parameter of unit (sec.)$^{-1}$.

The governing partial differential equation for longitudinal hydrodynamic dispersion with in a semi-infinite nonadsorbing porous medium in a unidirectional flow field is

$$D \frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} \quad (1)$$

in which D is the longitudinal dispersion coefficient, C is the average cross-sectional concentration, u is the unsteady seepage velocity, x is a coordinate parallel to flow and t is time.

The initial and boundary conditions are

$$C(x,0) = 0 \qquad x \geq 0 \quad (2)$$

$$C(0,t) = C_0 \exp(-qt) \qquad t > 0 \quad (3)$$

$$C(\infty,t) = 0 \qquad t \geq 0 \quad (4)$$

Ebach and White (1958) have shown for a broad range of Reynold's number, as for $R < 100$, that

$$\frac{D\rho}{\mu} \equiv a_0 \left(\frac{ud\rho}{\mu}\right)^{1.06} \quad (5)$$

where d is the particle size of porous medium, a_o is a dimensionless number, ρ is the density and μ is the dynamic viscosity. The exponent in Eq. (5) may be assumed as unity, then

$$D \equiv a_0 du \quad (6)$$

Eq. (1) becomes

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} \equiv (a_0 du) \frac{\partial^2 C}{\partial x^2} \quad (7)$$

Multiplying this equation by (u/u_o), where u_o is the initial seepage velocity, we get

$$\frac{1}{V}\frac{\partial C}{\partial t} + u_0 \frac{\partial C}{\partial x} = D_0 \frac{\partial^2 C}{\partial x^2} \quad (8)$$

where V stands for (u/u_o) and D_o for $(a_o\, du_o)$. Now by introducing a new time variable T by the transformation, Crank (1975)

$$T = \int_0^t V dt \quad (9)$$

Eq. (8) takes the form

$$\frac{\partial C}{\partial T} + u_0 \frac{\partial C}{\partial x} = D_0 \frac{\partial^2 C}{\partial x^2} \quad (10)$$

V may have a linear or exponential form as

$$V = 1 - mt \tag{11}$$

or

$$V = \exp(-mt) \tag{12}$$

The new time variable will be

$$T = t - \frac{1}{2} mt^2 \tag{13}$$

or

$$T = \frac{1}{m} [1 - \exp(-mt)] \tag{14}$$

where m is the flow resistance coefficient $(\text{sec.})^{-1}$. Eq. (12) is valid only for mt less than 1.0. Considering the exponential form for V, Eqs. (2) – (4) will become

$$C(x,0) = 0 \qquad x \geq 0 \tag{15}$$

$$C(0,T) = C_0(1-mT)^{q/m} \qquad T > 0 \tag{16}$$

$$C(\infty,T) = 0 \qquad T \geq 0 \tag{17}$$

To reduce the convective term in Eq. (10) we use the transformation

$$C(x,T) = K(x,t) \exp\left[\frac{u_0 x}{2D_0} - \frac{u_0^2 T}{4D_0}\right] \tag{18}$$

Eq. (10) assumes the form

$$\frac{\partial K}{\partial T} = D_0 \frac{\partial^2 K}{\partial x^2} \tag{19}$$

and Eqs. (15) – (17) take the form

$$K(x,0) = 0 \tag{20}$$

$$K(0,T) = C_0 (1-mT)^{q/m} \exp\left(\frac{u_0^2 T}{4D_0}\right) \tag{21}$$

$$K(\infty,T) = 0 \tag{22}$$

Applying Laplace Transformation with respect to T on the preceding boundary value problem using Eq. (20) yields

$$\frac{d^2 \overline{K}}{dx^2} = \frac{p}{D_0} \overline{K} \tag{23}$$

$$\overline{K}(\infty,p) = 0 \tag{24}$$

$$\overline{K}(0,p) = C_0 \int_0^\infty (1-mT)^{q/m} e^{-PT} dT \tag{25}$$

where

$$P = \left(p - \frac{u_0^2}{4D_0} \right)$$

and

$$\overline{K}(x,p) = \int_0^\infty K(x,T) e^{-pT} dT$$

p is a parameter.

