Bootstrapped discrete scale invariance analysis of geomagnetic dipole intensity

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SUMMARY

The technique of bootstrapped discrete scale invariance allows multiple time-series of different observables to be normalized in terms of observed and predicted characteristic timescales. A case study is presented using the SINT2000 time-series of virtual axial dipole moment, which spans the past 2 Myr. It is shown that this sequence not only bears a clear signature of a preferred timescale of about 55.6 Ka, but additionally predicts similar features (of shorter and longer duration) that are actually observed on the timescales of historical secular variation and dipole reversals, respectively. In turn, the latter two empirical sources both predict the characteristic timescale found in the dipole intensity sequence. These communal scaling characteristics suggest that a single underlying process could be driving dynamo fluctuations across all three observed timescales, from years to millions of years.

Key words: fractals, geomagnetic field, geomagnetic reversals, geomagnetic secular variation, palaeomagnetism, statistical methods.

INTRODUCTION

In a recent paper (Jonkers 2007), evidence is presented supporting the presence of power laws in geomagnetic fluctuations over a range of timescales, from years to millions of years. In addition, it is shown there that the power-law residuals exhibit a statistically significant modulation, resulting in log-periodic scaling in the probability distribution function (near-horizontal levels along the power-law slope), representing preferred temporal scales of the system. A detailed treatment of the actual technique developed to derive these results unfortunately fell outside the scope of that publication. This contribution can, therefore, be considered a companion piece, containing a step-by-step account of the analysis as applied to the SINT2000 data set published by Valet et al. (2005), followed by a comparison with results obtained from dipole reversals and historical secular variation.

Although this particular study focuses entirely on geomagnetic phenomena, it is important to stress that the presented technique is in principle applicable to any non-linear system that produces time-series, given the usual minimal conditions of sufficient span, range and accuracy. Such time-series need neither overlap temporally nor cover the same range of scales, and can even measure different observables altogether. The ultimate purpose is to enable comparison of scale-invariant features in different phenomena that are suspected to be driven by the same underlying process. Mutual cross-verification of predicted versus observed preferred timescales of all observables may identify the size and duration of the most important processes (i.e. constraints on the underlying physics), and may quantify their potential coherence.

Table 1 summarizes some relevant aspects of the three data sets examined here. The sequence of dipole reversals (a series of 332 dated events covering the Mesozoic and Cenozoic era) was gleaned from Gradstein & Ogg (1996). The dipole intensity time-series, expressed as virtual axial dipole moment (VADM, in $10^{19}$ A m$^2$, data variable mean palaeointensity used here), has been published by Valet et al. (2005). Thirdly, dipole-detrended secular variation time-series at the core–mantle boundary (CMB; X, Y and Z component in nanoteslas, at every 6° latitude and longitude, $N = 5400$ time-series) were extracted from historical field map gufm1 (Jackson et al. 2000; Jonkers et al. 2003). Not only do these three sets stem from completely independent sources (lava flows, ocean floor sediments and historical observations, respectively), they measure different geomagnetic phenomena, at different temporal resolution, and their time ranges overlap marginally at best (e.g. no reversal happened in the time-span covered by gufm1, and SINT2000 covers less than 2 per cent of the reversal sequence). In other words, only communal, scale-invariant features would be able to bridge these gaps. Given the highly non-linear nature of the geodynamo (Zhang 1999), and the growing consensus among geomagnetists that neither excursions (Langereis et al. 1997; Lund et al. 1998; Gubbins 1999) nor reversals (Sarson & Jones 1999; Coe et al. 2000) are exceptional dynamo states, a scale-free approach would potentially allow the entire gamut of internal field fluctuations from secular variation to (and perhaps even beyond) reversals to be described in a single overarching framework (Love 2000; Hollerbach 2003).

A salient difference between the sequence of dated reversals and the other two sets is that the former captures an irregularly spaced succession of similar field alterations, whereas the latter two track the fluctuations of a quantified variable at regular intervals. A dipole...
reversal can be interpreted as a threshold event; a global magnetic polarity change is a well-defined phenomenon, and intervals between successive ones are likely to be subject to a lower temporal bound of circa 10 Ka, due to the diffusion timescale of the inner core (Gubbins 1999; Sakuraba & Kono 1999). By contrast, a typical equidistantly sampled VADM time-series consists of small and larger wiggles and does not, at first sight, reveal which fluctuations or associated intervals, if any, are more significant than others. The technique presented here extracts such information, if present, by detecting signatures of discrete scale invariance (DSI).

The fractal concept of discrete scale invariance was formalized in the 1990s, and has already been successfully applied to such diverse phenomena as seismicity (Saleur et al. 1996), stock market fluctuations (Feigenbaum & Freund 1996), earthquake faults (Hännikä & Drossinos 1999), sandpile avalanches (Lee & Sornette 1999), evolutionary leaps (Nottale et al. 2000), turbulence (Novikov 1990; Johansen et al. 2000; Zhou et al. 2003), and fracturing (Kapiris et al. 2004; Kikuchi & Yamanaka 2005). A simple example of the principle, adapted from Sornette (1998), is given in the next section. This paper enhances the basic technique by incorporating it into a bootstrap process, which sequentially imposes on a time-series a vast range of artificial thresholds of change, producing large numbers of auxiliary data sets of size-ranked intervals, which are subsequently recombined into a single meta-data set with desirable properties. Its application to the SINT2000 time-series of dipole moment described below is primarily intended to illustrate the procedure, as well as its potential pitfalls and strengths. The stress on method implies that this contribution does not dwell on the background of the various source data sets, the general geophysical context, or specific theoretical implications for the convective regime of the geodynamo.

Finally, it deserves stressing that (bootstrap) DSI analysis examines a data set’s stationary probability distribution function for evidence of a particular kind of modulation spanning at least several orders of magnitude, which affects the likelihoods of size-specific fluctuations and their associated duration. In other words, it seeks to extract timescale-invariant statistics from time-invariant statistics. Consequently, although time-series do provide the initial input and periodograms are calculated at some stage, the actual processing here is no time-series analysis in the traditional sense at all, but an application of non-linear statistics. The extracted ‘characteristic’ or ‘preferred timescales’ likewise constitute a stationary statistical feature (see below), not to be confused with any kind of periodicity (peak in a Fourier spectrum), nor with statistical measures of central tendency or trend (e.g. median, mode and moving average), nor with a ‘saw-tooth pattern’ (Meynadier et al. 1994). Traditional methods such as Fourier analysis, that operate in the time and frequency domain, quantify a completely different type of signal within the data, the presence or absence of which has no bearing on the presence or absence of DSI, nor do they affect the latter’s detection. In addition, the bootstrap method presented here is able, unlike Fourier analysis, to associate a particular preferred magnitude of fluctuation (of the time-dependent variable tracked in the time-series) with a specific characteristic timescale. Of course, the above is not intended to disparage Fourier-like methods in any way, merely to underline that the bootstrapped DSI method measures something different and unrelated, providing a novel, scale-independent perspective that highlights a previously undisclosed non-linear property of the geodynamo.

