Progress of Theoretical Physics Vol. 17, No. 3, March 1957.

On the Wave Propagation in the Non-Linear Fields

Toshiya Taniuti

Department of Physics, Kobe University, Kobe

(Received October 27, 1956)

The wave propagation in the two non-linear fields is studied, one of which is the Born-type field and another the relativistic hydrodynamics. The discussion is restricted to one dimensional propagation; the formation of shock waves and the energy distribution are investigated with special attention on the basis of the method of characteristics.

§ 1. Introduction

The several non-linear field theories have been proposed by many authors in order to overcome the difficulties inherent in the field theories of elementary particles, such as the saturation of nuclear forces, the infinite self-energies, the mass spectrum of elementary particles, and the meson multiple production in high energy nuclear events. In the non-linear theories, the familiar law of superposition ceases to be valid, and dynamics of these fields in which the non-linear character of the basic equations plays a decisive role, is not yet in the mathematical perfection. This is one of the reasons why the attempts on the non-linear theories have been discouraged inspite of their physical realities, which might be expected by physicists. It seems to be desirable first to investigate the solutions of non-linear wave equations before we go over into the physical discussion on them and the problem of quantization.

However, discussions on general non-linear wave equations, even under the special physical conditions, are very difficult. Hence it seems to be most adequate in the present stage to discuss only some typical equations which have been considered in the field theories and to recapitulate the non-linear wave propagation in these equations.

In this paper, discussions will be done for the two special equations of the scalar variable $\phi$: one is the so-called Born type (§ 3), the other is the equation of the relativistic hydrodynamics proposed by Landau and Khalatnikov (§ 4). We moreover restrict discussions to propagation in a one-dimensional space of violent disturbances initially localized in a small spatial region and invading the quiet state (the vacuum). Corresponding to this physical phenomenon, we may consider the following initial condition for $\phi$: $\phi = \varphi(x)/x = 0$ at $t = 0$ for all $x$ and $\partial\varphi/\partial t$ is not equal to zero only in the small region.

For example, if the Born type equation is assumed to represent a complicated non-linear motion of some string and $\phi$ is interpreted as the displacement from the equilibrium, the above specified initial value problem turns out to be that of the motion of the string
struck suddenly at the center of it. In the case of the Landau-Khalatnikov's equation, using the relations between the scalar potential and the hydrodynamical quantities settled down by them, we can restate the above condition as follows: the fluid is initially at rest and localized in a small domain and proceeds to expand adiabatically into the vacuum.

It is worthwhile to explain the corresponding physical meanings of the above condition in the meson field theory. Heisenberg considered the Born-type equation to describe the motion of meson cloud suddenly excited by the nucleon-nucleon collision at extremely high energy and invading the vacuum from the nucleon source. The relativistic hydrodynamical equation was originally introduced into the field theory by Landau in order to account for the meson multiple production and its detailed physical correspondence has already been given in his paper.

The purpose of this paper is to obtain the solutions of these two equations satisfying the above specified initial conditions. Especially, the following two subjects will be investigated with the special attention:

(i) Do shock waves occur?

(ii) Energy distribution at each instance.

In linear wave motion any initial discontinuity is preserved and propagated along the characteristics (light cone). However, non-linear wave motion behaves in a different manner. Suppose there is an initial discontinuity between two regions of the field quantity. Then there are the following alternative possibilities: either the initial discontinuity is resolved immediately and the disturbance, while propagated, becomes continuous, or the initial discontinuity is propagated through one or two shock fronts. In fact, shock fronts are the most conspicuous phenomena occurring in non-linear wave propagation; even without being caused by initial discontinuities they may appear and be propagated; the underlying mathematical fact is that, unlike linear partial differential equations, non-linear equations often do not admit solutions which can be continuously extended whenever the differential equations themselves remain regular. In the succeeding investigations, these possibilities will be discussed for the above two equations.

The energy distribution is concerned with the structure of self-interaction which may give rise to the concentration or dissipation of wave energy during propagation and here is another striking difference between linear and non-linear waves. The following calculations will give for both equations the analytical expressions to the variation of energy distribution in the course of propagation. This result may serve as the criterion for the basic equation proposed if the comparison with experimental data is possible.*

§ 2. Linear equation as preliminaries

The propagation characters of the linear hyperbolic equations with constant coefficients have been known in detail. For the better understanding of the succeeding discussions

*) The unit $\hbar = c = \mu = 1$ is used, in which $\mu$ is the pion compton wave length.
we shall review, in this section, their main features in connection with the physical condition under consideration. Consider

\[ \square \phi = 0, \quad (2.1) \]

with the initial condition:

\[ \begin{align*}
\frac{\partial \phi}{\partial x} (= u) &= 0, \\
\frac{\partial \phi}{\partial t} (= -v) &= -\nu_0(x), \quad \text{at } t = 0,
\end{align*} \quad (2.2) \]

in which

\[ \nu_0(x) \begin{cases} 
= \alpha (= \text{const.}) & \text{for } |x| \leq a, \\
= 0 & \text{for } |x| > a.
\end{cases} \]

The solution of the above initial value problem may easily be obtained by the Fourier transformation. However, this method can not be applied to non-linear equations. In place of this approach, we shall use throughout this paper the method of characteristics which may be well applicable to both cases. The latter method is especially effective for the equations with two independent variables and not containing the potential \( \phi \) explicitly.

The equation (2.1) can easily be transformed into the following system of differential equations:

\[ \begin{align*}
u_a + v_t &= 0, \\
u_t + v_a &= 0; \quad (2.1a)
\end{align*} \]

or

\[ \begin{align*}
(u_a - u_t) - (v_a - v_t) &= 0, \\
(u_a + u_t) + (v_a + v_t) &= 0.
\end{align*} \]

By introducing a set of new variables

\[ x-t = \sqrt{2} \\eta, \quad x+t = \sqrt{2} \\xi, \quad (2.3) \]

the above equations become

\[ \begin{align*}
\eta_t - \xi_t &= 0, \quad (2.1a)'' \\
\eta_t + \xi_t &= 0. \quad (2.1b)''
\end{align*} \]

The two sets of lines in the \((x, t)\) plane specified by \( \xi = \text{const.} \) and \( \eta = \text{const.} \) are called \( C^- \) and \( C^+ \) characteristics respectively. The whole \((x, t)\) space can be divided into three kinds of regions \( \alpha, \beta \) and \( \gamma \) by the characteristics issuing from some special points on the initial line \( t = 0 \) as illustrated in Fig. (1.1).
Let us now consider the meaning of the equations (2.1a, b) in more detail. These equations lead to

\[ u - v = \text{const. along } C^- \]

and

\[ u + v = \text{const. along } C^+ , \]

when \( \xi \) and \( \eta \) are respectively kept constant.