As mt is less than 1.0, the value of m will be such as to satisfy also $mT < 1.0$. So for $q < 1.0$, we can take $(1\text{-}mT)^{q/m} = 1\text{-}qT$, neglecting other terms. So Eq. (25) yields

$$\overline{K}(0,p) = \frac{C_0}{P} - \frac{qC_0}{P^2} \tag{26}$$

so a general solution will be obtained as

$$\overline{K}(x,p) = C_0\left(\frac{1}{P} - \frac{q}{P^2}\right) \exp\left(-x\sqrt{\frac{P}{D_0}}\right) \tag{27}$$

Taking the Inverse Laplace Transformation of Eq. (27), and substituting the required values and using Eq. (18); we get the solution

$$C(x,t) = \frac{C_0}{2}\left[\left\{1 + q\sqrt{\frac{4D_0T}{u_0^2}}\left(\frac{x}{2\sqrt{D_0T}} - \sqrt{\frac{u_0^2T}{4D_0}}\right)\right\} \operatorname{erfc}\left(\frac{x}{2\sqrt{D_0T}} - \sqrt{\frac{u_0^2T}{4D_0}}\right) + \left\{1 - q\sqrt{\frac{4D_0T}{u_0^2}}\left(\frac{x}{2\sqrt{D_0T}} + \sqrt{\frac{u_0^2T}{4D_0}}\right)\right\} \cdot \exp\left(\frac{u_0x}{D_0}\right) \operatorname{erfc}\left(\frac{x}{2\sqrt{D_0T}} + \sqrt{\frac{u_0^2T}{4D_0}}\right)\right] \tag{28}$$

Case II – When the input concentration of the contaminants at $x = 0$ is C_o (1-exp (-qt)), the hydrodynamical dispersion problem is

$$D\frac{\partial^2 C}{\partial x^2} = u\frac{\partial C}{\partial x} + \frac{\partial C}{\partial t} \tag{29}$$

$$C(x,0) = 0 \qquad x \geq 0 \tag{30}$$

$$C(0,t) = C_0[1-\exp(-qt)] \qquad t > 0 \tag{31}$$

$$C(\infty,t) = 0 \qquad t \geq 0 \tag{32}$$

Proceeding in the similar fashion, we get the solution of the above dispersion problem as

$$C(x,t) \equiv \frac{C_0 q}{2}\sqrt{\frac{4D_0T}{u_0^2}}\,[(\frac{x}{2\sqrt{D_0T}} + \sqrt{\frac{u_0T}{4D_0}})\exp(\frac{u_0^2x}{D_0})\cdot$$

$$\cdot\operatorname{erfc}(\frac{x}{2\sqrt{D_0T}} + \sqrt{\frac{u^2T}{4D_0}}) - (\frac{x}{2\sqrt{D_0T}} - \sqrt{\frac{u_0^2T}{4D_0}})\cdot$$

$$\cdot\operatorname{erfc}(\frac{x}{2\sqrt{D_0T}} - \sqrt{\frac{u_0^2T}{4D_0}})] \tag{33}$$

PART II – Adsorbing Porous Media

Case 1 – If the source concentration of the contaminant is given by $C = C_o\exp(-qt)$.

The differential equation that describes the concentration distribution of a solute in one dimensional porous media (adsorbing) flow, Ebach and White (1958) has shown that

$$\frac{\partial C}{\partial t} + \frac{1-n}{n}\frac{\partial F}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} \tag{34}$$

A general expression for the rate equation that expresses the interphase transfer is of the form

$$\frac{\partial F}{\partial t} \equiv f(C,F) \tag{35}$$

Lapidius and Amundson (1952) considered the two cases

$$F = K_1 C + K_2$$

$$\frac{\partial F}{\partial t} = K_1 C - K_2 F \tag{36}$$

representing respectively equilibrium and nonequilibrium relationships between the concentration in two phases. For simplicity the former relation is adopted in the present analysis.

Defining the adsorption coefficient θ as

$$\theta = 1 + K_1\frac{1-n}{n} \tag{37}$$

From Eqs. (34) and (37) one obtains

$$\theta \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial x^2} \tag{38}$$

subject to the initial and boundary conditions on the concentration of the solute in the liquid phase given by Eqs. (2) – (4).

