Note on Models of Multiple Meson Production at Extremely High Energy*

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Takagi's model and Heisenberg's model (modified with regard to the total cross section) for multiple meson production are examined, and they are found to conform to two sources of experimental information on the properties of secondary particles in jet showers: their composition and the distribution of their transverse momenta. The main aspects of these models, as well as those of the Landau model, are discussed.

§ 1. Introduction and summary

1·1. In a previous note\(^1\) it has been remarked that two sources of experimental information concerning the properties of secondary particles can be regarded as important criteria for the theory of multiple production. The first is the ratio of heavy to light mesons,\(^2\) and the second is the distribution of their transverse momenta.\(^3\) Both of them appear to indicate the necessity of modifying Fermi's original treatment\(^4\) in favour of Landau's hydrodynamical theory,\(^5\)(6) and this theory implies the existence of a strong pion-pion interaction at a lower energy, say, near 1 Gev.

This way of reasoning, however, will not be the only one possible. The main purpose of the present note is to examine two plausible models of multiple production, different from both Fermi's and Landau's and to see whether they satisfy the criteria previously mentioned.

1·2. One of the promising modifications of the Fermi theory of multiple production was proposed by Takagi\(^7\) and his idea was further developed by Kraushaar and Marks.\(^8\) According to their model, the two colliding nucleons exchange a certain amount of momentum during the collision and become excited, and then each of them independently decays and emits secondary particles. The statistical treatment similar to Fermi's is applied to this second stage of particle formation; the equilibrium volume consists now of two separate spheres, each being of the radius of the order of \(1/\mu\), the Compton wave length of the pion\(^\ast\ast\), in the respective center of mass systems of these two excited

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** Units in which \(\hbar = c = k = 1\) are used throughout this note. (\(k\) is the Boltzmann constant.)
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Recent experimental data\(^9\) suggest that most of the high energy collisions are not completely inelastic; that is to say, an appreciable part of the energy available does not go over to the secondary particles, but remains in the original particle. One advantage of the above mentioned model is that it can take this circumstance into account in a rather natural way.

This model we shall investigate in \(\S\ 2\). The results can be reconciled with the two criteria referred to above, if we introduce \textit{a priori} a certain inelasticity factor, which does not depend much on the incident energy.

1.3. In \(\S\ 3\) we shall examine the predictions of the Heisenberg theory. Even before the discovery of mesons, Heisenberg has been discussing in a variety of ways the multiple formation of secondary particles by nuclear collisions at very high energy.\(^{13(14)}\) His fundamental point of view is that the nucleon-meson system is governed by a wave equation involving strong non-linearity,\(^*\) and as a characteristic feature of the latter a conspicuous energy dissipation, which corresponds to the formation of a large number of low energy mesons, takes place during the expansion process of the concentrated nucleon-meson cloud.

Many events have been recorded in nuclear emulsions with a large number of secondary particles, though most of them belong to a relatively lower energy. This fact may be regarded as supporting Heisenberg’s point of view.\(^{15(16)}\) At extremely high energy, on the other hand, his theory seems to give either too large a number of secondary particles or too large a cross section.\(^{20(21)}\)

The latest version of his theory has been worked out semi-quantitatively and can be compared with experimental data. In the present note I shall study this particular version of his theory, excepting, however, his arguments concerning the impact parameter, inelasticity and total cross section.\(^*\) Indeed this part of his work has been criticized because it appears to involve an extrapolation of the classical mechanics concept beyond its limit of applicability.

It appears possible to separate logically this particular part of Heisenberg’s work (for instance, III c) of the reference 14) from the rest of it. I shall assume in this note therefore that the total cross section of nucleon-nucleon collisions remains at a constant value \(\pi/\mu^2\), instead of increasing logarithmically with energy as in the original work.\(^*\)

\* The definition of the strong non-linearity in Heisenberg’s works appears somewhat unclear. It would be necessary to assume not only the presence of a parameter of the dimension of a positive power of a length, but also the presence of a derivative of the wave function in the non-linear term.

\*\* Heisenberg extends namely the integration with respect to the impact parameter \(r\) to such a value as to correspond to the minimum inelasticity: \(r_{\text{min}} = -1/\mu \log \tilde{\mu}_{\text{th}}\) with \(\tilde{\mu}_{\text{th}} = 2\tilde{p}_0/\tilde{W}\). Here \(\tilde{p}_0\) is the average energy of the secondary mesons in the center of mass system, which, in his theory, depends on the incident energy \(W\) in the center of mass system only logarithmically. Thus the total cross section turns out to be

\[
\sigma = \frac{\pi}{\mu^2} \left[ \log \left( \frac{W}{2\tilde{p}_0} \right) \right]^2.
\]  

\(1.1\)

\*\*\* See the expression in the above footnote.
This alteration will make it possible to apply Heisenberg’s theory to the case of nucleon-nucleus collisions at extremely high energy.*

The results can again be regarded as consistent with the two criteria referred to.