Then, we have the following relations for the values of \( u \) and \( v \) at any point \( G \) in \( \gamma \) (cf. Fig. 1.2)

\[ u - v = \text{const. along } \overline{GE} \ (C^-), \quad (2.4a) \]

\[ u + v = \text{const. along } \overline{GE'} \ (C^+). \quad (2.4b) \]

The initial condition yields

\[ u = v = 0 \quad \text{at } E \text{ and } E', \]

and consequently we have

\[ u = v = 0 \quad \text{in } \gamma \text{ region.} \]

Let us proceed to the analysis in "\( \beta \)-zone." This zone consists of two parts \( \beta^\pm \) specified by \( x \gtrless 0 \) respectively. However the present initial condition and the differential equation are invariant under the inversion of \( x \) and therefore \( v \) is symmetric and \( u \) is antisymmetric, i.e., \( v(x, t) = v(-x, t) \), \( u(x, t) = -u(-x, t) \). Hence it may be sufficient to consider only \( \beta^+ \) zone. In this zone, the initial condition gives

\[ u - v = 0, \]

\[ u + v = \alpha, \quad \text{at any point}, \]

namely
On the Wave Propagation in the Non-Linear Fields

\[ v = u = \alpha/2, \quad \text{throughout } \beta^+ \text{ zone.} \]

Similarly we have

\[ v = -u = \alpha/2 \quad \text{throughout } \beta^- \text{ zone.} \]

In a quite similar way, we have

\[ u + v = \alpha \quad \text{along } C^+, \]
\[ u - v = -\alpha \quad \text{along } C^-, \]

or, \( u = 0, \ v = \alpha \) in \( \alpha \)-region.

The energy density \( H = (1/2) (u^2 + v^2) \) has the following value in each region:

\[
H = \begin{cases} 
\alpha^2/2 & \text{in } \alpha \text{-region,} \\
\alpha^2/4 & \text{in } \beta \text{-zone,} \\
0 & \text{in } \gamma \text{-region.}
\end{cases}
\] (2.5)

The above result shows that the wave energy is always concentrated near the wave front, after the wave is swept away from \( \alpha \)-region. The initial discontinuities at \( A \) and \( A' \) are preserved and propagated with the light velocity.*

The essential point of the method developed here consists in the transformation of the original equation into the system of the ordinary differential equations with respect to the characteristic directions.

The transformations of this kind may be applicable to general differential equations. Since the characteristic lines are, in general, determined by the coefficients of the highest derivatives in the original equations, in the linear theories they are given without regard to the initial condition. However, in non-linear theories they may depend on the field quantities, accordingly on the initial conditions.

It will be shown in the succeeding discussions how the transformation into the characteristic direction can be performed in the non-linear differential equations and that the method of characteristics will be well applicable also to these non-linear theories.

§ 3. A Born type non-linear equation

Let us consider a non-linear field \( \phi \) defined by the Lagrangian:

\[ L = \frac{1}{2} \left\{ 1 + l^{-2} (\phi_x^2 - \phi_y^2) \right\}^{1/2} \] (3.1)

where \( l \) is a constant with the dimension of \((\text{length})^{-2}\).

Then, we have the following wave equation for \( \phi \):

* The method of solution exhibited here cannot be justified in the rigorous mathematical sense, because the relations (24. a; b) are valid only if \( u \) and \( v \) are continuous along \( C^\pm \). However this difficulty can easily be overcome if we start from the smooth initial distribution for \( \nu(x) \) and tend to the limit given by (2.2) after the all calculations were performed. This procedure, of course, gives the same result obtained here.
(1 - l^{-2} \phi^2) \phi_{ss} + 2l^{-2} \phi_s \phi_t - (1 + l^{-2} \phi^2) \phi_t = 0, \quad (3.2)

or

\begin{align*}
(\psi - l^2) u_s + (u_t - v_s) uv - (u^2 + l^2) v_t = 0, \\
u_t + v_s = 0,
\end{align*}

in which \( u \) and \( v \) are defined by

\begin{align*}
u &= \phi_s, & \psi &= -\phi_t.
\end{align*}

As was done in § 2, we shall transform (3·3a, b) into the two ordinary differential equations with respect to the characteristic directions. Multiplying them by indeterminate functions \( \lambda_1 \) and \( \lambda_2 \) respectively and adding, we have

\begin{align*}
\{ \lambda_1 (\psi - l^2) \partial/\partial s + (\lambda_1 uv + \lambda_2) \partial/\partial t \} u + \{ (\lambda_1 uv + \lambda_2) \partial/\partial s - \lambda_1 (u^2 + l^2) \partial/\partial t \} v &= 0.
\end{align*}

Hence the above equation may be transformed into

\begin{align*}
\lambda_1 (\psi - l^2) u_o + ( - \lambda_1 uv + \lambda_2) v_o &= 0, \\
\text{(3·5a)}
\end{align*}

or

\begin{align*}
(\lambda_1 uv + \lambda_2) u_o - \lambda_1 (u^2 + l^2) v_o &= 0, \\
\text{(3·5b)}
\end{align*}

if the characteristic direction at any point is defined by the following relations:

\begin{align*}
\lambda_1 (\psi - l^2) / (\lambda_1 uv + \lambda_2) = x_o / \tau_o \quad \text{(3·4a)}
\end{align*}

and

\begin{align*}
( - \lambda_1 uv + \lambda_2) / \{- \lambda_1 (u^2 + l^2)\} = x_o / \tau_o, \\
\text{(3·4b)}
\end{align*}

where \( \sigma \) means the length of the characteristics, and \( x_o = dx/d\sigma \), \( \tau_o = dt/d\sigma \).

