In-Medium Pions and Partial Restoration of Chiral Symmetry: 
A Model-Independent Analysis

Daisuke Jido,¹ Tetsuo Hatsuda² and Teiji Kunihiro¹

¹Yukawa Institute for Theoretical Physics, Kyoto University, 
Kyoto 606-8502, Japan 
²Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan

Exploiting operator relations in QCD, we derive a novel and model-independent formula relating the in-medium quark condensate \( \langle \bar{q}q \rangle^* \) to the decay constant \( F_t^* \) and the wave function renormalization constant \( Z^* \) of the pion in the nuclear medium. Evaluating \( Z^* \) at low density from the iso-scalar pion-nucleon scattering data, it is concluded that the enhanced repulsion of the \( s \)-wave isovector pion-nucleus interaction observed in the deeply bound pionic atoms implies directly the reduction of the in-medium quark condensate. The knowledge of the in-medium pion mass is not necessary to reach this conclusion.

§1. Introduction

Exploring possible evidence of partial restoration of chiral symmetry in nuclear medium is one of the hot topics in modern hadron physics. Systematic studies of deeply bound pionic atoms¹) and the low-energy \( \pi^- \)-nucleus elastic scattering²) have shown an enhancement of the repulsion in the \( s \)-wave isovector pion-nucleus interaction. This enhanced repulsion can be nicely accounted for by a reduction of the pion decay constant in nuclei provided that the \( s \)-wave isovector pion-nucleus interaction is given by the Weinberg-Tomozawa term with the in-medium pion decay constant \( F_t^* \).³) The reduction of \( F_t^* \) in nuclear medium is also argued to be the mechanism of the enhanced attraction of the in-medium \( \pi\pi \) scattering in the \( I = J = 0 \) channel (the \( \sigma \) channel).⁴) This has intimate relation to the near-threshold enhancement of the two-pion production off nuclei.⁵)

Recent theoretical works of chiral effective theory⁶) and of the pion optical potential⁷) have suggested that the in-medium renormalization of the pion wave function \( Z^* \) is responsible for the reduction of \( F_t^* \). The purpose of this work is to present a new scaling law relating \( F_t^* \) to the in-medium quark condensate \( \langle \bar{q}q \rangle^* \) and \( Z^* \).⁸) This relation is derived in a model-independent way based only on low energy theorems of QCD. Combining this relation and empirical observation of the pionic atoms and the \( \pi^- \)-nucleon scattering, we conclude the reduction of the quark condensate in the nuclear medium without recourse to the knowledge of the in-medium pion mass.

§2. Derivation of the scaling law

Let us consider the correlation functions of the axial current \( A_\mu \equiv \bar{\psi} \gamma_\mu \gamma_5 t^a \psi \) and/or the pseudoscalar density \( \phi_5^a \equiv \bar{\psi} i \gamma_5 t^a \psi \), where \( t^a \) is 1/2 times the Pauli matrix. To discuss how the in-medium quark condensate is expressed by hadronic quantities, we start with the following correlation function for the symmetric nuclear
where in the chiral limit:

\[
\Pi^{ab}(q) = \int d^4x e^{iq \cdot x} \partial^\mu \langle \Omega | T \{ A^a_\mu(x), \phi^b_5(0) \} | \Omega \rangle,
\]

(2.1)

where |\Omega\rangle denotes a nuclear matter state normalized as \(\langle \Omega | \Omega \rangle = 1\). Exploiting an operator relation, \(\partial^\mu T A^a_\mu(x) \phi^b_5(0) = T \partial^\mu A^a_\mu(x) \phi^b_5(0) + \delta(x_0) [A^a_\mu(x), \phi^b_5(0)]\) with the conservation of the axial current, one can express the correlation function (2.1) by the quark condensate:

\[
\lim_{q_\mu \to 0} \Pi^{ab}(q_\mu) = \int d^4x \delta(x_0) \langle \Omega | [A^a_\mu(x), \phi^b_5(0)] | \Omega \rangle = -i \delta^{ab} \langle \Omega | \bar{q} q | \Omega \rangle,
\]