1·4. In the final section, the main aspects of the three models for Landau, Takagi and Heisenberg are discussed. First of all, they differ in the prediction of the total number of secondary particles as a function of incident energy. If we assume an almost constant average inelasticity over a wide region of energy, we should be able to discriminate between them. Another difference is that Heisenberg’s and Landau’s models lead to the presence of pion-pion interaction, while Takagi’s model does not.

Concerning the mechanism of partially elastic collisions, there are at least two points of view possible: spatial localization of the self-energy of a nucleon and concentration of a large amount of energy on a small number of secondary particles.

§ 2. Predictions of Takagi’s model

2·1. First let us give a somewhat simplified formulation of multiple production in Takagi’s model, so as to see the qualitative aspects of this picture. The incident energy in the laboratory system will be denoted by \( E_0 \), and the total energy in the center of mass system by \( W \). The momentum carried away by each of the clouds in the center of mass system will be denoted by \( P \). We also introduce a parameter, \( M^* \), the fictitious mass of the excited nucleon-meson system,

\[
M^* = (W/2)^2 - P^2. \tag{2·1}
\]

In the case of completely inelastic collisions, when the inelasticity \( \eta = 1 \), the final momentum \( P \) vanishes and

\[
M^* = W/2. \tag{2·2}
\]

When \( \eta \leq 0.9 \), on the other hand, \( \eta \) and \( M^* \) are connected by an approximate relation

\[
\eta \approx 1 - M/M^*, \quad \text{for} \quad E_0 > 10^8 M, \tag{2·3}
\]

\( M \) being the nucleon mass. (See the appendix.)

The energy available for the emission of secondary particles in each cloud in its own center of mass system is equal to

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* It is to be remarked that also in this modified Heisenberg theory, the inelasticity has to be introduced without reference to the impact parameter, otherwise it would be unreasonable to assume as small a value for inelasticity as required from experiments in the case of nucleon-nucleus collision, because in the “tunnel” inside the nuclear matter (jet model) no extremely peripheral nucleon-nucleon collision can take place except the collision with a peripheral nucleon in the nucleus.

An attempt has been made to apply Heisenberg’s theory without any modification to the individual collision treatment of the jet model.22) This procedure does not seem to be quite consistent, however, because the concept of tunnel penetration (of the radius \( 1/\mu \)) and the small inelasticity (which implies a very large impact parameter in the original form of the theory) are used at the same time.

** In the original article of Takagi there is still another parameter available. Also our definition of “fictitious mass” is different from that given by Takagi.
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\[ M^* - M = \begin{cases} \left(\frac{W}{2}\right) - M & \text{for } \eta = 1, \\ \frac{\eta}{(1-\eta)} M & \text{for } \eta \leq 0.9, \end{cases} \] (2.4)

so that the energy density \( \varepsilon \) is given by*\n
\[ \varepsilon = \begin{cases} \left(\frac{W}{2}\right) - M \mu^3, & \text{for } \eta = 1, \\ \frac{\eta}{(1-\eta)} M \mu^3, & \text{for } \eta \leq 0.9. \end{cases} \] (2.5)

Notice that in the case of inelasticity not very close to 1, the energy or energy density available for the secondary particle emission does not depend on the initial energy explicitly.** The energy density determines the corresponding temperature, which, in turn, gives the number of secondary particles.

Table 2.1. Equilibrium temperatures (in \( \mu \)) in Fermi's and Takagi's models for nucleon-nucleon collision.*

| Incident energy (in \( M \)) | Temperature in Fermi model** | Temperature in Takagi model |  \\
|-----------------------------|-----------------------------|-----------------------------|  \\
| \( r = 0 \) \( r = 0.9/\mu \) | \( \eta = 1 \) \( \eta = 0.5 \) \( \eta = 0.1 \) | \( \eta = 1 \) \( \eta = 0.5 \) \( \eta = 0.1 \) |  \\
| \( 10^3 \) | 6 | 4 | 2.5 | 1.3 | 0.9 |
| \( 10^4 \) | 11 | 8 | 3.3 | 1.3 | 0.9 |
| \( 10^5 \) | 19 | 14 | 4.4 | 1.3 | 0.9 |

* The temperatures have been estimated by using the formulae (2.5), (2.9) and (2.10).
** In Fermi's case the results for head-on collision (impact parameter \( r = 0 \)) and peripheral collision (\( r = 0.9/\mu \)) are given.

As is seen in table 2.1 the equilibrium temperature in the Takagi model is much reduced compared with the original Fermi model. This is due to two reasons. i) A part of the available energy is used as mass motion and not as thermal motion. ii) The equilibrium volume is not Lorentz-contracted. The conspicuous feature of this model is, as was pointed out above, that one has a definite temperature if one specifies an inelasticity not very close to one.