FROM DATA TO METADATA

The lower part of Fig. 1 depicts a sequence familiar to palaeomagnetists: a stacked, optimally correlated estimate of relative geomagnetic dipole intensity, converted into virtual axial dipole moments (VADMs) using absolute palaeointensities from volcanic records (Valet et al. 2005). It has a temporal resolution of 10\(^3\) yr (N = 2000), is estimated to be accurate to within about 20 kyr, and spans the period from 2 Ma to the present day. During this period, several full polarity reversals took place, recognizable as intensity drops to or below 0.2 (\(\times\) 10\(^{-23}\) A m\(^2\), omitted from here on), followed by a steeper recovery phase. Although the recorded range in VADM spans over an order of magnitude (from 0.09 to 1.56), most coherent fluctuations are of intermediate size, and no specific associated duration stands out as particularly important.

The technique of bootstrapped discrete scale invariance is a three-stage process (Fig. 2). In the first stage, the original data set (a time-series) is used to generate a multitude of auxiliary data sets mapping a stationary probability distribution function, each of which is tested in stage two for the presence of discrete scale invariance (defined below). Those that do, contribute a single data point to a meta-data set, which, when complete, is itself tested for discrete scale invariance in stage three. Lastly, a comprehensive investigation of a complex system will involve several of these bootstrap DSI analyses of different variables, derived from independent sources. Establishing the degree of coherence of individual results may hold information regarding which observables might be linked across, and even beyond, their respective timescales.

In the absence of a geophysically unambiguous ‘event’ on the resolved timescale of typical dipole intensity variability, one can artificially create such criteria by imposing a threshold of absolute exceedance of some arbitrary limit of change. The original time-series is thereby converted into a data set like the dipole reversals, that is, a sequence of dated events with known properties, easily converted into a size-ranked set of intervals, which constitutes one auxiliary data set. The problem of choosing a ‘suitable’ threshold is obviated by defining not one, but a vast range of incremental thresholds (in this case, 2000 were used, from 0.0005 to 1.0000, step 0.0005). This division is here applied in a linear sense (constant increments), which will warrant a weighting scheme in the log-domain in stage three. Alternatively, no weighting is to be applied there if thresholds applied in stage one are instead made to increase with a constant increment on a log-scale.

Regarding the total range, absolute bounds are given by the range of the original data (here: 1.46), although usually the largest threshold that still produces statistically significant information will be substantially smaller than that. Ideally, the range should span...
The three stages of bootstrapped discrete scale invariance analysis. In stage one, the imposition of a range of thresholds of absolute change generates a large number of auxiliary subsets. Each of these subsets of intervals is size-ordered, and subjected to DSI analysis in stage two. For each statistically significant DSI result, the largest scaling strata is associated in a meta-data set with the threshold that generated the subset. DSI analysis of the metadata constitutes stage three (see also Fig. 7).

Formalizing the above, starting with a time-series $T$ of length $n$:

$$T = \{T_i\}, \; i = 1, 2, \ldots, n, \; T_{\text{range}} = \max(T) - \min(T), \quad (1)$$

a range of $m$ thresholds $\lambda$ is defined:

$$\lambda = \{\lambda_j\}, \; j = 1, 2, \ldots, m, \; \lambda_1 > 0, \; \lambda_m \leq T_{\text{range}} \quad (2)$$

of which successive members increase with small, (linear- or log-scaled) constant step size $\lambda_{\text{step}}$:

$$\lambda_{j+1} = \lambda_j + \lambda_{\text{step}}, \; T_{\text{range}} \gg \lambda_{\text{step}} > 0. \quad (3)$$

For each threshold $\lambda_j$, the first point of the original time-series is initially taken as a reference value:

$$T_{r, \text{ref}} = T_1 \quad (4)$$

after which the following algorithm is applied repeatedly to build an auxiliary intervals data set $K$ (with index $v$), for a given constant threshold $\lambda_{j}$:

$$T_i = |T_i - T_{r,\text{ref}}| > \lambda_j, \; i = 2, 3, \ldots, n$$

$$K_v = i - i_{\text{ref}}, \; v \leftarrow v + 1, \; v = 1, 2, \ldots, k$$

$$T_{r, \text{ref}} = T_i \quad (5)$$

yielding a set of $k$ intervals per set. The above algorithm thus calculates the absolute difference in VADM between the reference value and each successive value until it finds one that exceeds the current threshold; the difference in time is then stored, the current value becomes the new reference value, and this process is repeated until the series is exhausted. One important caveat concerns temporal resolution: since any fluctuation that takes less time than the distance between two successive data points (here: 1000 yr) will be coarse-grained to that minimum duration, these intervals are invalid estimates, and are thus to be excluded, but afterwards, since rejection during compilation would spuriously extend the next valid interval recorded. This exclusion causes the initial steep increase of the total number of intervals per subset (as increasingly fewer inter-point durations are identified), followed by a slow descent. Upon completion, each auxiliary data set is labelled with the appropriate threshold value for future reference. Finally, the current threshold is incremented by $\lambda_{\text{step}}$ for the next run.

To illustrate this first stage, the results of three arbitrary sample thresholds applied in the SINT2000 example are shown in the top half of Fig. 1, sharing the same temporal axis, with intervals depicted as horizontal bars bounded by vertical lines. The thresholds

Several orders of magnitude; a simple test to confirm an adequate range of thresholds is supplied further down. Less clear-cut is the choice of threshold increment. A workable step size will be as much dependent on the variability of the original data as on the available computing resources. A starting point could be a division result of several hundred to one thousand thresholds, with optional fine-graining if the initial results are too poorly behaved to warrant conclusions at a reasonable (statistical) confidence level.

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Figure 1. The SINT2000 data set of geomagnetic dipole intensity (Valet et al. 2005), a series of fluctuations spanning 2–0 Ma ($N = 2000$), ranging from 0.0977 to 1.5586 $\times$ $10^{23}$ A m$^2$. Above it are marked the boundaries of intervals during which an absolute change takes place exceeding 0.02 (marked A, $N = 428$), 0.22 (B, $N = 95$) and 0.42 (C, $N = 26$) units, respectively.

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The number of intervals (larger than one time step) per threshold ($N = 1188$); the three example thresholds (A–C) are circled. The 68 points preceding the peak are to be discarded (including example subset A).

These intervals thus do not contribute to this particular subset.

In general, interval density clearly decreases with increasing threshold, as would be expected; whereas auxiliary set A contains 428 members larger than a single time step, set B has only 95, and C a mere 26. Taking all 1188 thresholds that eventually produced statistically significant results (see below), the number of intervals larger than inter-point distance can be seen in Fig. 3 to rise sharply initially, followed by a gradual decline, as larger thresholds produce ever fewer, but longer intervals. This is likewise attested in the histograms in Fig. 4, which mark the size and number of intervals in each of these three samples. Another aspect highlighted in the histograms is the relatively constant ratio of larger to smaller intervals; a small minority on the left (high end of the scale) is balanced by a large majority on the right, regardless of absolute size. Note in passing that a similar pattern is in evidence in the distribution of chrons, that is, the intervals between successive dipole reversals (Jonkers 2003).