The direction \( x_o / \tau_o \) in (3·4a, b), however, is restricted by the following equation:

\begin{align*}
( - l^2 + \psi^2) \tau_o^2 - 2uv \tau_o x_o + (l^2 + u^2) x_o^2 = 0, \\
\text{(3·6)}
\end{align*}

which is derived by eliminating \( \lambda_1 \) and \( \lambda_2 \) from (3·4a; b). Corresponding to two real roots \( \tau_o / x_o = \lambda_+ \) and \( \lambda_- \) of (3·6), we have two characteristic curves with the length \( \xi \) and \( \eta \) respectively. Namely, these curves \( C_+ \) and \( C_- \) are defined by

\begin{align*}
t_\xi - \lambda_+ x_\xi = 0 : & \quad C_+, \\
t_\eta - \lambda_- x_\eta = 0 : & \quad C_-,
\end{align*}

where

\begin{align*}
\lambda_\pm = (uv \pm l^2 s) / (\psi - l^2), \\
s = L \tau_o^{-2}.
\end{align*}

Now the original equations become a couple of the ordinary differential equations with respect to these characteristic directions \( \sigma : (\xi, \eta) \). In fact, eliminating \( \lambda_1 \) and \( \lambda_2 \) in (3·4a), (3·5a) and (3·4b), (3·5b), we obtain the equations:
On the Wave Propagation in the Non-Linear Fields

\[ u_t - \lambda_+ v_x = 0, \quad (3.9a) \]
\[ u_\eta - \lambda_+ v_\eta = 0. \quad (3.9b) \]

The solutions of the above equations give the relations between \( u \) and \( v \) and these correspond to the families of curves in the \((u, v)\) plane. These curves are denoted by \( I^{\pm} \) characteristics corresponding to the \( C^{\pm} \) characteristics in the \((x, t)\) plane respectively.

Finally, we have the following system of equations equivalent to the original equation;

\[ I^+ : \quad u_t - \lambda_+ v_x = 0, \]
\[ C^+ : \quad t_x - \lambda_+ x_t = 0; \]
\[ I^- : \quad u_\eta - \lambda_+ v_\eta = 0, \]
\[ C^- : \quad t_\eta - \lambda_+ x_\eta = 0. \]

It follows also from the above equations that

\[ t_x v_\eta - x_t u_\eta = 0, \quad (3.10) \]
\[ t_\eta v_t - x_\eta u_t = 0. \]

This relation indicates that the angle between \( C^+ \) and \( x \)-axis and the angle between \( I^- \) and \( u \)-axis are complementary to each other and the similar relation holds for \( I^+ \) and \( C^- \), and the knowledge of \( I \) characteristics may give the behaviours of \( C \) characteristics in the \((x, t)\) plane and vice versa. However, it should be kept in mind that this relation of representation has its meaning only if the transformation Jacobian is not zero and this condition can not necessarily be satisfied for the physically realizable state (i.e., the constant state and the simple wave region which will be explained in the succeeding discussion).

In the following treatment, it may be more convenient to refer to the equation

\[ (\varepsilon^2 - u^2) u_\eta^2 + 2uw u_\xi v_\eta - (\varepsilon^2 + v^2) v_\eta^2 = 0, \]
\[ \varepsilon^2 = p^2 + u^2 - v^2, \]

or

\[ \varepsilon^2 (du^2 - dv^2) = (udu - vdv)^2; \quad \text{on } I \text{-characteristics}, \quad (3.11) \]

which can be derived from \((3.5a)\) and \((3.5b)\). The integration of \((3.11)\) is easy and it gives the general solution:

\[ v = mu \pm pl, \quad (3.12) \]

in which \( p = \sqrt{1 - m^2} \), and \( m \) is an arbitrary real constant whose absolute value is smaller than unity. The above relation shows that \( I^{\pm} \) are straight lines. The somewhat detailed calculations show that

\[ v = mu + pl \quad \text{corresponds to} \]
\[ I^-, \text{ if } pu - lm > 0 \quad (\lambda^- = 1/m), \]
\[ I^+, \text{ if } pu - lm < 0 \quad (\lambda^+ = 1/m), \quad (3.13a) \]
and \( v = mu - pl \) corresponds to
\[
\begin{align*}
I^+ & \quad \text{if} \quad pu + lm > 0 \quad (\lambda_- = 1/m), \\
I^- & \quad \text{if} \quad pu + lm < 0 \quad (\lambda_+ = 1/m).
\end{align*}
\]

The above relations indicate that \( \lambda^+ \) is constant along \( C^- \), therefore all the \( C^+ \) curves are parallel to each other at their intersecting points with an arbitrary \( C^- \) curve. The same relation also holds for \( C^- \) curves. This leads to the remarkable character of the solutions that the curves \( C^- \) can never cross each other and the same is true for \( C^+ \), namely \( C^+ \) and \( C^- \) families are those composed of parallel curves.*

Let us now consider the general method of solution for an arbitrary initial value problem. When the initial conditions: \( v = f(x), u = g(x) \) at \( t = 0 \), are given, we always have the two values of \( m \), say \( m^\pm(x_0) \), from the equations
\[
f(x_0) = m(x_0)g(x_0) \pm p(x_0)l,
\]
for each point \( x_0 \) on the surface \( t = 0 \). These values of \( m \) determine the \( I^\pm \) families in the \((u,v)\) plane and accordingly the \( C^\pm \) families in the \((x,t)\) plane can be drawn on the basis of the relation (3·10). On the other hand, the relations (3·9a, b) or (3·13a, b) are valid along \( C^\pm \), hence the values of \( u \) and \( v \), at the cross point of the two curves \( C^+ \) and \( C^- \), can easily be obtained from (3·13a, b).

It is especially interesting to consider the initial condition in which \( f(x) \) and \( g(x) \) are not arbitrary functions, but satisfy a relation
\[
f(x) = pg(x) + pl, \quad p = \sqrt{1 - \mu^2},
\]
with a parameter \( \mu \); in other words we are considering the initial distribution, in which one of the \( m(x_0)'s \), say \( m^-(x_0) \), is reduced to a given constant \( \mu \). If, moreover, \( pg - l\mu > 0 \) is assumed, it turns out that the relation for \( I^- \) is valid on the initial line \( t = 0 \). Accordingly, the relation
\[
v(x, t) = \mu u(x, t) + pl,
\]
holds over all \((x,t)\) space, because the relation for \( I^- \) is preserved along \( C^- \). On the other hand, we have
\[
v(x, t) = m^+(x_0)u(x, t) - p^+(x_0)l
\]