Multiplying Eq. (38) by $(u_o/\theta u)$ and using the notations adopted in Eq. (8), we get

$$\frac{1}{V}\frac{\partial C}{\partial t} + \frac{u_0}{\theta}\frac{\partial C}{\partial x} = \frac{D_0}{\theta}\frac{\partial^2 C}{\partial x^2} \tag{39}$$

Using Eqs. (9), (14) the dispersion system is

$$\frac{\partial C}{\partial T} + \frac{u_0}{\theta}\frac{\partial C}{\partial x} = \frac{D_0}{\theta}\frac{\partial^2 C}{\partial x^2} \tag{40}$$

along with Eqs. (15) – (17).

With the help of transformation

$$C(x,T) = K(x,T)\exp\left[\frac{u_0 x}{2D_0} - \frac{u_0^2 T}{4D\theta}\right] \tag{41}$$

and proceeding as before we may get the final result $C(x,T)$ for concentration distribution in adsorbing porous media as

$$\begin{aligned} C(x,T) = \frac{C_0}{2}\Big[&\Big\{1 + q\sqrt{\frac{4D_0 T\theta}{u_0^2}}\Big\}\Big(\frac{x}{2\sqrt{D_0 T/\theta}} - \sqrt{\frac{u_0^2 T}{4\theta D_0}}\Big)\cdot \\ &\cdot \operatorname{erfc}\Big(\frac{x}{2\sqrt{D_0 T/\theta}} - \sqrt{\frac{u_0^2 T}{4\theta D_0}}\Big) \\ &+ \Big\{1 - q\sqrt{\frac{4D_0 T\theta}{u_0^2}}\Big\}\Big(\frac{x}{2\sqrt{D_0 T/\theta}} + \sqrt{\frac{u_0^2 T}{4\theta D_0}}\Big)\cdot \\ &\cdot \exp\Big(\frac{u_0 x}{D_0}\Big)\operatorname{erfc}\Big(\frac{x}{2\sqrt{D_0 T/\theta}} + \sqrt{\frac{u_0^2 T}{4\theta D_0}}\Big)\Big] \end{aligned} \tag{42}$$

Case II – If the source concentration of the contaminant at $x = 0$ is given by $C = C_o\,(1\text{-}\exp(-qt))$, the distribution of concentration $C(x,T)$ in the adsorbing porous media will be given by

$$C(x,T) = qC_0\sqrt{\frac{D_0\theta T}{u_0^2}}\,[(\frac{x}{2\sqrt{D_0T/\theta}} + \sqrt{\frac{u_0^2T}{4\theta D_0}})\exp(\frac{u_0x}{D_0})\cdot$$

$$\operatorname{erfc}(\frac{x}{2\sqrt{D_0T/\theta}} + \sqrt{\frac{u^2T}{4\theta D_0}}) - (\frac{x}{2\sqrt{D_0T/\theta}} - \sqrt{\frac{u^2T}{4\theta D_0}})\cdot$$

$$\cdot\operatorname{erfc}(\frac{x}{2\sqrt{D_0T/\theta}} - \sqrt{\frac{u^2T}{4\theta D_0}})] \qquad (43)$$

Numerical Example and Discussion

The graphical solutions of Eqs. (28) and (42) are presented in Fig. (2) and that of Eqs. (33) and (43) are presented in Fig. (3). The dotted lines represent adsorbing porous media whereas full lines represent non-adsorbing. The values are taken as: $u_o = 1.0$ cm/sec., $a_o = 1.92$, $d = 0.546$ cm (for gravel medium), $q = 0.008$ (sec.)$^{-1}$, $m = 0.004$ (sec.)$^{-1}$. The adsorbing coefficient $\theta = 1.42$. The new time variable has its values 15, 45, 75 and 105 (secs.). For these values the old time variable, the seepage velocity u are computed and presented in Table 1. From the table it is observed that the condition $mt < 1.0$ is very well satisfied. Also T can not assume much greater values for the sake of expansion $(1\text{-}mT)^{q/m} = 1\text{-}qT$.

It is also observed that for this set of numerical data the second term of Eqs. (28) and (42) and the first term of Eqs. (33) and (43) have negligible values as compared to the rest terms.