After the range of thresholds is explored to a $\lambda$-value whose associated auxiliary data set would count 10 members or fewer, post-processing of each set with $N > 10$ members requires intervals per set to be re-ordered and ranked by size. Thus the largest duration is assigned rank 1, the next largest obtains rank 2, and so on. The log$_{10}$ values of each rank and its associated duration are subsequently stored as that subset’s $x$ and $y$ coordinates, respectively. This is followed by testing for scale-invariance, first by fitting a power-law distribution, which is most easily performed in the log–log domain. The threshold methodology (developed by the author) thus functions as a necessary pre-processing step to turn a time-series into multiple subsets containing time-invariant statistics (of the kind that DSI analysis requires), since size-ranking destroys all time-dependence. The potential bias of unequal data density in case of linearly increasing ranks (low ranks being wide apart, high ones clumping together) is commensurately counteracted by log-weighting the points with individual weights $w$:

$$w_i = \log_{10}(i + 1) - \log_{10}(i).$$  \hspace{1cm} (6)

Naturally, a statistical confidence level has to be determined for this attribution. A thorough approach will put a suite of alternative probability hypotheses (e.g. Gaussian, gamma, exponential and Poisson) through the same paces, and accept only those sets for which the power-law attribution not only performs best, but also at a reasonable alpha (i.e. statistical confidence) level. The peculiarities of power-law distributions in this respect are adequately described in many recent textbooks on non-linear statistics (e.g. Adler et al. 1998; Boccara 2004; Sornette 2004); the only additional remark in this context is to avoid setting too stringent a confidence criterion here on power-law acceptance, since the potential presence of discrete scale invariance could result in substantial differences between a pure power law and a modulated one. Regarding the quantitative implications of this distinction, the following formal treatment of (discrete) scale invariance was largely gleaned from Sornette (1998) and Abed-Pour et al. (2003).
An $x$-dependent observable $O$ is scale-invariant under arbitrary magnification $\lambda x$ if a number $O(x)$ exists that satisfies:

$$O(x) = \mu O(\lambda x)$$

yielding a power law:

$$O(x) = C x^\alpha$$

with exponent $\alpha$:

$$\alpha = \frac{\log \mu}{\log \lambda}$$

Power laws are the hallmark of continuous scale invariance, since the ratio:

$$\frac{O(\lambda x)}{O(x)}$$

does not itself depend on $x$. Note that combining [7] rearranged:

$$\frac{O(x)}{O(\lambda x)} = \frac{\mu O(\lambda x)}{\mu O(x)}$$

with [10] yields:

$$1 = \mu \lambda^\alpha.$$  

Scale invariance is a fractal property; it encodes geometrical self-similarity of a set divisible into subsets that (approximately) resemble the whole over a range of scales. Interestingly, many natural fractals obey scale invariance only for specific choices of magnification (or resolution). This so-called discrete scale invariance is thus a weaker form of scale invariance, describing parts self-similar to the whole only at resolutions equal to some multiple of a characteristic scale. Obviously, such discreteness in scaling will alter the simple power-law relationship as expressed in (8). Given some characteristic scale. Obviously, such discreteness in scaling will alter the simple power-law relationship as expressed in (8). Given some magnification factor $x$, call $N_s(n)$ the number of discernable subsets at magnification $n$. In the earlier, continuous case, when the magnification increases by a factor $\lambda$, $N_s(n)$ increases by a factor $\mu$, regardless of absolute scale. The ratio between the change in discernable elements and the change in resolution can be expressed through the fractal capacity dimension $D_c$, which can be defined:

$$D_c = \lim_{x \to \infty} \frac{\log N_s(n)}{\log x} = \lim_{x \to 0} \frac{\log N_s(n)}{\log x}.$$  

In the discrete case, however, if magnification increases continuously over an interval from $x = \lambda^p$ to $x = \lambda^{p+1}$, the number of discernable subsets will increase at these specific points only, while remaining constant over the interval between these points, causing the capacity dimension to decrease in-between successive increments. For continuous values of $x$, the scaling relationship thereby becomes modulated:

$$N_s(n) = N_s(n)x^{D_c} p \left( \frac{\log x}{\log \lambda} \right)$$

with $P$ a function of period one. This encodes the lacunarity of the fractal structure, and the more general form of (8) analogously becomes:

$$O(x) = C x^\alpha p \left( \frac{\log x}{\log \lambda} \right).$$

which can be expressed alternatively as a phasor Fourier series. A phasor is here defined as a complex exponential function of time:

$$x(t) = a \exp[i(\omega t + \theta)],$$

that is, a vector in the complex plane with length $a$ and angle $(\omega t + \theta)$ at time $t$, that rotates with fixed angular frequency $\omega = 2\pi f$ and phase $\theta$. Given Euler's identity:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

the real part of a phasor can be written:

$$x(t) = \Re[a \exp[i(\omega t + \theta)]] = \Re[a \cos(\omega t + \theta) + i \sin(\omega t + \theta)] = a \cos(\omega t + \theta).$$

An even Fourier series (with wave number index $k$) can thus be represented as a phasor sum:

$$x(t) = \sum_{n=-\infty}^{\infty} a_k \exp[i(k \omega t + \theta_k)] = \sum_{n=-\infty}^{\infty} a_k \exp[i(\omega n t + \theta_k)].$$

By expanding $P$ in (15) into a Fourier series:

$$P \left( \frac{\log x}{\log \lambda} \right) = \sum_{n=-\infty}^{\infty} a_k \exp \left[ i2\pi n \left( \frac{\log x}{\log \lambda} \right) \right]$$

$O(x)$ becomes a sum of power laws with $n$ an arbitrary integer and an infinite set of discrete complex exponents. Neglecting (for now) the higher terms in the Fourier expansion, and setting $a_1 = a_{-1}$ to ensure a real solution, and recalling (12):

$$1 = \mu \lambda^\alpha \exp[i2\pi n].$$

the power-law exponent $\alpha$ can be restated:

$$\alpha = \frac{\log \mu}{\log \lambda} + i \left( \frac{2\pi n}{\log \lambda} \right),$$

through which continuous scale invariance is recovered for the special case $n = 0$. The complex solution of $n \neq 0$ implies a characteristic scaling factor $\lambda$, expressed as a log-periodic correction to pure power-law scaling. The empirical detection of discrete scale invariance will, therefore, entail analysis of power-law residuals in terms of a (series of) coherent modulation(s).