2-2. The distribution of the transverse momenta of secondary particles in this model depends on the following two factors. First, the transverse component of the momentum exchange between the two nucleons will give the mean value of transverse momenta of the particles in the diffuse and narrow cones. Second, the thermal motion corresponding to the equilibrium temperature will be superposed on this average value. The first term

* For simplicity the volume is taken to be equal to \( 1/\mu^3 \),
** This can be understood as follows. Let us assume, for the moment, that a metastable excited state of the nucleon with energy \( kM \) exists, and further, that the production of secondary particles takes place exclusively through this intermediary stage. The total energy of the system is expressed by \( 2\Gamma kM \), \( \Gamma \) being the Lorentz factor of the outgoing excited cloud in the center of mass system; the energy of the original nucleons in the final state is \( 2\Gamma M \). The inelasticity in this case is obviously \( (k-1)/k \), if we identify the total energy with the total available energy. (This is permissible if \( k \) is not too large.) In this approximation, therefore, the inelasticity is a function of \( k \) only, and vice versa.
does not seem, however, to play an essential role. If it should be of any importance, one could find an asymmetry of narrow and diffuse cones with respect to the direction of primary incidence.

Further, the ratio of heavy to light mesons is determined by the equilibrium temperature just before particles fly out, as was explained previously.\(^1\)

It is expected, therefore, that the low temperature equilibrium predicted by this model for \(\eta < 1\) will imply reasonable values with regard to these two criteria, and it will be worthwhile to study this model in more detail.

2.3. Since most of the experimental data are concerned with nucleon-nucleus collisions, we shall now extend the model to this more complicated case. At extremely high energy (\(\gtrsim 10^3\) Gev), the nucleon-nucleus collision can be fairly well approximated by the so-called jet model;\(^2\)\(^3\) the incident nucleon interacts only with the nuclear matter in the “tunnel” which it penetrates into the nucleus, and the remaining part of the nucleus suffers little excitation. In this approximation we can still employ two opposite points of view: composite collision\(^4\) and individual collision,\(^5\) but the former appears to be the more adequate description when the energy of the incident particle is very large.

2.4. We shall treat, therefore, the composite collision model. As regards the final division of the nucleons and the total energy among the two clouds, there is no \textit{a priori} criterion that would come out of a consideration of the nucleon-nucleus collision, but, for simplicity, we shall treat the following two cases.

A) Case of \(n : 1\) division.

We assume that, after exchange of momentum and energy, two excited clouds containing \(n\) and 1 nucleons (or, more precisely, “nucleons minus antinucleons”) respectively, will appear, the ratio of fictitious masses \(M_1^*\) and \(M_2^*\) of these clouds also being \(n : 1\). This means that, when referred to their own center of mass systems, both the energy and the equilibrium volumes of these clouds are in the ratio \(n : 1\).

If the collision is completely inelastic, \(\eta = 1\),

\[
M_1^* = \left(\frac{n}{n+1}\right)W, \quad M_2^* = \left(\frac{1}{n+1}\right)W,
\]

\((2.6)\)

\(W\) being the total energy in the center of mass system.

If the inelasticity is not close to 1, it can be shown that the approximate relation

\[
\eta = 1 - \left(\frac{M}{M_1^*}\right), \quad \eta < 0.8
\]

\((2.7)\)

for \((E_0/M) \gtrsim 10^8\) and \(n \leq 5\)

holds in this case, too. Thus the density of available energy, which turns out common to both clouds, is

\[
\varepsilon = \begin{cases} 
(W/(n+1) - M) \mu^2, & \text{for } \eta = 1, \\
(\eta M/(1-\eta)) \mu^2, & \text{for } \eta < 0.8.
\end{cases}
\]

\((2.8)\)
The temperature corresponding to this energy density can be determined by the formulae:

\[ \varepsilon = \varepsilon(\pi) + \varepsilon(\theta) + \varepsilon(N) \]

\[ \varepsilon(\pi) = \frac{3T^4}{2\pi^2} G\left(\frac{\mu}{T}\right), \quad \varepsilon(\theta) = \frac{4T^4}{2\pi^2} G\left(\frac{m_\theta}{T}\right), \quad \varepsilon(N) = \frac{8T^4}{2\pi^2} G^*(\frac{M}{T}), \]  

(2.9)

with

\[ G(x) = x^n \sum_{i=0}^{n} \frac{1}{1+\nu} \left\{ \frac{3K_{3,1}(1+\nu)x}{1+\nu} + xK_{1,1}(1+\nu)x \right\}, \]  

(2.10)

\[ G^*(x) = x^n \sum_{i=0}^{n} (-1)^i \frac{1}{1+\nu} \left\{ \frac{3K_{3,1}(1+\nu)x}{1+\nu} + xK_{1,1}(1+\nu)x \right\}. \]

\( m_\theta \) means the mass of the heavy meson. Here we have assumed the statistical weight of the heavy mesons to be equal to 4.*

Once the temperature is determined, we can derive the number density for each kind of particles and multiplying it by the volume we get the total number.