* This important character may largely exclude the possibility of the occurrence of shock waves (cf. (4)).
along $C^+$. From these, $u$ and $v$ take some constant values along $C^+$ and $C^+$ becomes a straight line. As was already stated, the lines $C^+$ are parallel to each other, therefore the initial shape of distribution is shifted along $C^+$ without any deformation while propagated, that is, the solution is the permanent wave propagating with the phase velocity $\mu$. (cf. Fig. 3.2, in which the lines denoted as $C^+$ are not the true $C^+$ but their projections on $(x-t)$ plane are true ones.) In the above example, contrary to the general one to one correspondence between $\Gamma^+$ and $C^-$-characteristics, all the $C^-$-curves in $(x-t)$ plane correspond to a single $\Gamma^-$ by virtue of our special initial relation between $u$ and $v$. In general, such a domain of the $(x, t)$ space, say $D$, as those corresponding to one $\Gamma^+$ or $\Gamma^-$ line in the $(u-v)$ plane is called a simple wave region. As is obvious from the above discussion, the simple wave has the remarkable property that the characteristic line $C^+$ or $C^-$ contained in its region is surely straight. However, in general, it is not true that those $C^+$ or $C^-$ are parallel and the simple wave preserves its initial shape during propagation.**

In most cases of non-linear equations the $C$ characteristics in the simple wave region are the families composed of the straight lines with different slopes, and in those cases shock waves may occur.6 We also mention the following important theorem concerning the simple wave.

**Theorem**:

The region adjacent to a region of constant state is that of simple wave.·······(A)***

On the basis of the preceding discussions, the solution satisfying the following initial conditions will be given without any approximation:

at $t=0$, \[ \phi = u = 0, \]

\[
\begin{align*}
  v &= v_0(x) \quad (>0) & \text{for} & \quad a \leq |x| \leq a+b, \\
  &= \alpha \quad (>0) & \text{for} & \quad |x| \leq a,
\end{align*}
\]

* In the present example this can be seen more easily as follows: Using the relation for $\Gamma^-$, we have $v_0=\mu v_0$, then it results immediately from (3.3b) that $u_0+\mu v_0=0$, or $u=f(x-\mu t)$.

** In the hydrodynamics the peculiar example of this kind has also been acknowledged as the Kármán-Tsien's gas, which is fictitious and may not physically be realizable.

*** cf. R. Courant and K. O. Friedlichs loc. cit.; especially chapt. II. § 29.
in which $\alpha$ is a constant and $v(x)$ is a smooth function as a whole and symmetric with respect to $x$. The limit $b \to 0$ will be taken after all calculations are performed. In the first place, the $C^-$ issuing out of $(a+b, 0)$ and $C^+$ out of $(-a-b, 0)$ are the straight lines whose tangents are $+1$ or $-1$ respectively, because in the vacuum $(u=v=0)$, $\lambda = \pm 1$. Referring to the procedure employed in § 2, we divide the $(x, t)$ space into the three regions $\alpha$, $\beta$, $\gamma$, by the $C^\pm$ characteristic curves mentioned above (cf. Fig. 3.3).

(i) $\gamma$-zone

Any point in this region is connected with the vacuum (I) and (I)' respectively by $C^+$ and $C^-$ running across the $\beta$-zones.

Since

$$u=v=0 \text{ in (I)}$$
$$u=-v=0 \text{ in (I)'}$$

we see that

$$u=v=0$$

at every point in $\gamma$, and that $PQ$ and $PQ'$ in Fig. 3.3 are the straight lines of gradient $\pi/4$.

(ii) $\alpha$-region

We subdivide this region into the four regions $\alpha_1$, $\alpha_{11}$, $\alpha_{111}$ and $\alpha_{111}$ illustrated in Fig. 3.4.

(a) $\alpha_1$-region

In this region, both $I^\pm$ characteristics should be given by $v=mu+pl$ so as to be continuously connected with the initial conditions $v=\alpha>0$, $u=0$, at $t=0$.

Using the initial conditions, we have

$$m_1 = \pm \sqrt{1-(\alpha/l)^2} \ (\text{=const.})$$

and

$$v=m_1 u+pl$$

representing

$$I^+ \quad \text{for } m_1>0,$$

and

$$I^- \quad \text{for } m_1<0.$$
On the Wave Propagation in the Non-Linear Fields

\[ v = -|m_1|u + P_1 l \quad \text{along } C^- , \]
\[ v = |m_1|u + P_1 l \quad \text{along } C^+ , \]

which give \( v = P_1 l, u = 0 \) in the \( \alpha_I \)-region and the straight characteristics \( AD, A'D \) with \( \lambda_\pm = \mp |m_\mp| \).

(b) \( \alpha_{II} \)-region (cf. Fig. 3.5)

By the similar argument, we have
\[ v = m_{II}(x_0)u + P_{II}(x_0)l , \]
with
\[ m_{II}(x_0) = \pm \sqrt{1 - \left(\frac{v(x_0)}{l}\right)^2} ; \]

or
\[ v = -|m_{II}(x_0')|u + P_{II}(x_0')l \quad \text{along } C^- , \]
\[ v = |m_{II}(x_0'')|u + P_{II}(x_0'')l \quad \text{along } C^+ . \]

From the above equations, \( u \), for example, becomes
\[ u = l \cdot \frac{P_{II}(x_0') - P_{II}(x_0'')}{|m_{II}(x_0')| + |m_{II}(x_0'')|} , \]
which shows the curved \( C^\pm \) characteristics.

(c) \( \alpha_{III} \)-region

This region is adjacent to the constant state \( \alpha_I \); therefore, according to the theorem (A), it is the region of simple wave. We have, in this case, the relations \( v = -|m_1|u + P_1 l \) along \( C^- \) prolonged from the \( \alpha_I \)-region and \( v = |m_{II}(x_0)|u + P_{II}(x_0)l \) along \( C^+ \) prolonged from the \( \alpha_{II} \)-region.

The values of \( u \) and \( v \) at a cross point between \( C^- \) and \( C^+ \) can be given as follows;
\[ v = l \cdot \frac{P_{II}(x_0) - P_{II}(x_0')}{|m_{II}(x_0)| + |m_{II}(x_0')|} , \]
\[ u = l \cdot \frac{P_{II}(x_0') - P_{II}(x_0'')}{|m_{II}(x_0)| + |m_{II}(x_0'')|} . \]

The values of \( u \) and \( v \) are, of course, not constant; however they are constant along \( C^+ \). Hence, \( C^+ \)'s are the straight lines parallel to \( AP \) and \( C^- \)'s are the curves parallel to \( AE \).

(d) \( \alpha_{IV} \)-region

The values of \( u \) and \( v \) in this region can be settled down at each cross point between \( I^+ (C^+) \) issuing out of \( \alpha_{III}(x > 0) \) and \( I^- (C^-) \) out of \( \alpha_{III}(x < 0) \).