The triadic Cantor set provides a simple example of log-periodic scaling (Fig. 5, left-hand panel, top right). This well-known fractal is generated from a single line segment by iterative removal of the middle third, leaving two smaller segments one-third in length. Consequently, for each increase in observational resolution by a factor

![Figure 5](https://academic.oup.com/gji/article-abstract/169/2/646/625332/646)

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three, the number of discernible elements doubles. Assume first that
this division is re-iterated a sufficient number of times to generate
a substantial number of segments (lower bounds on data set size
for successful DSI analysis are discussed below) and, secondly, that
each segment contributes one length datum to the set. The compiled
lengths are subsequently size-ranked (original segment = rank 1,
the next two segments span ranks 2–3, the next four segments span
ranks 4–7, and so on). Plotted on a log–log scale of rank versus seg-
ment length (Fig. 5 left-hand panel), the resulting power-law slope
\( \alpha \) is, to a first approximation:

\[
\alpha = -\frac{\log 2}{\log 3}
\]  

(23)

However, the actual point distribution of sizes consists of a
number of distinct stratae along the power-law slope. Following
(22), the power-law exponent can be restated more accurately as:

\[
\alpha = -\frac{\log 2}{\log 3} + \frac{2\pi n}{\log 3}
\]  

(24)

with the imaginary part directly controlled by the preferred scal-
ing ratio three under which the set is exactly self-similar. Note that
this ‘pure’ fractal is artificial in consisting only of stratae, and
could alternatively be described by a monotonically descending step
function.

Contrastingly, natural observables, being affected by noise, mea-
surement error, and external processes, exhibit intermediate sizes
(and durations) everywhere along the observed range. This is il-
lustrated in the right-hand panel of Fig. 5, where all segment
sizes have been contaminated with 30 per cent white noise. In-
stead of the points forming distinct stratae, they now cover much
of the intermediary range along the modulation. However, the
statistical significance of the captured DSI signal in the noise-
contaminated Cantor set is hardly affected, even though the quantifi-
cation of the preferred length scales is degraded slightly by the added
noise.

Instead of attempting to recognize particular horizontal subsets
of points along the best-fitting power law, preferred length scales
are here identified through specific nodes (i.e. the near-horizontal
intersections of modulation and power law, defined below), based
on the wave function that best fits all points, a far more robust
approach. Admittedly, mapping a superposed modulation, rather
than a step function, may introduce slight artefacts when applied
to synthetic fractals, in that the resulting function may no longer
be monotonically descending. However, not only is a modulation a
more appropriate choice, both in an empirical and a formal sense
(e.g. see 15 and 22), but even for artificial cases such as the Cantor
set it correctly identifies the preferred scales, in that each strata in
Fig. 5 coincides exactly with a near-horizontal intersection of power
law and modulation.

Returning to stage one of the bootstrap analysis of geomagnetic
data, Fig. 4 plots the departure (data points) from a pure power law
(the straight sloping line on a log–log scale) for the three sample
sets introduced earlier. In addition to the saw-wave-like appearance
of set A, all panels show coherent modulated residuals, followed
in panels B–C by a cut-off towards the right. Since these residuals
are unequally spaced, traditional spectral techniques such as Fourier
are inappropriate. An alternative is provided by the Lomb-Scargle
normalized periodogram (Lomb 1976, Press et al. 1997), which
expresses spectral power as a function of angular frequency \( \omega \), based
upon the actual time of measurement \( t \):

\[
P(\omega) = \frac{1}{2\sigma^2} \left\{ \frac{\sum_{k} (T_k - \bar{T}) \cos \omega(t_k - \tau(\omega))}{\sum_{k} \cos^2 \omega(t_k - \tau(\omega))} \right. \\
\left. + \frac{\sum_{k} (T_k - \bar{T}) \sin \omega(t_k - \tau(\omega))}{\sum_{k} \sin^2 \omega(t_k - \tau(\omega))} \right\}
\]  

(25)

with \( \bar{T} \) and \( \sigma^2 \) representing data mean and variance, respectively,
and the offset \( \tau(\omega) \) defined by:

\[
\tau(\omega) = \frac{1}{2\omega} \tan^{-1} \frac{\sum_{k} \sin 2\omega t_k}{\sum_{k} \cos 2\omega t_k}
\]  

(26)

The resulting best-fitting modulations have likewise been drawn in
Fig. 4. As a by-product, the periodogram algorithm yields the like-
lihood of unwarranted attribution to any particular frequency (the
type II-error, a tiny value indicating a significant periodic signal). If a
frequency of sufficient significance is obtained (in this case, 0.1 was
used as upper bound for acceptance), the remaining best-fitting wave
parameters (phase and amplitude) can be derived straightforwardly
through standard methods such as matrix inversion. Whereas the
DSI frequency quantifies the logperiodicity of the preferred scales,
the phase constitutes a time normalization used to determine their
coordinates; the amplitude can be interpreted as a measure of the
intensity of symmetry-breaking (i.e. the extent of the departure from
pure power-law scaling).

In the ranked representation of Fig. 4, the first node at which the
modulation intersects the power law near-horizontally equals the
highest observed scaling strata, that is, the largest observed charac-
teristic duration for that particular threshold. Let

\[
X_0 = \min \{ \log_{10}(x) \}
\]

\[
X_{\text{range}} = \max \{ \log_{10}(x) \} - X_0.
\]  

(27)

Within the log–log domain, given power-law intercept \( C \) and slope
\( \alpha \) and modulation parameters frequency \( f \), phase \( \phi \), and amplitude
\( \beta \), and recalling eqs (15), (18), (21) and (22), DSI can be expressed as:

\[
Y = C + \alpha x + \beta \cos \left( \phi + 2\pi \frac{\log_{10}(x) - X_0}{X_{\text{range}}} \right)
\]  

(28)

which allows the \( \log_{10}(x) \) values of the two nodes \( N_1 \) and \( N_2 \) within
one wavelength to be defined:

\[
N_1 = X_0 + (0.25 - f/\beta)(X_{\text{range}}/f)
\]

\[
N_2 = X_0 + (0.75 - f/\beta)(X_{\text{range}}/f)
\]  

(29)

which, when plugged into (28) yield the temporal scaling coor-
dinate. Other nodes within observed range are found by adding
terger values to the numeric term (0.25 c.q. 0.75). Only one of
these two nodes represents a strata; the other one coincides with
the near-vertical intersection of modulation and power law, that is,
the least-preferred scales. Proper identification relies upon the com-
bined signs of power-law slope and amplitude (both non-zero by
definition). Given positive amplitude, the strata is associated with
\( N_1 \) for a positive slope, and with \( N_2 \) for a negative slope; given
negative amplitude, these associations are reversed. Lastly, when
multiple stratae are found, the largest preferred scale within ob-
served range is the strata closest to \( X_0 \) for a negative slope, but
closest to \( X_0 + X_{\text{range}} \) for a positive slope.