The results of the computation are shown in the following table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( E_0/M )</th>
<th>( T/\mu )</th>
<th>Total number of secondary particles</th>
<th>Composition (( \pi : K : N ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 10^2 )</td>
<td>1.9</td>
<td>( 1.0 \times 10 )</td>
<td>1:0.7:0.5</td>
</tr>
<tr>
<td></td>
<td>( 10^3 )</td>
<td>2.6</td>
<td>( 3 \times 10 )</td>
<td>1:0.9:0.8</td>
</tr>
<tr>
<td></td>
<td>( 10^4 )</td>
<td>3.3</td>
<td>( 8 \times 10 )</td>
<td>1:1:1</td>
</tr>
<tr>
<td></td>
<td>( 10^5 )</td>
<td>4.4</td>
<td>( 2 \times 10^3 )</td>
<td>1:1:1:1.4</td>
</tr>
<tr>
<td></td>
<td>( 10^6 )</td>
<td>5.7</td>
<td>( 5 \times 10^3 )</td>
<td>1:1:1:1.6</td>
</tr>
<tr>
<td>3</td>
<td>( 10^2 )</td>
<td>1.9</td>
<td>( 1.9 \times 10 )</td>
<td>1:0.7:0.5</td>
</tr>
<tr>
<td></td>
<td>( 10^3 )</td>
<td>2.5</td>
<td>( 6 \times 10 )</td>
<td>1:0.9:0.8</td>
</tr>
<tr>
<td></td>
<td>( 10^4 )</td>
<td>4.2</td>
<td>( 4 \times 10^3 )</td>
<td>1:1:1:1.4</td>
</tr>
<tr>
<td></td>
<td>( 10^6 )</td>
<td>5.5</td>
<td>( 9 \times 10^3 )</td>
<td>1:1:1:1.6</td>
</tr>
<tr>
<td></td>
<td>( 10^7 )</td>
<td>1.8</td>
<td>( 2 \times 10 )</td>
<td>1:0.7:0.4</td>
</tr>
<tr>
<td>5</td>
<td>( 10^2 )</td>
<td>2.4</td>
<td>( 7 \times 10 )</td>
<td>1:0.9:0.8</td>
</tr>
<tr>
<td></td>
<td>( 10^3 )</td>
<td>3.1</td>
<td>( 1.8 \times 10^3 )</td>
<td>1:1:1</td>
</tr>
<tr>
<td></td>
<td>( 10^4 )</td>
<td>4.1</td>
<td>( 5 \times 10^3 )</td>
<td>1:1:1:1.3</td>
</tr>
<tr>
<td></td>
<td>( 10^6 )</td>
<td>5.4</td>
<td>( 1.3 \times 10^3 )</td>
<td>1:1:1.6</td>
</tr>
</tbody>
</table>

* That is to say, we have assumed only one kind of heavy meson with isospin 1/2 and spin 0. This is the least possible value for the statistical weight of the heavy meson.
Z. Koba

(ii) $\psi < 1$ (independent on incident energy)

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$T/\mu$</th>
<th>$n=1$</th>
<th>$n=3$</th>
<th>$n=5$</th>
<th>Composition $(\pi : K : N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>1.7</td>
<td>6</td>
<td>13</td>
<td>19</td>
<td>1 : 0.6 : 0.3</td>
</tr>
<tr>
<td>0.5</td>
<td>1.3</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>1 : 0.5 : 0.1</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1 : 0.3 : 0.08</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>0.3</td>
<td>0.6</td>
<td>0.9</td>
<td>1 : 0.2 : 0.03</td>
</tr>
</tbody>
</table>

† In the case of $E_0/M = 10^2$, the jet model is probably not valid, and in the cases, where the number of secondary particles turns out very small, the thermodynamical approximation obviously fails. Such cases are listed only for rough orientation.

B) Case of symmetrical (1 : 1) division

We assume that after collision two excited clouds fly out in perfect symmetry. This may seem less plausible than the case A), if one adheres to the original picture of momentum exchange. It could be regarded, however, as the opposite extreme case to A); and so it would give a rough idea of how much the results depend on the assumption concerning the division of energy and momentum between two clouds.