We have
\[ v = |m_{II}(x_0')|u + P_{II}(x_0')l \quad \text{along } C^+ , \]
\[ v = -|m_{II}(x_0'')|u + P_{II}(x_0'')l \quad \text{along } C^- , \]

with the solution.
\[ v = l \{ P_{ij}(x'_0) | m_{ij}(x''_0) | + P_{ij}(x'_0) | m_{ij}(x''_0) | / (| m_{ij}(x''_0) | + | m_{ij}(x''_0) |) \}, \]
\[ u = l \{ P_{ij}(x'_0) - P_{ij}(x'_0) \} / (| m_{ij}(x''_0) | + | m_{ij}(x''_0) |). \]

\( C^+ \)'s and \( C^- \)'s are the families of the curves parallel to \( AE \) and \( A'E' \) respectively.

(iii) \( \beta \)-zone

We consider the solutions in the region \( \beta^+(x > 0) \) only; the solutions in another region, \( \beta^-(x < 0) \), can easily be obtained from the symmetry. In this case, we also have the three regions \( \beta_1, \beta_{II}, \) and \( \beta_{III} \) as are shown in Fig. 3.6.

(a) \( \beta_1 \)-zone

We have \( v = u \) along \( C^+ \) prolonged from the vacuum (I) and also
\[ v = - m_{II}(x_0) u + P_{II}(x_0) l \]
along \( C^- \) issuing out of \( \alpha_{II} \). The solution is
\[ v = u = l P_{II}(x_0) / (1 + | m_{II}(x_0) |). \]

These values are constant along \( C^- \)'s and give \( \lambda^- = 1 \) along them. \( C^- \)'s are of course parallel straight lines of gradient \( \pi / 4 \), while \( C^+ \)'s are parallel to the curve \( BE \). This region is, of course, that of simple wave.

(b) \( \beta_{II} \)-zone

By using the similar argument, we have in this case
\[ v = u \]
along \( C^+ \),
\[ v = - m_j u + P_j l \]
along \( C^- \),
and
\[ v = u = l P_j / (1 + | m_j |) \quad (= \text{constant}). \]

\( C^- \)'s are straight lines whose slopes are equal to unity and \( C^+ \)'s are also straight and parallel to \( EF \).

(c) \( \beta_{III} \)-zone

In this case, we see that
\[ v = u \]
along \( C^+ \),
On the Wave Propagation in the Non-Linear Fields

Fig. 3·7
The characteristics of the equation (3·2).

Fig. 3·8
The C characteristics and the spatial distribution of \( u \) and \( v \) in the limiting case: \( b \to 0 \).

\[
v = -|m_{11}(x_0)|u + P_{11}(x_0)l \quad \text{along } C^{-},
\]

and

\[
v = u = lP_{11}(x_0)/[1 + |m_{11}(x_0)|].
\]

C\(^{-}\)'s are straight, C\(^{+}\)'s are parallel to PF and this region is of simple wave. Let us now consider the limit \( b \to 0 \). In this limit, as is obvious in the Fig. 3·7, the regions \( \alpha_{II} \) and \( \alpha_{II'} \) and the regions of simple wave tend to vanish and there remain only the regions of constant state, and on the boundaries between these, the values of \( u \) and \( v \) are discontinuous (cf. Fig. 3·8). It is quite interesting to compare this result with that of the linear equation (1·1) illustrated in Fig. 1. At a glance we see that the solution of the non-linear equation under consideration is quite similar to that of the linear equation. The differences are as follows:

1. \( \Delta APP' \) is not equilateral, hence \( C^{\pm} \) are discontinuous at \( P \).
2. the breadth of the \( \beta \)-zone and the values of \( u \) and \( v \) in it are different from those of the linear case.

As for the energy distribution \( H \), we have

\[
H = |l^2 + u^2| \cdot [1 + l^{-2} (u^2 - v^2)]^{-1/2} - l^2 \begin{cases} 
= (1 - (\alpha/l)^{1/2} - 1) l^2 & \text{in the } \alpha\text{-region}, \\
= \alpha^2/[1 + \{1 - (\alpha/l)^{1/2}\}^{1/2}] & \text{in the } \beta\text{-zone}, \\
= 0 & \text{in the } \gamma\text{-region}.
\end{cases}
\]

Let us now proceed, on the basis of the results obtained here, to discuss the two subjects cited in § 1: the occurrence of shock waves and the energy distribution.

As for the former, this non-linear equation yields the similar result as that of linear equation and the peculiar phenomena characteristic to most non-linear equations, the formation of shock waves, does not appear in this case, so far as the present initial conditions are concerned. Hence it may be said that this model belongs to the special class
of non-linear equations (for example, such as Kármán-Tsiên's gas in the hydrodynamics). However the so-called Born-type equations are rich in variety of the form of Lagrangian and it may be possible that the modification of Lagrangian destroys the above special character entirely.

We also investigate the difference between the non-linear and the linear equation existing in the energy distribution. The spatial breadth of β zone (=a+OP) increases as α increases, (α cannot exceed l), and a considerable amount of energy remains near the origin until the critical time \( t_c = a/\sqrt{1-(\alpha/l)^2} \) which is larger by the factor \((1-(\alpha/l)^2)^{-1/2}\) than that of the linear equation. Therefore, if the value of α is so large as to be comparable to \( l \), these effects may be expected to be observed. However, the following numerical calculation shows that these are, in general, too small to be recognized.

Let us assume that \( l=6 \), the initial energy \( \tilde{H}=360 \), and \( a=0.07 \). The value of \( l \) can be interpreted as the inverse of the nucleon compton wave length. And the above values of \( a \) and \( \tilde{H} \) stand for the corresponding quantities in the event where a nucleon of energy \( 10^{11} \) ev collides with a nucleon at rest giving rise to a sudden multiple production of mesons which carry away a half of the total energy. For these values, we have

\[ \alpha \approx 5.970, \]

hence

\[ OP \approx 0.61 \approx 0.85 \times 10^{-12} \text{cm}. \]

§ 4. Relativistic hydrodynamics

Consider the equation

\[ \frac{\partial}{\partial x_i} \left[ \frac{\partial \phi}{\partial x_i} \left( \frac{\partial \phi}{\partial x_j} \right)^2 \right] = 0, \tag{4.1} \]

which may be derived from the Lagrangian

\[ L = \frac{1}{2} (\phi_j \cdot \phi_j)^2. \tag{4.2} \]