A characteristic scale thus constitutes an adjustment to pure
power-law scaling everywhere (a time-invariant feature). As previ-
ously stated, the notion has nothing to do with traditional statistical
measures of central tendency such as mode, median, or (moving or
global) average, nor with peaks in the time-series’ Fourier spectrum.
Although the remaining points plotted in Fig. 6 (left-hand panel) do exhibit coherence, two distinct elongated clusters can be made out, but these confusingly overlap at the high end of the timescale. This is a problem of ranking. Unlike the time axis, the $x$-coordinate of rank is geophysically meaningless. The reader may wish to convince themselves by log–log plotting rank versus size of Cenozoic chron, and comparing it with a plot containing Cenozoic and Mesozoic chron combined. Both display a similar modulated power law (with different slope), but the latter one is horizontally stretched with regard to the former, since the data span a larger range of ranks.

To solve the problem of overlap in Fig. 6, a new, physically meaningful $x$-coordinate is introduced, that of the threshold $\lambda$, associated with each subset. The result is depicted on the right-hand side (labelled ‘after’); the two subsets now run parallel. This alignment stems from the fact that for a minority of thresholds ($N = 328$), the resulting spacing of intervals has missed the largest typical durations, capturing instead the next (log-periodic) preferred scaling level. Rather than resorting to some artificial means of upscaling beyond the observed range to the next-higher node, a sufficiently large majority ($N = 792$) of auxiliary data sets that do map this point allows the lower ones to be simply discarded. The remainder constitutes the final meta-data set, completing the first DSI pass of the analysis (i.e. stage two), as well as the bulk of the computing. Unlike the sets of ranked intervals before, the metadata re-associate the timescale with the original observable, in the form of thresholds expressed in the original units of measurement.

This step in the analysis highlights the fundamental connection between DSI as observed in artificial, size-ranked subsets, and the main empirical DSI signal of interest that identifies characteristic-scale pairs of time and absolute fluctuation magnitude. The relationship is schematically represented in Fig. 7. It depicts the (much simpler) forward problem: given a known modulation affecting the power-law probability distribution of threshold versus duration (left side), the imposition of a particular threshold (the vertical dashed line) allows a small window of durations to be extracted (right

The necessary and sufficient conditions for extracting a characteristic timescale (in terms of DSI) can be summarized as: (1) the data span several orders of magnitude in fluctuations and time and (2) a statistically significant power law plus statistically significant modulation are quantified.

The entire procedure is repeated for each auxiliary data set, and any that lack sufficient statistical significance in either the power-law distribution fit, DSI distribution fit, or the periodogram type II-error are rejected. Furthermore, chi-squared residuals are calculated for both fits, and only those sets whose DSI fit outperforms the power law (i.e. the former’s chi-squared residuals are smaller than the latter’s) are accepted. Other quality criteria to reject outliers are described below. In the present case, out of an initial 2000 auxiliary sets, only a small majority of 1188 passed all tests, and these have been plotted (as rank versus duration) in the left-hand panel of Fig. 6 (labelled ‘before’). The lowest durations (roughly below 10 kyr) are clearly heavily contaminated by noise. Being associated with the smallest thresholds of absolute change, inspection of Fig. 3 reveals that all of these points occur prior to the peak. Since this curve represents intervals larger than interpoint distance, the auxiliary sets before the peak still contain substantial gaps where unit intervals have been discarded, with concomitant loss of signal. In other words, these erroneous estimates stem from undersampling, to be distinguished from empirical uncertainties in the original data (treated further on). A clear-cut solution readily dispenses with every noisy set, by simply discarding all 68 points to the left of the peak in Fig. 3. This figure also serves to confirm that the choice of threshold range was adequate for the given fluctuations, as both extremes of the curve have such a low number of intervals that no statistically significant results can reasonably be expected in subsequent analysis.
side). If the threshold were smaller, then some of the larger intervals would be cut into two or more shorter ones over which that smaller threshold was already exceeded. Conversely, if the threshold were larger, some of the shortest durations would become subsumed in a longer interval. However, regardless of threshold choice, the stationary statistics that describe the probability distribution of all durations remains applicable within any (sufficiently large) window. In other words, even a truncated distribution curve will still be affected by the presence of preferred scales in the underlying dynamics. This constitutes the key principle that underpins bootstrapped DSI analysis.

The inverse problem, of reconstructing the main DSI curve of interest from a collection of user-defined subsets, based on a time-series of limited accuracy and known error margins (see below), is of course much harder. Nevertheless, application of DSI analysis to each subset will yield an estimate (including quantified statistical significance) of the most reliable part of its modulation, that is, the stratae, the near-horizontal parts of the curve. These exhibit the largest positive departure from power law-based likelihood, and are thus most likely to be recovered statistically, even given input time-series contaminated with noise and error. Taking this highest ‘fixed’ point along the modulation of each subset, a meta-sample of (statistically significant) durations is gradually compiled. The pervasive nature of DSI ensures that their basic power-law probabilities (when plotted against their original subset threshold) are adjusted upward and downward, which at long last allows the main curve itself to be quantified.

Stage three of the bootstrap process thus revisits the same techniques applied earlier to auxiliary data sets, presently employed to examine the metadata for evidence of (discrete) scale invariance. At no stage in the proceedings is any kind of data binning performed, which could degrade the DSI signal. The main difference between auxiliary and metadata (other than ranking being replaced by threshold) is a sign change of the power-law slope, negative in the former case and positive in the latter. Again, a power law is clearly attested in the data (see Fig. 8, left-hand panel), and despite having a small amplitude, a highly significant low-frequency fit is obtained (see Fig. 8 inset; type II-error: 3.967e-66), yielding a mapped scaling strata at a threshold of about $0.1 \times 10^{23}$ A m$^2$, associated with a typical duration of about $10^{7.745}$ yr or ca. 55.6 kyr.

This brings up the question whether the SINT2000 data set, a ‘tuned’ composite of several different palaeomagnetic ocean cores, is sufficiently accurate to support such a value, given known remaining noise levels due to rock magnetic and measurement errors in palaeointensity, and age errors due to finite sampling, interpolation approximations, variations in sedimentation rates, and incomplete or inaccurate tie point information. Extensive simulation studies on the various errors plaguing these measurements (Guyodo & Channell 2002; McMillan et al. 2004) have found that age errors are the dominant source of decorrelation of multiple stacked cores. Although Guyodo & Channell (2002) optimistically estimate 4 kyr as the shortest achievable wavelength, more conservative estimates by Valet et al. (2005) and McMillan et al. (2004) place the current recovery cut-off at about 20 kyr, that is, less than half the found DSI result.

The latter value is supported by a series of tests in which increasing percentages of white noise were added to either of SINT2000’s coordinates (time or dipole moment), after which the entire bootstrap process was repeated to examine how the outcome was affected. The introduced temporal distortions become noticeable at noise magnitudes of 10–20 kyr, independently validating the findings of McMillan et al. (2004). Secondly, white noise contamination of the dipole moment fluctuations seriously degraded the DSI signal only once it reached the same order of magnitude as the overall mean VADM. These results confirm that the found characteristic scale of circa 56 kyr is robust and lies well above the minimum criteria imposed by the remaining empirical errors. A different, independent source of confirmation is introduced in the next section. Finally,
the entire procedure was repeated for temporal windows 2-1 and 1-0 Ma \( (N = 1000) \) yielding highly similar results, differing from the original DSI parameters by about 5 per cent or less.