In this case we have

$$W_1 = W_2 = W/2,$$

$$M_1^* = M_2^* = \{ (W/2)^2 - P^2 \}^{1/2}.\tag{2.11}$$

For a completely inelastic collision

$$M_1^* = W/2,$$

$$\varepsilon = \left( \frac{W}{2} - \frac{n+1}{2} M \right) \frac{2\mu^2}{n+1},$$

while in the case of $\eta < 0.8$ (See appendix.)

$$\eta = 1 - (n+1) M/2 M^*,$$

so that

$$M^* = (n+1) M/2 (1 - \eta),$$

$$\varepsilon = [\eta/ (1-\eta)] M \mu^2.$$  

The case of completely inelastic collision is of course, identical with the case A), because two clouds have zero momentum and two ways of division do not yield any difference. The case of $\eta < 1$ is, as can be expected, the same as the case of nucleon-nucleon collision with mass and volume multiplied by a factor of $(n+1)/2$.

The results are given in the following table.
Table 2·3. Temperature, total number, and composition. Case of 1:1 division.*

(i) \( \eta = 1 \)

The results are the same as A) (i).

(ii) \( \eta < 1 \)

(Independent on incident energy)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( T/\mu )</th>
<th>( n=1 )</th>
<th>( n=3 )</th>
<th>( n=5 )</th>
<th>Composition ( (n : K : N) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>1.7</td>
<td>6</td>
<td>13</td>
<td>19</td>
<td>1:0.6:0.3</td>
</tr>
<tr>
<td>0.5</td>
<td>1.3</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>1:0.5:0.1</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1</td>
<td>0.8</td>
<td>1.6</td>
<td>1.4</td>
<td>1:0.2:0.08</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>0.4</td>
<td>0.8</td>
<td>1.2</td>
<td>1:0.2:0.03</td>
</tr>
</tbody>
</table>

* Cf. the footnote to table 2·2.

2·6. This model may be regarded as an extreme idealization. It is seen, nevertheless, from the above numerical values that it reproduces some of the general tendencies of jet showers. It predicts a very small average number of secondary particles, if one assumes a small inelasticity. This point may be fatal. Stars with quite a large number of prongs observed in nuclear emulsions could not be explained except by considering the fluctuations in the inelasticity, or by taking into account some additional mechanism which would result in the formation of a large number of low energy secondary particles.

§ 3. Predictions of Heisenberg’s theory

3·1. In the Heisenberg theory the transverse component of the secondary particles is essentially determined by the Fourier component of the transverse dimension of the initial excited state (Lorentz-contracted disk), and this dimension is of the order of \( 1/\mu \). Thus the transverse momenta will be of the order of pion rest mass. The energy dissipation characteristic of the Heisenberg theory is concerned only with the longitudinal component. In both Heisenberg and Landau theories the essential feature of the angular distribution is determined by its one-dimensional character in the expanding process. This feature will not be changed in a nucleon-nucleus collision, either.

3·2. The problem of the composition of the secondary particles will be considered next. Since most of the experimental data are concerned with nucleon-nucleus collision at extremely high energy \( (>10^3 \text{ Gev}) \), we must slightly modify the original formulae, so that they can be applied to the general picture of the jet model.

Instead of two flat disks of the same size colliding, we have now a contracted nucleon and a contracted tunnel-like part of the nucleus containing \( n \) nucleons, \( n \) being less than 6 for the nuclear emulsions. This picture is compatible with our modified assumption of the geometrical cross section at extremely high energy.

Thus we should put

\[
p_0^{\text{max}} = \mu \Gamma' / n,
\]

where \( p_0^{\text{max}} \) is the maximum energy of a secondary particle and \( \Gamma' \) is a certain average of the Lorentz factors in the center of mass system. This expression corresponds to the
so-called composite collision model.

As is explained in detail in Heisenberg’s works, if the strong non-linearity* of the meson wave field is assumed, the energy spectrum of the emitted particles will be of the form,

\[ dN \propto \frac{dp_0}{p_0^2}, \quad (3\cdot2) \]

\( p_0 \) being the energy of the secondary particle; the total number of a certain kind, \( j \), of particles will be given to a rough approximation by

\[ N_j \propto g_j \int_{m_j}^{p_{0\text{max}}} \frac{dp_0}{p_0^2}, \quad (3\cdot3) \]

g\( j \) and \( m_j \) being respectively the statistical weight** and the mass of that particle.

From (3·2) and (3·3) one has

\[ N_j \propto g_j \left( \frac{1}{m_j} - \frac{n}{\mu l^0} \right), \quad (3\cdot4) \]

whence one gets easily the number ratio for a given incident energy and a given value of \( n \). The results are shown in the Table 3·1. These values are found, considering the rough approximation we have used to be consistent with the experimental informations.

