Discussion

W. J. Carter. This fine paper presents yet another method for acceleration determination of linkage elements. The use of derivatives which are only geometrically dependent may very well be a more direct way of analyzing motions of linkages than by working with time derivatives. There is no reason why this scheme cannot be applied to higher derivations of motion. The superiority of this method for complex mechanisms is very evident.

F. Freudenstein. An examination of existing methods of acceleration analysis leads to the conclusion that the author is to be congratulated on an ingenious and substantial contribution to the systematic acceleration analysis of complex mechanisms, as anyone who has struggled with six-link chains can appreciate. The evaluation of the first geometrical derivative as a velocity ratio forms the basis of the Hartmann construction which, however, has no other relation to the present work. The author's derivations lend themselves readily to generalization to the higher geometrical derivatives and thus also to the analysis of second accelerations discussed by Wolford and Hall. The writer looks forward to the extensive use of this new technique.

A. S. Hall. The writer considers this paper valuable enough to add to the rather short list of references which are required reading for his students. The reasons are as follows:

1. The paper emphasizes the idea of using inversions. We tend to lose sight of the essential sameness of all mechanisms formed from a particular one-degree-of-freedom kinematic chain. It is a great waste of time and effort to treat these as different mechanisms. The author has expressed this very well in his paragraph directly above Fig. 3.

2. The author supplies ammunition to support a pet contention of the writer; i.e., that it is frequently a sensible idea to determine quantities which are strictly geometric by means of a kinematic inversion. In this paper the idea is used to evaluate the first and second derivatives of one angle with respect to another. The writer uses the idea as follows:

(a) To derive the well-known Euler-Savary equation.
(b) To evaluate partial derivatives of the form dR/2p, where R is the output displacement of any mechanism and p is one of the adjustable constants of the mechanism (a link length, for example). It is a great waste of time and effort to treat these as different mechanisms. The author has expressed this very well in his paragraph directly above Fig. 3.
(c) To prove the Bobillier theorem.
(d) To derive an expression for the radius of curvature of the polodes (centrodes) in the four-bar mechanism.

G. A. Nothmann. The author is to be congratulated for having made a useful and elegant contribution to the subject of mechanism analysis.

For completeness, the author's use of the coefficient r (Equation [6]?) is possibly desirable. However, it would seem that the velocity scale can always be adjusted such that r = 1, after the velocity analysis has been completed, so that Equation [9] can be used in simplified form.

The author's method is not restricted to analyses based on angular velocities and accelerations but can be used, with a minor modification, for analyses using linear quantities only. For example, in the first problem (Fig. 1), if $V_{A1}$ and $A_{A1}$ are prescribed, then Equation [6] can be rewritten for point $E$, for example, as follows (with $i, j, k, l$ being identified as $a, f, g, f$, respectively, and with $r = 1$)

$$\frac{E\dot{S}(\alpha_{ij} - \alpha_{il})}{A\dot{Q}_{a}} A\dot{Q}(\alpha_{ij} - \alpha_{il}^*)$$

or

$$A_{E1}^T - A_{E1}^T = \frac{V_{B1}}{V_{A1}} (A_{A1}^T - A_{A1}^T)$$

where the superscript $T$ designates the tangential acceleration component. If the author's first example were stated in terms of linear rather than angular velocities and accelerations, one would still proceed similarly, find the velocity and acceleration-vector diagrams with $V_{B1}$ and $A_{B1}$ arbitrarily assumed, and calculate $A_{E1}^T$ from the foregoing equation, since all the other terms become available. The graphical and numerical labor appears to be the same, and the choice between working with linear or angular quantities would depend on the requirements of the problem. This modification of the author's method also can be extended to mechanisms, like that of Fig. 2, requiring an inversion.
Thus paragraphs directly above Fig. 3.

larly, when a mechanism is investigated for a range of positions, the angular accelerations for the whole range of motion can be found approximately by graphical or tabular differentiation of angular velocities, and then the positions of greatest interest can be investigated more precisely by the method of the paper. For a complex mechanism, displacement and velocity analysis, as well as acceleration analysis, can be greatly simplified by use of the simplest inversion of the mechanism, as indicated in the two paragraphs directly above Fig. 3.