**COMPARING CHARACTERISTIC TIMESCALES**

Because discrete scale invariance is a log-periodic phenomenon, it is possible to compare predicted and observed characteristic scales in the time domain between data sets, even if the temporal range and/or resolution of the observables differ. For earlier work along similar lines (detecting self-similarity in the form of 1/f power spectra in dipole moment fluctuations generating power-law statistics in chron intervals), see Pelletier (1999). Expanding the current investigation to durations of millions of years, DSI analysis of the ranked Mesozoic and Cenozoic reversal intervals yields a much longer preferred timescale, either \( 10^5 \) or \( 10^6 \) yr, depending on whether the Cretaceous superchron is included or excluded, respectively. The results for dipole intensity and chron intervals (superchron included) are illustrated in Fig. 8. Note again that because the chron data set is a series of dated intervals rather than an equidistantly sampled time-series, bootstrap DSI analysis cannot be performed, and a link between the observed preferred duration and some physically meaningful observable of energy cannot be made in this case.

Table 2 lists observed and predicted scaling strata for dipole intensity fluctuations, reversal intervals without the superchron, and reversal intervals with the superchron included. In the first column are the results obtained from secular variation data, following a nearly identical procedure as for dipole intensity. The only differences with the latter were using multiple time-series of local, dipole-detrended intensity (instead of a single one of dipole moment), and exploration of a limited number of approximately logarithmically (rather than linearly) spaced thresholds. Due to modest computing resources and a much larger range of fluctuations than observed in SINT2000, the implementation of a full-range, small-step exploration of thresholds regrettably proved prohibitively expensive, since the analysis would have to be performed 5400 times (three orthogonal geomagnetic vectors for each of 1800 grid cells) per threshold. On the positive side, those thresholds that were explored yielded highly consistent results with small error bounds, based on very large samples.

Several different approaches can be taken to obtain a global average from a field map like gyrm1, providing that the thresholds applied to each time-series for all locations are identical. DSI analysis of a multidimensional data set, therefore, requires an extra preprocessing pass over all time-series to determine the global range of fluctuations, from which a single, sufficiently large set of thresholds is derived. The results from bootstrap stage two are subsequently collected for all time-series, prior to averaging all statistically significant estimates for each threshold, to generate standard metadata (one timescale per threshold per point). More sophisticated schemes can first apply cluster analysis to the data cloud of stage two results to eliminate outliers. Such data clouds are produced by spatially heterogeneous DSI, a subject to be explored in future work. For the current purpose of comparison with other, global geomagnetic data sets, attention is here restricted to the global scaling profile.

As can be gleaned from Table 2, each observed scaling strata is accompanied by a two standard deviation error margin that is smallest within the observed range and increases linearly with distance therefrom. Note that the rms difference between the DSI fit and the empirical data represents the amplitude error, and cannot be used to obtain an estimate of the confidence of the frequency fit. Instead, recourse is taken to the commonly used technique of bootstrapped Monte Carlo (Press et al. 1997), made possible because the Lomb periodogram algorithm is insensitive to the order of the points. At least several hundred such simulations should be run for each data set; in this particular example, \( N = 1000 \) set was chosen. Plotting the fitted frequencies as histograms provides an initial impression of their distribution, central tendency, and spread, which can be further quantified using probability distribution fitting tools as offered in statistical software packages. One should expect approximately Gaussian-distributed errors around a mean within one standard deviation away from the frequency obtained from analysing the original, full set of points. Poorly constrained results may arise if the source data set is smaller than one hundred points, which constitutes another reason to define at least several hundred thresholds in the bootstrap DSI process, if possible. Note furthermore that a departure from a Gaussian probability distribution would not be disastrous; it merely indicates that the calculation of the standard deviation should be adjusted to reflect the observed distribution of errors. Once obtained, the standard deviation \( \sigma \) of the frequency estimate has to be converted to a standard deviation \( \sigma_f \) along the time axis for each of the various stratae found. Since the modulation is superposed upon a power-law baseline with slope \( \alpha \), this involves:

\[
\sigma_f = \sin\sin^{-1}(x) \sigma
\]

and for any point \( T \), with x-coordinate outside the observed range between \( T_{\text{min}} \) and \( T_{\text{max}} \), the temporal standard deviation is adjusted through:

\[
\sigma_f|_c = (1 + R) \sigma_f
\]

Table 2. Characteristic geomagnetic timescales as observed and predicted by discrete scale invariance, in powers of 10 yr with 2\( \sigma \) error margins.

<table>
<thead>
<tr>
<th>Secular variation</th>
<th>Dipole intensity fluctuations</th>
<th>Reversal intervals ((CNS\ included))</th>
<th>Reversal intervals ((CNS\ excluded))</th>
</tr>
</thead>
<tbody>
<tr>
<td>*0.502827 ± 0.0463</td>
<td>1.627087 ± 0.0938</td>
<td>0.778666 ± 0.6207</td>
<td>0.642603 ± 0.5829</td>
</tr>
<tr>
<td>*1.328296 ± 0.0457</td>
<td>*4.744981 ± 0.0356</td>
<td>2.571215 ± 0.4493</td>
<td>2.575871 ± 0.4016</td>
</tr>
<tr>
<td>2.153765 ± 0.0648</td>
<td>3.186034 ± 0.05811</td>
<td>4.363764 ± 0.2779</td>
<td>4.509139 ± 0.2203</td>
</tr>
<tr>
<td>2.979234 ± 0.0975</td>
<td>*6.156313 ± 0.2228</td>
<td>6.303929 ± 0.0488</td>
<td>*6.442407 ± 0.1632</td>
</tr>
<tr>
<td>3.804703 ± 0.1302</td>
<td>6.039299 ± 0.0488</td>
<td>7.862876 ± 0.0844</td>
<td>7.948861 ± 0.2877</td>
</tr>
</tbody>
</table>

Note: * = observed; CNS = Cretaceous Normal Superchron.
with
\[
R = \frac{\min(|T_{\min} - T_x|, |T_{\max} - T_x|)}{|T_{\max} - T_{\min}|}.
\]