This equation was proposed by Khalatnikov\(^3\) as describing a fluid with an extremely high temperature. In the present paper we shall investigate the mathematical behaviour of the above equation with the same initial condition as that of § 3 without regard to its physical background.* Putting

\[ \phi_x = u, \ \phi_t = -v, \]

we have

*) The result obtained in this paper is fairly good, even in the region of low temperature, compared with what is derived by using another equation which is confirmed to be valid in the region of low temperature.
On the Wave Propagation in the Non-Linear Fields

\[
\begin{align*}
(3u^2-v^2)u_x+2uv(u_x-v_x)+(u^2-3v^2)v_x &= 0, \\
(u_x+v_x) &= 0.
\end{align*}
\]

(4.3a)

(4.3b)

The above equation is analogous to (3.3a, b), hence, applying the same method as was used in § 3, we can solve this equation. For the C characteristic equation, we have

\[
(3u^2-v^2)t_x^2-4uvt_xx_x-(u^2-3v^2)x_x^2 = 0,
\]

(4.4)

or

\[
dt/dx = \lambda_\pm = \{2uv \pm \sqrt{3} (u^2-v^2)\} / (3u^2-v^2).
\]

(4.5)

We have also the relation

\[
u^2-v^2 = -(T^2)^0
\]

(4.6)

in which \( T \) is the temperature; then in the quiet gas of temperature \( T_0 \), \( \lambda \) reads

\[
\lambda_\pm = \mp \sqrt{3}.
\]

(4.7)

This may, of course, be interpreted as the sound velocity in the rest medium. It should be noted that in the vacuum \( T_0 = 0 \) and \( \lambda_\pm \) is indefinite, therefore the vacuum is the singular region of the original equation.** On the other hand, the \( I' \) characteristics may be given by

\[
I'^+ : u_\eta - \lambda_\eta v_\eta = 0 \quad \text{along } C^+,
\]

(4.8a)

\[
I'^- : u_\eta + \lambda_\eta v_\eta = 0 \quad \text{along } C^-.
\]

(4.8b)

Hence, also in this case, we have the following relations of representation

\[
t_\eta v_\eta - x_\eta u_\eta = 0, \quad t_\eta v_\eta - x_\eta u_\eta = 0.
\]

(4.9)

Corresponding to (3.11), the equation for \( u \) and \( v \) turns out to be

\[
2(udu-vdv)^2 + (u^2-v^2)(du^2-dv^2) = 0.
\]

(4.10)

This equation may easily be solved by putting \( u = T \sinh \theta \) and \( v = T \cosh \theta \). The result is

\[
Y (1 - \sqrt{3}) - CX (1 + \sqrt{3}) = 0,
\]

(4.11a)

and

\[
Y (1 + \sqrt{3}) - C'X (1 - \sqrt{3}) = 0,
\]

(4.11b)

in which

*) As for the general scheme to transform the equations of these types into the characteristic equations we can, for example, refer to R. Courant and K. O. Friedrichs, loc. cit. The succeeding equations may directly be obtained, if the formulae cited there are used.

**) In the region in which the relation \( u = v \) holds, we have \( \lambda_+ = \lambda_- \) and the equation is parabolic.
\(X=v-u,\) \hspace{1cm} (4.12a)

\(Y=v+u,\) \hspace{1cm} (4.12b)

and \(C\)'s are the integration constants. The correspondence of (4.11a, b) to the \(I^{\pm}\) characteristics can be given by calculating \(dv/du\) on the basis of the above solutions. For this purpose, it is convenient to write \(\lambda^\pm\) as

\[
\lambda^\pm = -\frac{[X^2-Y^2 \pm 2\sqrt{3}XY]}{[X^2+Y^2-4XY]}, \quad (4.5)'
\]

which will often be used in the following arguments. In this way we have

\[
\begin{align*}
I^+: Y'(t-V) - C X'(t-V) &= 0, \quad \text{along } C^+: dt - \lambda_+ dx = 0; \\
I^-: Y'(t-V) - C X'(t-V) &= 0, \quad \text{along } C^-: dt - \lambda_- dx = 0.
\end{align*} \quad (4.13a, b)
\]

Let us investigate, in some illustrative examples, the peculiar characters of the simple wave. This may, on the one hand, be the answer to the subject (i) in § 1 and, on the other, be of use for the succeeding discussions.

We consider firstly the solution satisfying the following conditions: at \(t=0,\)

\[
\begin{align*}
u &= 0 \quad \text{for all } x, \\
v &= \alpha_0 \quad (= \text{const., not zero}) \quad \text{for } x > a, \\
v &= \alpha (x) \quad \text{for } x \leq a,
\end{align*}
\]

in which \(v(x, 0)\) is smooth as a whole (cf. Fig. 4.1). As the region (I) is a constant state of temperature \(T_0,\) the region (II) is that of simple wave. According to (4.7), we have in (I),

\[
C^+: t = \sqrt{3}x + \text{const.}
\]

and in (II),

\[
\begin{align*}
Y'(t-V) - C(\eta) X'(t-V) &= 0 \quad \text{along } C^+, \\
Y'(t-V) - C_0 X'(t-V) &= 0 \quad \text{along } C^-.
\end{align*} \quad (4.14a, b)
\]

Inserting the initial conditions, we have

\[
C(\eta) = \alpha(\eta)^2V, \quad C_0 = \alpha_0^{-2V},
\]

and

\[
\begin{align*}
X &= \alpha(t-V)^2 \alpha_0(t+V)^2, \\
Y &= \alpha(t+V)^2 \alpha_0(t-V)^2.
\end{align*} \quad (4.15a, b)
\]

Accordingly \(\lambda_+\) becomes...
On the Wave Propagation in the Non-Linear Fields

\[ \lambda_+ = \frac{V_+}{V_{+0}} = \frac{V - (2 - \sqrt{3})}{V - (2 - \sqrt{3})}, \]

with

\[ V = (\alpha/\alpha_0)^{\frac{1}{2}} \neq (2 + \sqrt{3}). \]

There exist the following two cases concerning the behaviour of \( \lambda_+ \), according as \( \alpha \) is larger or smaller than \( \alpha_0 \).