Dr. Nothmann has pointed out that the author's use of the term relative, as applied to velocities and accelerations, appears to differ from that of other authors. The reason for this apparent difference is that all quantities used in the paper are scalars rather than vectors. When the links of a mechanism all move in the same or parallel planes, their relative angular displacements, velocities, and accelerations may be treated as scalars, and the relationships among these quantities for any three links (k, l, and m) can be determined by simple algebraic addition or subtraction. Thus

\[ \omega_{kl} + \omega_{lm} = \omega_{km} \] \[ \alpha_{kl} + \alpha_{lm} = \alpha_{km} \]

Similarly, if two links have only translational motion relative to each other, the magnitudes of the relative translational displacements, velocities, and accelerations may be treated as scalars. Time is also a scalar quantity, and the essence of the method presented in the paper is that a scalar relative displacement (either a rotational displacement \( \theta_{ij} \) or a translational displacement \( s_{ij} \)) of two links is found in place of time as the independent variable which specifies the motion of a constrained mechanism. In the example given by Dr. Nothmann, the angular velocities and accelerations may still be thought of as scalars (angular velocities and accelerations multiplied by the scalar lengths \( BS \) and \( AQ \)).

While the independent variable (\( \theta_{ij} \) or \( s_{ij} \) in the equations of the paper; \( A\theta_{ij} \) in Dr. Nothmann's equations) must necessarily be a scalar, the dependent variables (for example, \( V_{kl} \) and \( A_{kl} \) in Equations [11] through [13]) may also be treated as vectors. Using bold-faced symbols to denote vectors and defining \( \mathbf{R}_{kl} \) as the position vector from a point fixed on link \( l \) to a point \( B \) moving with link \( k \), equations for the vector velocity and acceleration of point \( B \) may be derived in a manner similar to the derivation of Equations [11] through [13]:

\[ \mathbf{V}_{kl} = \frac{d\mathbf{R}_{kl}}{dt}, \quad \frac{dV_{kl}}{dt} \]

\[ \mathbf{A}_{kl} = \frac{d^2\mathbf{R}_{kl}}{dt^2}, \quad \frac{dA_{kl}}{dt} \]

Identifying the links \( i, j, k, l \) with \( d, f, c, f \), respectively, Equations [17] and [18] can be rewritten to apply specifically to point \( B \) of Fig. 1:

\[ \mathbf{V}_{Bdf} = \frac{d\mathbf{R}_{Bdf}}{dt}, \quad \frac{dV_{Bdf}}{dt} \]

\[ \mathbf{A}_{Bdf} = \frac{d^2\mathbf{R}_{Bdf}}{dt^2}, \quad \frac{dA_{Bdf}}{dt} \]

These vector equations have the same advantages as the scalar equations [2] and [6]. The values of \( V_{Bdf} \) and \( A_{Bdf} \) can be found once and for all by assuming the simplest possible values for \( \omega_{ij}^p \) and \( \alpha_{ij}^p \) (for example, +1.00 rad/sec and 0, respectively), and then Equations [19] and [20] can be used to compute the actual values of \( V_{Bdf} \) and \( A_{Bdf} \) for any actual values of \( \omega_{df} \) and \( \alpha_{df} \) with no further graphical constructions except the vector addition indicated by Equation [20].

Extreme caution must be used, however, in using vector quantities with inversions of a mechanism. For any three links \( k, l, m \),

\[ \mathbf{V}_{kl} + \mathbf{V}_{lm} = \mathbf{V}_{km} \]

but, unfortunately, in general

\[ \mathbf{A}_{kl} + \mathbf{A}_{lm} \neq \mathbf{A}_{km} \]

because of the Coriolis component of acceleration (1, 2).

The scalar relationships given by Equations [14] and [15], however, always hold for plane mechanisms.