Examining the contents of Table 2, and limiting attention initially to chron, there are a number of aspects worth noting. First, the in or exclusion of the superchron has only a minor effect on reversal-based predicted durations of shorter preferred timescales. A more pronounced difference in the observed strata (6.16 versus 6.44) is largely due to a change in the (log-weighted) power-law slope (from $-0.92$ with, to $-0.69$ without the superchron). Secondly, when compared with the SINT2000 metadata, the stratae at the high end (large timescales) align quite well, but when extrapolated down to timescales of centuries or less, a shift is becoming more apparent. Thirdly, in terms of cross-validation, the observed strata in SINT2000 is predicted somewhat better by reversals when excluding the superchron, whereas vice versa, the dipole intensity data-based prediction of the characteristic reversal timescale falls almost exactly between the two reversal data alternatives. Extending the temporal window to encompass the smaller timescales of secular variation, the discrepancy between dipole intensity fluctuations and chron in predicted short-term scales can be seen to map reasonably well onto successive secular variation stratae, although in terms of cross-validation, the latter are only moderately well predicted by any of the others ($0.50$ versus $0.64$ and $0.78$, and $1.33$ versus $1.63$). By contrast, secular variation itself does far better predicting the observed longer-term scales, and the same can be stated more generally regarding all bottom-up predictions (i.e. based upon DSI modulations extrapolated to larger timescales than observed) versus top-down ones (i.e. based upon DSI modulations extrapolated to smaller timescales than observed). Yet despite a certain degree of coherence between these three independent data sets, the worrying fact remains that the power-law modulation observed in secular variation (based on a vastly larger sample size of $N = 61$ 166 time-series with statistically significant DSI) has a much higher frequency (i.e. more scaling stratae) than the palaeomagnetic sets. Could additional, still undisclosed information potentially alleviate this discrepancy?

A closer look at Fig. 8 indicates that this may indeed be the case. Just as discrete scale invariance was revealed by spectral analysis of power-law residuals, the remaining residuals of power law plus modulation can themselves also be subjected to spectral analysis. Theoretically, it could be argued that a single periodogram suffices to obtain all relevant frequencies in a single pass. In practice, however, it can be very hard to pick additional periodogram peaks when the fundamental frequency is poorly constrained, and a far clearer, ‘cleaner’ periodogram is obtained (at relatively small extra cost) by running a separate Lomb analysis of the DSI residuals proper. These residuals have been plotted in Fig. 9 for dipole intensity fluctuations on the left, and for reversal intervals on the right; the horizontal baseline represents the subtracted primary DSI signal. A statistically significant secondary modulation is evident in both sets, even though the dipole intensity DSI residuals are subject to increasing noise for larger thresholds (due to smaller sample size, right-hand side). These secondary signals are not unexpected (recall eq. 20), since discrete scale invariance comprises a theoretically infinite set of discrete complex exponents, of which the higher (smaller-amplitude) terms have up to now been neglected (Sornette 1998).

The residual frequencies are roughly double those of the original modulation, supporting the notion that this analysis simply exposes the second term in the expansion. Note that phase-fitting the residual modulation is performed independently of prior results for the fundamental frequency, so nodes do not necessarily coincide exactly. The fact that they generally do agree quite well with those derived from the primary data lends further support to the hypothesis of pervasive DSI affecting these observables. After calculation of all relevant parameters, the new scaling stratae are listed in Table 3. Here each secular variation scaling strata has a close counterpart in dipole intensity DSI residuals and reversal DSI residuals. Again, the reversal set without the superchron is more congruent with secular variation at the low end of the timescale, whereas its counterpart with superchron does better at the top. Meanwhile, the agreement of the dipole moment residuals with all neighbours has improved markedly.

The two tables additionally shed some light on the long-standing question whether or not superchrons are part of the geodynamo’s ordinary gamut of fluctuations (Hulot & Gallet 2003; Jonkers 2003; Zhu et al. 2004). Regrettably, the current palaeomagnetic record is simply too short to instigate a full DSI analysis of superchron statistics. However, it is possible to study how well the few observed superchrons fit the emerging picture; even though their number is too small for significance levels to be established, qualitative statements are possible as to whether or not their durations are approximatively of the right size agree with the nearest preferred timescale based upon the DSI profile as derived from records spanning shorter timescales.

In Table 2 is shown that when extrapolated to the next-higher strata beyond the one mapped by reversals, all three data sets agree in a remarkably close estimate of a characteristic timescale of circa $10^{7.91}$ yr, somewhat longer than the Kiaman reversed superchron (recent estimates range from $10^{7.69}$ to $10^{7.80}$ yr; Eide & Torsvik 1996; Torcq et al. 1997; Buchan & Chandler 1999; Erwin 1999; Opdyke et al. 2000). In addition, the duration of the Cretaceous superchron ($10^{7.567}$ yr) is predicted quite accurately by both sets of
reversal residuals, and almost exactly by the dipole intensity residuals. In other words, if the series of log-periodic preferred timescales as gleaned from palaeomagnetic residuals can be continued to durations longer than chron, the next characteristic timescale would neatly coincide with the observed duration of the CNS. From the perspective of scale invariance, no reason, therefore, exists to treat superchrons differently from ordinary chron and subchrons, which are not thought to require an external triggering mechanism either (McFadden & Merrill 1995; Sarson & Jones 1999; Coe et al. 2000; Hollerbach 2003; Hulot & Gallet 2003).

The potential implications for the role of the mantle on these very long timescales are noteworthy. Attribution of phenomena such as perceived changes in reversal rate (e.g. prior to, and following the CNS included) (CNS excluded)

Table 3. Characteristic timescales derived from palaeomagnetic residuals compared to secular variation, in powers of $10^3$ yr with $2\sigma$ error margins.

<table>
<thead>
<tr>
<th>Secular variation</th>
<th>Dipole intensity residuals</th>
<th>Reversal residuals (CNS included)</th>
<th>Reversal residuals (CNS excluded)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.502287 \pm 0.0463$</td>
<td>$0.412407 \pm 0.8532$</td>
<td>$0.761987 \pm 0.6216$</td>
<td>$0.542206 \pm 0.7835$</td>
</tr>
<tr>
<td>$1.328296 \pm 0.0457$</td>
<td>$1.061969 \pm 0.7490$</td>
<td>$1.732949 \pm 0.5289$</td>
<td>$1.411077 \pm 0.6756$</td>
</tr>
<tr>
<td>$2.153875 \pm 0.0548$</td>
<td>$2.236191 \pm 0.5040$</td>
<td>$2.703914 \pm 0.4361$</td>
<td>$2.281047 \pm 0.5677$</td>
</tr>
<tr>
<td>$2.979234 \pm 0.0795$</td>
<td>$3.010653 \pm 0.4362$</td>
<td>$3.151018 \pm 0.4598$</td>
<td>$3.020899 \pm 0.3519$</td>
</tr>
<tr>
<td>$3.804703 \pm 0.1302$</td>
<td>$3.660213 \pm 0.3319$</td>
<td>$3.674878 \pm 0.3434$</td>
<td>$3.020899 \pm 0.3519$</td>
</tr>
<tr>
<td>$4.309776 \pm 0.2502$</td>
<td>$4.959337 \pm 0.2502$</td>
<td>$4.645480 \pm 0.2507$</td>
<td>$4.890959 \pm 0.2440$</td>
</tr>
<tr>
<td>$5.455641 \pm 0.1955$</td>
<td>$5.608898 \pm 0.2502$</td>
<td>$5.616805 \pm 0.2225$</td>
<td>$5.760930 \pm 0.2158$</td>
</tr>
<tr>
<td>$6.281111 \pm 0.2282$</td>
<td>$6.258460 \pm 0.3354$</td>
<td>$6.587769 \pm 0.2225$</td>
<td>$6.630901 \pm 0.2158$</td>
</tr>
<tr>
<td>$7.106580 \pm 0.2609$</td>
<td>$6.908898 \pm 0.4396$</td>
<td>$7.557582 \pm 0.5439$</td>
<td>$7.558732 \pm 0.2500$</td>
</tr>
</tbody>
</table>

Note: $* = $ observed; CNS = Cretaceous Normal Superchron [observed duration: $7.567 \log_{10}(yr)$].