(2.2)

where we have used the commutation relation \([Q^5, \phi^b_5] = -i \delta^{ab} \phi\) with the generator \(Q^5(x_0) = \int d^3x A^a_0(x_0, x)\) and the scalar density \(\phi = \bar{\psi}(I/2)\psi\). In such a soft limit, the correlation function (2.1) is saturated by zero modes whose contents in the chiral limit are the pionic mode and particle-hole excitations in nuclear matter. These modes are coupled and mixed with each other in nuclear matter. Here we are interested in the zero mode which is continuously connected to the in-vacuum pion state. With the four vector \(n_\mu\) characterizing the nuclear matter, the conservation of the axial current and Lorentz invariance lead to the following relations for the matrix elements of \(A^a_\mu\) and \(\phi^b_5\) taken by the states |\Omega\rangle and |\Omega^5\rangle:

\[
\langle \Omega | \phi^b_5(x) | \Omega^5\rangle = \delta^{ab} Z^{1/2} e^{-iq \cdot x},
\]

(2.3)

\[
\langle \Omega | A^a_\mu(x) | \Omega^5\rangle = \delta^{ab} i \left( -\frac{q^2}{(q \cdot n)} n_\mu + q_\mu \right) F^* e^{-iq \cdot x}.
\]

(2.4)

The matrix elements \(Z^*\) and \(F^*\) are functions of \(q \cdot n\) and \(q^2\). Taking the frame \(n_\mu = (1, 0, 0, 0)\) and using the in-medium dispersion relation \(q_\pi^2 - v_\pi^2 q^2 = 0\) with the velocity \(v_\pi\), we obtain linear dependence of the matrix elements on energy as follows:

\[
\langle \Omega | A^a_0(x) | \Omega^5\rangle = \delta^{ab} i q_0 \frac{1}{v_\pi^2} F^* e^{-iq \cdot x}, \quad \langle \Omega | A^a_\mu(x) | \Omega^5\rangle = \delta^{ab} i q_\mu F^* e^{-iq \cdot x}.
\]

(2.5)

Writing the temporal and spacial components of the decay constant separately as \(F^*_t = F^*/v_\pi\) and \(F^*_s = F^*/v_\pi\), one can see that the relation \(F^*_s / F^*_t = v_\pi^2\) is automatically satisfied in this derivation. Hereafter we refer to \(F^*_s\) and \(Z^*\) as the quantities in the soft limit \(q_\mu = 0\). The pion contribution to the correlator Eq. (2.1) can be isolated in the soft limit and is rewritten as

\[
\lim_{q_\mu \to 0} \Pi^{ab}(q_\mu) = \lim_{q_\mu \to 0} \left( i \delta^{ab} \frac{q_0^2 F^*_s - q^2 F^*_t}{q_0^2 - v_\pi^2 q^2} Z^{1/2} + \cdots \right) = \delta^{ab} i F^*_t Z^{1/2}.
\]

(2.6)

The ellipsis in Eq. (2.6) denotes terms without singularities at \(q_\mu = 0\), which are strongly suppressed in the soft limit due to the factor \(q_\mu\) in the numerator. Combining Eqs. (2.2) and (2.6), we finally arrive at an exact relation in the chiral limit

\[
F^*_t Z^{1/2} = -\langle \Omega | \bar{q} q | \Omega \rangle.
\]

(2.7)
Taking a ratio with the relation in the vacuum, we find the following novel scaling law connecting the pion decay constants and the quark condensate through the pion wave function renormalization constant:

\[
\frac{F^*_t Z^{1/2}}{F^* Z^{1/2}} = \frac{\langle \Omega | \bar{q} q | \Omega \rangle}{\langle 0 | \bar{q} q | 0 \rangle}.
\] (2.8)

This relation implies that one can deduce the in-medium reduction of the quark condensate solely from the decrease of the pion decay constant, once one knows the wave function renormalization of the pion in medium.

The in-medium Weinberg-Tomozawa relation can be also derived in a similar way as Eq. (2.7). The essential point is that the renormalization of the pion wave function for the isoscalar part is already taken into account and the isovector part is treated as a perturbation.\(^8\) The in-medium Gell-Mann–Oakes–Renner relation is also derived in this line with the PCAC relation under the assumptions of the small quark mass and the linear density approximation.\(^8\)

§3. Pion wave function renormalization in nuclear medium

For examining the renormalization of the pion wave function in the low density limit, let us consider the following correlation function of the pseudoscalar density in symmetric nuclear matter in the chiral limit:

\[
D^*_\pi(q) = \int d^4x e^{iq \cdot x} \langle \Omega | T[\phi_5(x)\phi_5(0)] | \Omega \rangle,
\]

which has the pion pole at \(q^2 - v_\pi^2 q^2 = 0\) with the residue \(Z^*\) defined in Eq. (2.3). Collecting all the in-medium corrections to the self-energy \(\Sigma_\pi(q_0, \vec{q})\), or the optical potential, we can express the in-medium normalization constant by the self-energy as

\[
Z^* = Z \left(1 - \frac{\partial \Sigma_\pi(q_0, \vec{q} = 0)}{\partial q_0^2} \bigg|_{q_0=0} \right)^{-1}.
\] (3.1)