1. \( \alpha > \alpha_0 \) (i.e. \( V > 1 \))

\( \lambda_+ \) decreases monotonously from \( \sqrt{3} \) as \( V \) increases monotonously and \( \lambda_+ \) tends to unity in the limit of \( V \to \infty \). (cf. Fig. 4·2)

This means that the velocity profile gradually steepens and it gives rise to discontinuity surfaces, that is, shock waves.\(^*\)

2. \( \alpha < \alpha_0 \) (i.e. \( V < 1 \))

\( \lambda_+ \) increases as \( V \) decreases monotonously

from unity, and tends to \( +\infty \) as \( V \) approaches \( 2 - \sqrt{3} \), then jumps discontinuously to \( -\infty \) as \( V \) exceeds \( 2 - \sqrt{3} \) and increases monotonically as \( V \) decreases. (cf. Fig. 4·3).

Contrary to the case (1), we have, in this case, \( C^+ \)'s spreading out of (III) as \( t \) increases.

We are especially interested in the discontinuity limit of \( B \to A \) in which \( v(x) \) decreases suddenly from \( \alpha_0 \) to \( \alpha_1 \). It may easily be understood that in this case, the \( C^+ \) characteristic lines become a bundle of lines issuing out of \( A \) as are illustrated in Fig.

\(^*\) In fact, this corresponds to the compression wave in the non-relativistic hydrodynamics.
4.4 and hence the region (II) is smoothly connected with the constant regions (I) and (III). This indicates the remarkable fact that the initial discontinuity at \( A \) is immediately resolved. The simple wave, which was discussed in the above example (2), corresponds to the rarefaction wave in the non-relativistic case which can be produced by the motion of receding piston. We illustrated in Fig. 4-5 the characteristics of a rarefaction wave in which \( AB \) is the trajectory of the piston with a constant terminal velocity. We shall encounter this case, when we shall deal with the initial value problem under the physical conditions explained in § 1. The analytical character of such centered rarefaction waves will be given later.

Now we are ready to answer the question (i) in § 1. As can be seen in the above example, the equation (4·1) has such conspicuous properties characteristic to non-linear equations that the initial discontinuity is resolved immediately and the disturbance, while propagated, becomes continuous while the discontinuities, shock fronts, may appear and be propagated even without being caused by initial discontinuities. It should, however, be noted that we have also the same characters of solutions in the non-relativistic hydrodynamics.\(^6\)

Let us now consider the case of a flow invading the vacuum. The initial condition is supposed to be

\[
\begin{align*}
\alpha &= \alpha (\text{const.}) \quad &|x| \leq a, \\
v &= v_0(x) \quad &a \leq |x| \leq a+b, \\
u &= 0 \quad &|x| > a+b, \\
\end{align*}
\]

(4·18)

that is, the initial velocity of fluid is zero and the enthalpy (or temperature) is given only in the region : \(|x| \leq a+b\). It was already stated that the vacuum is the singular region contrary to the case illustrated in Fig. 3.3. Hence, there exists the essential difference between the following two cases, the one that of invading the vacuum and the other the quiet gas.

(i) \( \alpha \)-region (cf. Fig. 4·6)

The definition of \( \alpha \)-region and \( \beta \)-zone are somewhat different from those given in § 3, and the \( \alpha \) region is defined by the characteristic lines \( C^- \) and \( C^+ \) issuing out of two points \( A, (0, a) \) and \( A', (0, -a) \), respectively (cf. Fig. 4·6).

In this region, we have

\[
Y^{(t+V^-)} - \alpha^2 V^- X^{(t-V^-)} = 0 \quad \text{along } C^+,
\]
On the Wave Propagation in the Non-Linear Fields

\[ Y^{(1+Y_{3}^{-})} - \alpha^{-2Y_{3}^{+}} X^{(1+Y_{3}^{-})} = 0 \] along \( C^{-} \);

hence, we obtain the solutions

\[ X = \alpha, \quad Y = \alpha, \quad (4.19) \]

or

\[ v = \alpha, \quad u = 0, \]

and

\[ \lambda^{\pm} = \pm 3. \]

(ii) \( \beta_{II} \)-zone

\[ \beta_{II} \)-zone is defined, in this case, as the region being adjacent to \( \alpha \)-region and bounded by the characteristics \( C^{+} \) and \( C^{-} \) issuing out of \( P \) and \( A \) respectively. (The definition of \( \beta_{II} \) may be obvious from this). (cf. Fig. 4.7)

This region is that of simple wave and we have

\[ Y^{(1+Y_{3}^{-})} - \alpha^{-2Y_{3}^{+}} X^{(1+Y_{3}^{-})} = 0 \]

along \( C^{+} \), and consequently over all \( \beta_{II} \); and

\[ Y^{(1-Y_{3}^{-})} - C(\xi) X^{1+Y_{3}^{-}} = 0 \]

along \( C^{-} \) issuing out of \( AB \). In this equation \( C(\xi) \) can be given as \( C(\xi) = v_{0}(\xi)^{-3/2} \) from (4.8).

These two equations give

\[ Y = \alpha^{(1+Y_{3}^{-})/2} v_{0}^{(1-Y_{3}^{-})/2}, \]

\[ X = \alpha^{(1-Y_{3}^{-})/2} v_{0}^{(1+Y_{3}^{-})/2}, \]

or

\[ v = (1/2) (\alpha v_{0})^{3/2} \left\{ (\alpha/v_{0}) Y_{3}^{-} + (v_{0}/\alpha) Y_{3}^{+} \right\}, \]

\[ u = (1/2) (\alpha v_{0})^{3/2} \left\{ (\alpha/v_{0}) Y_{3}^{-} - (v_{0}/\alpha) Y_{3}^{+} \right\}. \]

The values of \( u \) and \( v \) are constant along the straight \( C^{-} \)'s with gradient
\[ \lambda_\pm = \frac{V + (2 + \sqrt{3})}{V - (2 + \sqrt{3})}, \quad (V \neq 2 - \sqrt{3}) \]

in which \( V = \left(\frac{a}{u_0}\right) V_t^* \) (cf. Fig. 4·7'). In this region \((x > 0)\), where \( V \geq 1 \), the slope of \( C^- \) is equal to \(-\sqrt{3}\) for \( V = 1 \) (corresponding to \( AP \)) and becomes steeper as \( V \) increases up to \( 2 + \sqrt{3} \), is parallel to \( t \)-axis for \( V = 2 + \sqrt{3} \) and then tends to the line of gradient \( \pi/4 \) as \( V \) increases up to \( \infty \). On the other hand, the value of \( \lambda^+ \) tends to unity as \( V \to \infty \), hence the \( C^- \) starting from \( B \) tends to the \( C^+ \) issuing out of \( A \) in the limit of \((x, t) \to \infty\). Hence we have the infinitely extended region \( ABQ \) which is called \( \beta_1 \)-zone, corresponding to the discussions in § 3.