TOWARDS A SINGLE GEOMAGNETIC DSI SIGNATURE

The preceding deluge of scaling stratae may have obscured the fact that the various power-law modulations each appear to have rather different frequencies, superposed on power laws with different slopes (e.g. 0.53 for secular variation, 1.26 for dipole intensity fluctuations). Do these discrepancies undermine the fundamental hypothesis of a single underlying process? Naturally, analyses of different phenomena (dipole versus non-dipole part of the field, polarity flips versus dipole intensity changes) would be expected to yield different suites of fluctuation scaling parameters. Nevertheless, if these could be shown to be mere permutations of a single suite, this would lend powerful support for the aforementioned hypothesis.

Returning to the tables, additional features beckon for attention. First, secular variation stratae in Table 2 appear approximately twice as often as do palaeomagnetic ones. Furthermore, predicted preferred palaeomagnetic durations agree much better, both with one another and with secular variation stratae, near their own observed ranges than on historical timescales. By contrast, in Table 3, a palaeomagnetic predicted scale exists within close proximity for each scaling level in secular variation. Nevertheless, strata alignment is still not exact: for three secular variation stratae, dipole fluctuation DSI residuals yield two prospective candidates, and for one pair of secular variation stratae, chron residuals (+CNS) offer only one estimate. Yet overall agreement is much improved. Given the knowledge that DSI is theoretically predicted to exhibit higher frequencies (recall 22), it does not appear unreasonable to investigate the collapse of all observed modulations onto a single DSI signal, consisting of a primary frequency expressed in palaeomagnetic primary data, and a secondary, doubled frequency expressed in secular variation and palaeomagnetic residuals. This hypothesis would imply that all perceived types of geodynamo fluctuations are merely differently scaled surface expressions of the same scale-invariant process, and that a single suite of constraints affects all. This notion is further explored in Jonkers (2007).

Another tantalizing clue in this regard can be gleaned from a comparison of the temporal interscale distances. The log-periodic nature of discrete scale invariance implies that scaling stratae are equidistant from their immediate neighbours on a log–log scale. In the communal temporal domain, the temporal interscale distance (TID, or $\delta_t$) can be determined for any data set for which power-law modulation parameters have been quantified. In analogy to eq. (30), given a wavelength $\delta$ along the power-law slope $\alpha$, the vertical distance $\delta_v$ between successive stratae (here along the time axis, see also Fig. 5, right-hand panel) is derived by:

$$\delta_v = \sin[(\tan^{-1}(\alpha))\delta]$$

and by analogy, the fluctuation interscale distance (FID, i.e. the horizontal displacement between stratae nodes) can be found by...
variances (μ error margins can be improved by incorporating all estimates into compared to those of ranked chrons. The latter's relatively large moreover, double the frequency (half the wavelength) displayed in magnetic residual DSI share the same frequency (wavelength), which is, assumption to be explored is that secular variation and palaeomag-variation and palaeomagnetic residuals, therefore, their TID has to
inputs does not affect the final outcome. A joint temporal interscale paired with the third one, and so on. Note that the particular order of
power-law modulations above and below the observed range, more-
expressed in the original units of measurement. The extrapolation of other major advantage of the bootstrapping process lies in the as-
series that capture different observables on different timescales. The
ance, although computationally expensive for large data sets, can
of magnitude in time, giving rise to a number of system-preferred
timescales, allowing mutual cross-validation. A comparison of tem-
cause the initial departure from power-law statistics, which then reverberates and resonates at other scales along the line. As far as
gemagnetism is concerned, preliminary indications, both from DSI analysis (Jonkers 2007) and geodynamo cascade models (Narteau & Mouël 2005), suggest that this fundamental level might be found at the lower end of the scale, in tiny processes (e.g. of turbulence) that spontaneously recombine at higher levels (emergence) to form an upward cascading series of ever larger coherent units, exhibiting notable jumps and gaps in progressive sizes and lifetimes, that give rise to the witnessed log-periodic profile.

The answer to the ultimate question what constitute the fundamen-
tal scale(s) of the geodynamo will require much additional work, in
theoretical physics generally and fluid dynamics in particular. The
method of bootstrap DSI can make a modest contribution to ad-
cess this quest empirically, by identifying which observed parts of
a complex system are dynamically connected below the empirical
surface. In the geomagnetic case, observed preferred scales might
furthermore be applied as constraints in future geodynamo simula-
tion studies (Christensen & Olson 2003; Wicht & Olson 2004), and
may help guide research into the existence of particular thresholds
affecting the dynamo process. More generally, it is hoped that the
method can be brought to bear on many other complex systems,
both inside and outside of geophysics, in order to bring their hidden
characteristic scales into sharp relief, so as to provide a scaffolding
for an improved quantified understanding.

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ieu Dumberry, Gauthier Halot, Anders Johansen, Mike Kendall, Clément Narteau, Jon Pelletier, Tine Thomas, and Johannes Wicht for comments.

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Table 4. Temporal interscale distance (TID) per data set, with 2σ error margins.

<table>
<thead>
<tr>
<th>Data set</th>
<th>TID ± 2σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Secular variation</td>
<td>1 × 0.825468 ± 0.0457</td>
</tr>
<tr>
<td>Dipole intensity fluctuation</td>
<td>2 × 0.779474 ± 0.0356</td>
</tr>
<tr>
<td>Chrons (+CNS)</td>
<td>2 × 0.896274 ± 0.2228</td>
</tr>
<tr>
<td>Chrons (–CNS)</td>
<td>2 × 0.966634 ± 0.1632</td>
</tr>
<tr>
<td>Dipole DSI residuals</td>
<td>1 × 0.649560 ± 0.2502</td>
</tr>
<tr>
<td>Chron DSI residuals (+CNS)</td>
<td>1 × 0.970964 ± 0.2225</td>
</tr>
<tr>
<td>Chron DSI residuals (–CNS)</td>
<td>1 × 0.869971 ± 0.2158</td>
</tr>
<tr>
<td>Bayesian combined estimate</td>
<td>1 × 0.808496 ± 0.0135</td>
</tr>
</tbody>
</table>

Note: CNS = Cretaceous Normal Superchron.