In the linear density approximation, the self-energy in symmetric nuclear matter is given by \(\Sigma_\pi(q_0) = -\rho T^{(+)}_{\pi N}(q_0)\) with an iso-singlet \(\pi N\) scattering amplitude:

\[
T^{(+)}_{\pi N}(\nu, \vec{\nu}; k^2, k'^2) = iZ^{-1}k^{2}k'^2 \int d^4x \, e^{ik \cdot x} \langle N | T[\phi_5(x)\phi_5(0)] | N \rangle
\]

with the outgoing (incoming) pion momentum \(k\) (\(k'\)), and kinematical variables defined by \(\nu \equiv p_N \cdot (k + k')/(2M_N)\) and \(\vec{\nu} \equiv -k' \cdot k\). The off-shell extrapolation of this amplitude is defined by the above reduction formula and is consistent with the low energy theorems obtained from the commutation relation involving the pseudoscalar density \(\phi_5\). The contributions to the off-shell amplitudes are represented by the diagrams in which the pseudoscalar density couples directly to the nucleon states.\(^9\) The \(\pi N\) scattering amplitude with \(\vec{q} = 0\) is a function of \(q_0; \nu^2, \vec{\nu}, k^2, k'^2 \rightarrow q_0^2\). The chiral expansion is valid for the low energies and we have

\[
T^{(+)}_{\pi N}(q_0) = \alpha + \beta q_0^2
\] (3.2)

up to the higher order terms. The intercept \(\alpha\) is given by the explicit chiral symmetry breaking proportional to the quark mass, while the slope \(\beta\) is a value in the chiral limit. From Eq. (3.1), one sees that \(Z^*\) is given by the slope \(\beta\) in the low density limit.
The sign of $\beta$ can be extracted as follows: (i) In the Weinberg point, $\nu, \bar{\nu}, k^2, k'^2 = 0$, we have $T^{(+)}_{\pi N}(0) = -\sigma_{\pi N}/F^2 = \alpha$ with the $\pi N$ sigma term $\sigma_{\pi N}$.\textsuperscript{10,11} (ii) At the threshold, the amplitude is identified as the scattering length $a_{\pi N}$: $T^{(+)}_{\pi N}(m_\pi) = 4\pi(1 + m_\pi/m_N)a_{\pi N} = \alpha + \beta m_\pi^2$. The pionic hydrogen atom data suggest a very small scattering length $a_{\pi N} = (0.0016 \pm 0.0013)m_\pi^{-1}$,\textsuperscript{12} which is consistent with zero. Combining these facts, we find $\beta \simeq \sigma_{\pi N}/(F^2 m_\pi^2)$. Since the value of the $\pi N$ sigma term is positive, we can conclude that

$$Z^{1/2}/Z^{1/2} \simeq F^2 m_\pi^2/(F^2 m_\pi^2 + \rho \sigma_{\pi N}) < 1.$$ \hfill (3.3)

This conclusion is not altered as long as the scattering length is larger than $-0.050m_\pi^{-1}$ for $\sigma_{\pi N} \simeq 45$ MeV and $-0.067m_\pi^{-1}$ for $\sigma_{\pi N} \simeq 60$ MeV.

§4. Summary

On the basis of exact low-energy theorems in QCD, we have derived a new scaling law which relates the in-medium quark condensate to the in-medium pion decay constant and the in-medium pion wave function renormalization constant. We have utilized operator relations and chiral symmetry in QCD so that an explicit analysis of complicated dynamics of the pion in nuclear matter is partially circumvented. Combining the new relation with experimental data of the pionic atoms and $\pi N$ scattering, we conclude the reduction of the quark condensate in nuclear medium.

Acknowledgements

D. J. thanks Professor W. Weise and Professor M. Harada for fruitful discussions on this work during YKIS2006. This work was supported in part by Grant-in-aid for Scientific Research of Monbukagakusho of Japan (Nos.17540250, 18042001, 18540253) and by the 21st Century COE “Center for Diversity and Universality in Physics” of Kyoto University.

References