Moreover, minor discrepancies, even if they should be found, would never be fatal for the Heisenberg theory, because the detailed behaviour of the low energy part of the energy spectrum of secondary particles and consequently the number ratio of various particles is rather sensitive to the particular properties of the non-linear term involved in the wave equation. This situation might be regarded as corresponding to the sharp dependence on the critical temperature in the Landau theory. A rough idea of this fact will be given by the last column of the Table 3·1.*** These values have been computed using a slightly more plausible spectrum given by Heisenberg:**

\[ N_j \propto g_j \left( 1 + 2 \left( \frac{m_j}{p_{0\text{max}}} \right)^2 - 2 \frac{m_j}{p_{0\text{max}}} \sqrt{1 + \left( \frac{m_j}{p_{0\text{max}}} \right)^2} \right). \quad (3\cdot5) \]

One can conclude, therefore, that the composition of the secondary particles cannot discriminate between Landau’s, and Heisenberg’s theories.

3·3. A somewhat subtler distinction may be possible, however, if one can detect distribution of heavier particles. In the Landau picture the critical temperature \( T_c \) is determined by the pion-pion interaction and is common to all the elements of the expanding nucleon-meson cloud, so that one can expect the same number ratio of heavy to light mesons in all directions. In the Takagi picture, too, the situation is similar. In

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* See the footnote on page 289.

** Heisenberg does not necessarily identify \( g_j \) with the statistical factor. Here I have assumed this for simplicity.

*** The results are much more sensitive to a change in the value of \( g_j \).
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Table 3-1. Number ratio of various particles predicted by the Heisenberg theory

<table>
<thead>
<tr>
<th>Incident energy</th>
<th>Number of nucleons in the &quot;tunnel&quot;</th>
<th>Maximum momentum of secondary particles</th>
<th>Number ratio by (3-4) $N_n : N_K : N_M$</th>
<th>Number ratio by (3-5) $N_n : N_K : N_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0/M$</td>
<td>$n$</td>
<td>$p_0^{max}/\mu$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^2$</td>
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</tr>
<tr>
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<td>1 : 0.38 : 0.39</td>
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<td>63</td>
<td>1 : 0.38 : 0.40</td>
<td>1 : 0.38 : 0.40</td>
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the Heisenberg picture, on the other hand, the transverse energy is so low from the outset that few heavy mesons can appear in the perpendicular direction in the center of mass system.

This difference is due to the fact that Heisenberg deals with a well-defined wave motion, in which, so to speak, all the particles are found in a definite phase relation, while Takagi and Landau regard the nucleon-meson system as a thermodynamical system without any definite phase relation between constituent particles. The thermal motion in the latter pictures can furnish enough energy to create heavy mesons independently of the direction of the "mass" motion.

§ 4. Discussion

4·1. As we have examined in the previous\(^1\) and the present notes, three model for multiple meson production, namely those of Landau, Takagi and Heisenberg, give similar results, compatible with experimental information now available, concerning the composition\(^2\) and the distribution of transverse momenta\(^3\) of secondary particles.

The main difference between the predictions of these theories can be found in the dependence of the total number of created particles on the incident energy. The distinctions are obscured, however, by the intervention of another parameter, inelasticity. Indeed the results are expressed by
If we assume for the present that the average inelasticity is practically constant and much less than 1 over a wide range of incident energy, as is suggested by the analysis of latitude dependence of various components and extensive air showers, then we could discriminate between these theories by examining a large number of events statistically. Such a treatment is necessary because the present theoretical predictions are concerned with the average behaviour only and fluctuations from the latter are expected to be quite large.** In analyzing emulsion data we should further notice that events with small numbers of secondary particles are likely to escape detection. Thus the arguments put forward so far** which infer the relation of total number to incident energy from a few events of conspicuous stars, do not appear very convincing.

4·2. One difference between these three models, which may have some bearing on the analysis of the phenomena at lower energy, is that both Landau’s and Heisenberg’s pictures lead naturally to the presence of a pretty strong pion-pion interaction at lower energy, near 1 Gev, while in Takagi’s picture it is not necessary to introduce such an interaction.

The role of the pion-pion interaction in the nucleon-nucleon and nucleon-meson reactions is reported to be not yet quite clear. It has also been suggested that this interaction, if it exists, will be strongly dependent on energy and isotopic spin. Such a circumstance might have slight influence on the results of Landau and Heisenberg.

Takagi’s model, too, has its analogue in the Gev energy region: one can assume the existence of a similar mechanism in the one-or two pion production or strange particle production by nucleon-nucleon collision.

4·3. The point which is left open is how to incorporate the small value of inelasticity into the theory.