Table 1

T	15	30	45	60	75	90	105	120
mt	0.0618	0.1278	0.1984	0.2744	0.3566	0.4462	0.5447	0.6539
t	15.46	31.95	49.61	68.60	89.16	111.57	136.18	163.48
u	0.94	0.88	0.82	0.76	0.70	0.64	0.58	0.52

Conclusion

Mathematical and graphical solutions have been developed for predicting the possible concentration of a given dissolved substance in unsteady unidirectional seepage flows through semiinfinite, homogeneous, isotropic porous media subject to the source concentrations that vary exponentially with time. The porous media

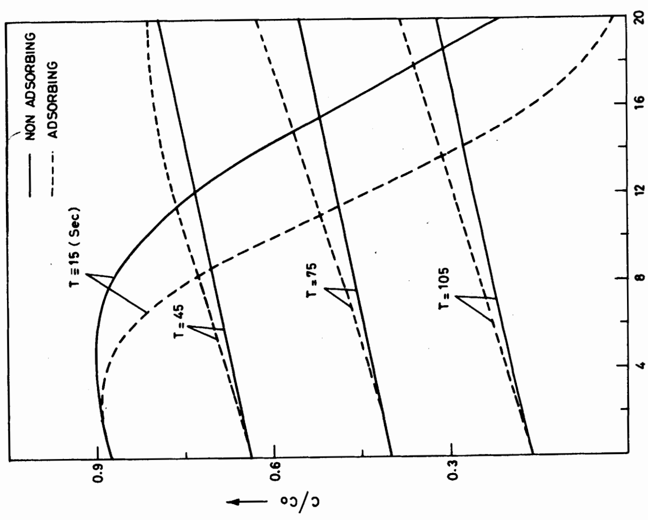

Fig. 2. Concentration distributions for an input concentration decreasing exponentially in time from C_0.

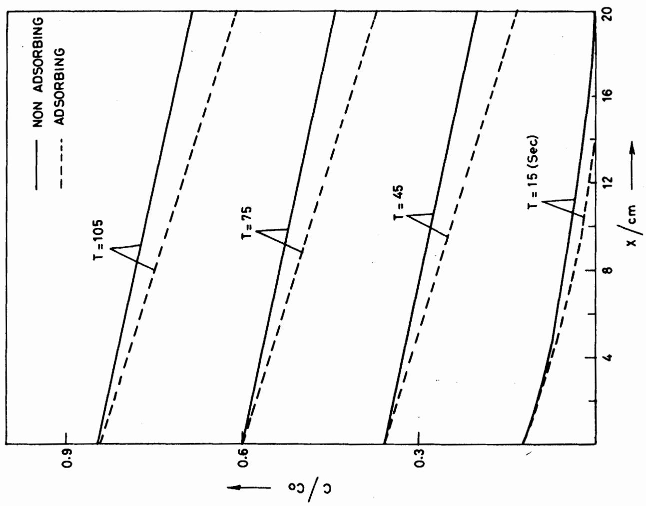

Fig. 3. Concentration distributions for an input concentration increasing exponentially in time to C_0.

is first considered nonadsorbing. Also the expressions take into account the mass transfer from liquid matrix to solid matrix due to adsorption. The concentration distributions as the flow advances, are well expressed in Figs. (2) and (3).

The analytical expressions obtained here are useful to the study of salinity intrusion in groundwater, helpful in making quantitative predictions on the possible contamination of groundwater supplies resulting from groundwater movement through burried wastes. In addition, they should prove useful for other processes such as ion exchange in soils and decay of organic substances.

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Notations

The following symbols are used in this paper.

a_o – dimensionless constant
C – concentration of solute in liquid phase
C_o – Initial concentration of solute in liquid phase
D – dispersion coefficient based on u
D_o – dispersion coefficient based on u_o
d – particle size (diameter) of porous material
F – concentration of solute in solid phase
K_1,K_2 – constants in adsorption equation
m – flow resistance coefficient
n – porosity of porous medium
R – Reynolds number ($R = ud\,\varrho/\mu$)
t – variable time
T – new variable time
u – seepage velocity
u_o – initial seepage velocity
x – longitudinal direction
V – dimensionless velocity (u/u_o)
θ – adsorption coefficient
q – a parameter
μ – dynamic viscosity coefficient
ϱ – mass density

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