Bhabha has once proposed to take into account the possible localization of a certain part of nucleon self-energy in a small central region of the extended nucleon. Although this idea, if taken too literally, is subject to criticism in the same way as Heisenberg’s argument, it may have something to do with the real situation. Indeed the concept of pion-pion interaction seems to be in a sense consistent with this picture of energy localization.

\[
N \propto \gamma E_0^{3/2} \quad \text{Heisenberg,}
\]

\[
N \propto \gamma E_0^{1/4} \quad \text{Landau,}\ *
\]

\[
N \propto \begin{cases} 
E_0^{3/8} & \text{for } \gamma = 1 \\
(\gamma/(1-\gamma))^{3/4} & \text{for } \gamma \lesssim 0.8
\end{cases} \quad \text{Takagi.}
\]

* This relation has been derived assuming that the equilibrium volume in the center of mass system also proportional to $\gamma$. If we presume this volume to be independent of $\gamma$, we have: $N \propto \gamma^{3/4} E_0^{1/4}$

** Podgoreckij, Rozental, and Černavskij have estimated the fluctuations based on statistical considerations and found them fairly large. Since the statistical treatment itself is sometimes to be regarded as a rough approximation, the deviations could be still larger.
It may be worth noting that the concentration of an appreciable amount of the available energy on one or two secondary particles can be regarded as producing inelasticity. In Landau's theory, in particular, this kind of concentration can really take place, as has been pointed out by Gerasimova and Černavskij.\textsuperscript{29} If this mechanism for inelasticity is true, one might expect an anomalous behaviour corresponding to the case that this particular secondary particle happens to be a \( \pi^0 \)-meson or some other particle which decays into \( \gamma \)’s immediately.

Acknowledgments

This work has been performed during my stay at Yale University and at Brookhaven National Laboratory, U.S.A. I should like to express my sincere gratitude to Prof. Gregory Breit and to Prof. George B. Collins for their hospitality extended to me, and especially to Prof. Breit for careful inspection of the manuscript and for comments. I am also indebted to a number of my colleagues in Japan, especially to Dr. S. Kaneko (Osaka) for his comments and discussions on Heisenberg’s theory. Further, I am indebted to Dr. K. Gottstein (Göttingen) for his comments and information on Heisenberg’s theory and on recent German research, and to Dr. V. P. Silin (Moscow) for information on recent Soviet works.

Appendix

Definition and evaluation of inelasticity in Takagi’s model

When a nucleon with energy \( E_0 \) collides with an \( n \)-nucleon nuclear aggregate at rest, the velocity of the center of mass \( v \) is given by

\[
v = \sqrt{E^2 - 1} / (n + \varepsilon),
\]  

(A.1)

with

\[
\varepsilon = \frac{E_0}{M}.
\]  

(A.2)

The total energy, the energies of the first and the second clouds in the center of mass system will be denoted by \( W \), \( W_1 \), and \( W_2 \) respectively. Then we have

\[
W = (n^2 + 2n\varepsilon + 1 \varepsilon / M, 
\]  

(A.3)

\[
W_1 + W_2 = W,
\]  

(A.4)

and define \( M_1^* \) and \( M_2^* \) by

\[
W_1^* - P^2 = M_1^*,
\]  

(A.5)

\[
W_2^* - P^2 = M_2^*.
\]  

In the case A) we have

\[
M_1^* = nM_2^*,
\]  

(A.6)
and so

\[ W_1 = (W/2) \left[ 1 + (n^2 - 1) \left( \frac{M_2^*/W}{M_1^*} \right)^2 \right] \]
\[ W_2 = (W/2) \left[ 1 - (n^2 - 1) \left( \frac{M_2^*/W}{M_1^*} \right)^2 \right] \]  \hspace{1cm} (A·7)

and in the case B),

\[ W_1 = W_2 = W/2, \]  \hspace{1cm} (A·8)
\[ M_1^{*2} = M_2^{*2} = (W/2)^2 - P^2. \]  \hspace{1cm} (A·9)

In the theoretical treatment we generally define the inelasticity by

\[ \eta_{th} = \frac{\text{energy of all the secondary particles in the center of mass system}}{\text{kinetic energy of incident (n+1) particles in the center of mass system}}. \]  \hspace{1cm} (A·10)

Experimentally it is usual to put

\[ \eta_{exp} = \frac{\text{energy of all the secondary particles in the laboratory system}}{\text{kinetic energy of the incident nucleon in the laboratory system}}. \]  \hspace{1cm} (A·11)

In this case the final energy of the target nucleon (or the nuclear matter containing n nucleons) is sometimes included in the secondary energy, because it may not be distinguished from created particles.

In the case A) we have

\[ \eta_{exp} = \frac{E_0 - \frac{M}{M_2^*} \sqrt{1 - \frac{v^2}{c^2}} (W_2 + vP)}{E_0 - M}, \]  \hspace{1cm} (A·12)

because the final energy and momentum of the incident nucleon in the center of mass system are given by \( W_2 (M/M_2^*) \) and \( P(M/M_2^*) \) respectively. Thus

\[ \eta_{exp} = \frac{1 - \left( \frac{M}{M_2^*} \right) \alpha}{1 - 1/c}, \]  \hspace{1cm} (A·13)

with

\[ \alpha = \left( 1 + \frac{n}{\varepsilon} \right) \left[ \frac{1}{2} \left( \frac{n^2 - 1}{n^2 + 2n \varepsilon + 1} \right) \right] \]
\[ + \frac{1}{2} \left( 1 + \frac{1}{\varepsilon^2} \right)^{1/2} \left( \frac{n}{\varepsilon} \right)^{-1} \left[ \frac{2(n^2 + 1)}{n^2 + 2n \varepsilon + 1} \right] \]
\[ + \frac{(n^2 - 1)^2}{(n^2 + 2n \varepsilon + 1)^2} \left( \frac{M_2^*}{M} \right)^4 \]  \hspace{1cm} (A·14)

\( \varepsilon \) is very large and \( n \lesssim 5 \), so that for \( M_2^*/M \lesssim 5 \) we get \( 0.9 < \alpha < 1 \). We have then approximately (within 2% error)

\[ \eta = 1 - \frac{M}{M_2^*} \]  \hspace{1cm} for \( n \lesssim 0.8. \]  \hspace{1cm} (A·15)
In the case of nucleon-nucleon collision this relation is valid for \( \gamma \leq 0.9 \).

On the other hand

\[
\eta_{\text{th}} = \frac{W - ((nM/M_\ast)W_1 + (M/M_\ast)W_2)}{W - (n+1)M} = \frac{1 - M/M_\ast}{1 - \frac{n+1}{\sqrt{n^2 + 2n\varepsilon + 1}}}.
\]

Again this can be approximated within 6% error by

\[
\eta_{\text{th}} \approx 1 - M/M_\ast \quad \text{for} \quad \varepsilon \geq 10^3, \quad n \leq 5
\]

in agreement with \( \eta_{\exp} \) for \( \gamma \leq 0.8 \). So we have used in the text a conventional combination* of \( \eta_{\text{th}} \) and \( \eta_{\exp} : \eta = \eta_{\text{th}} \) for \( \gamma = 1 \), and \( \eta = \eta_{\exp} \approx 1 - M/M_\ast \) for \( \gamma \leq 0.8 \).

In the case B) it would be reasonable to define the elastic part of the energy as that carried away by \((n+1)/2\) nucleons in the incident direction.

\[
\begin{align*}
\eta_{\text{th}} &= \frac{W - 2 \left( \frac{W}{2} \frac{(n+1)M}{2M^\ast} \right)}{W - (n+1)M} \\
&= \frac{1 - \frac{n+1}{2} \frac{M}{M^\ast}}{1 - \frac{n+1}{(n^2 + 2n\varepsilon + 1)^{1/2}}} \\
&= \frac{1 - \frac{n+1}{2} \frac{M}{M^\ast}}{1 - \frac{n+1}{(n^2 + 2n\varepsilon + 1)^{1/2}}} \\
&\approx \eta_{\exp} = \frac{E_0 - \frac{1}{\sqrt{1 - \nu^2}} \left( \frac{W}{2} \frac{M}{M^\ast} \frac{n+1}{2} + \nu \sqrt{\left( \frac{W}{2} \right)^2 - M^\ast} \frac{(n+1)M}{2M^\ast} \right)}{E_0 - \frac{1}{2} M^\ast} \\
&\approx \frac{1 - \frac{n+1}{2} \frac{M}{M^\ast} \left( 1 + \frac{n}{\varepsilon} \right) \left[ \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^2} \left( 1 - \frac{1}{\varepsilon^2} \right)^{1/2} \left( 1 + \frac{n}{\varepsilon} \right)^{-1} \left( 1 - \frac{4}{n^2 + 2n\varepsilon + 1} \left( \frac{M^\ast}{M} \right)^2 \right)^{1/2} \right]}{1 - 1/\varepsilon}.
\end{align*}
\]

When \( \varepsilon \geq 10^3, \quad n \leq 5, \quad \eta_{\exp} \) can be expressed within 5% error by

\[
\eta = 1 - \frac{n+1}{2} \frac{M}{M^\ast} \quad \text{for} \quad \gamma \leq 0.8.
\]

\( \eta_{\text{th}} \) can also be approximated by

* The maximum value of \( \eta_{\exp} \) is not exactly one:

\[
(\eta_{\exp})_{\text{max}} = \frac{1 - \frac{1}{(2n\varepsilon)^{1/2}} \left( 1 + \frac{n}{\varepsilon} \right) \left( 1 + \frac{n^2 + 1}{2n\varepsilon} \right)^{-1/2}}{1 - \frac{1}{\varepsilon}}.
\]
\[ \eta = 1 - \frac{(n+1)M}{2M^*} \]

within 6\% error for \( n \leq 5, \varepsilon > 10^3 \).

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