However, we are interested only in the limiting case \( b \to 0 \), or \( B \to A \). Noting the fact that the \( \beta_{12} \)-zone is just the region of rarefaction wave produced by the piston receding to the positive \( x \) direction whose trajectory is given by the boundary characteristic line \( AQ_{\text{lim}} \) (Fig. 4·7), we can easily perform this limiting process, the result of which is shown in Fig. 4·8 (the terminal velocity of piston, in this case, is the light velocity). It can be seen from the above example that the wave invading the vacuum is not the compression wave but the rarefaction wave, contrary to that invading the quiet gas. This indicates that the energy distribution near the wave fronts is entirely different from that of linear equation.

We investigate, now, the analytical behaviour in the \( \beta_{12} \)-zone introduced above. In the \( \beta \)-zone (Fig. 4·7'), we have:

\[ \lambda_\pm = \text{const.} \quad \text{along } C^- , \]

or

\[ t = \lambda^- \cdot (x - a), \quad (4·20) \]

and

\[ Y^{(u + V \beta)} - \alpha^2 V \beta X^{(u - V \beta)} = 0 \]

over all \( \beta \)-zone.

Using the above relations and the expression \( (4·5)' \) for \( \lambda^- \), we can express \( X \) and \( Y \) in terms of \( \lambda^- \) as follows*:

\[ X = \alpha \left[ \frac{(\lambda - 1)}{(2 + \sqrt{3}) (\lambda + 1)} \right]^{(u + V \beta)/V \beta}, \quad (4·21a) \]

\[ Y = \alpha \left[ \frac{(\lambda - 1)}{(2 + \sqrt{3}) (\lambda + 1)} \right]^{(u - V \beta)/V \beta}, \quad (4·21b) \]

*) For the sake of convenience, we write hereafter \( \lambda \) in place of \( \lambda_- \).
On the Wave Propagation in the Non-Linear Fields

\[ X = a \left[ \frac{(t - x + a)}{(2 + \sqrt{3})(t + x - a)} \right]^{(1 + \sqrt{3})/2 \sqrt{3}}, \quad (4.21a) \]
\[ Y = a \left[ \frac{(t - x + a)}{(2 + \sqrt{3})(t + x - a)} \right]^{(1 - \sqrt{3})/2 \sqrt{3}} \quad (4.21b) \]

This gives \( u \) and \( v \) in terms of \( x \) and \( t \), and the solutions in \( \beta \)-zone can be settled down.

On the other hand, the \( C^+ \)'s are naturally not straight lines and they are given by the differential equation

\[ \frac{dt}{dx} = \frac{(X^2 - Y^2 + 2 \sqrt{3} XY)}{(X^2 + Y^2 - 4 XY)}. \]

Inserting (4·21 a, b) into this equation we have

\[ \frac{dt}{dx} = \frac{(2t + \sqrt{3}x - \sqrt{3}a)}{(\sqrt{3}t + 2x - 2a)}, \]

or

\[ \frac{(t + x - a)^{2 - \sqrt{3}}}{(t - x + a)^{2 + \sqrt{3}}} = \text{const.} \quad (4.22) \]

Introducing the new coordinates

\[ t + x' = \sqrt{2} \eta, \quad t - x' = \sqrt{2} \xi, \quad x' = x - a, \]

we have finally

\[ \eta = \text{const.} \cdot \xi^{1 + \sqrt{3}}. \]

This equation gives the graph of \( C^+ \) in the \((x, t)\) space: (Fig. 4·8). The integration constants can be given by the values at the cross points of \( C^+ \)'s and \( AP \). The boundary lines between \( \beta \) and \( \gamma \) regions are thus settled down: they are illustrated in Fig. 4.10.

(iii) \( \gamma \)-zone

In this region, we can transform the original
equation into the linear equation by means of the Legendre transformation, which could not be used in the region of constant state or simple wave. It was shown by Khaltmikov\(^2\) that by this transformation the original equation is reduced to the linear one with constant coefficients, which is easy to handle aside from the complexities in boundary conditions. By this procedure, he has obtained the solution corresponding to the flow invading vacuum. Employing the same method, Amai, Fukuda et al.\(^7\) have also obtained the solutions under more general conditions, which include our initial conditions as the special case. They also investigated the solution in this \(\gamma\) region on the method of characteristics, but in some graphical approximation. In the present paper, we shall try to solve analytically by means of the same method.

In this region, we have

\[
Y^{(1-V_3^\pm)} - C(\lambda) X^{(1+V_3^\mp)} = 0
\]

along \(e \ C^-\) which is the prolongation of that in the \(\beta\)-zone, say \(AB\) (cf. Fig. 4·10). Inserting the relations (4·21a, b), which are valid in the \(\beta\)-zone into the above equation, we have

\[
C(\lambda) = (2 + \sqrt{3})^2 \alpha^{-2V_3^-} (\lambda + 1)^2 (\lambda - 1)^{-2}.
\]

(4·23a)

According to the similar calculation, we also have

\[
Y^{(1+V_3^-)} - C(\lambda') X^{(1-V_3^-)} = 0 \quad \text{along } C^+,
\]

in which \(\lambda' = t/(x+a)\),

\[
C(\lambda') = (2 + \sqrt{3})^{-2} \alpha^{2V_3^-} (\lambda' + 1)^2 (\lambda' - 1)^{-2},
\]

(4·23b)

and the symmetry \(v(x) = v(|x|)\) \(u(x) = -u(|x|)\) for \(x < 0\), has been used. Hence we obtain the solutions

\[
Y = \alpha (2 + \sqrt{3})^{-1/V_3^+} [(\lambda+1)/(\lambda-1)]^{(\alpha-V_3^-)/6} [(\lambda'+1)/(\lambda'-1)]^{(\alpha+V_3^-)/6},
\]

\[
X = \alpha (2 + \sqrt{3})^{-1/V_3^-} [(\lambda-1)/(\lambda+1)]^{(\alpha+V_3^-)/6} [(\lambda'+1)/(\lambda'-1)]^{(\alpha-V_3^-)/6}.
\]

(4·24b)

The temperature \(T\) is given by

\[
T' = \alpha^2 (2 + \sqrt{3})^{-2V_3^-} [(\lambda-1)/(\lambda+1)]^{V_3^-} [(\lambda'+1)/(\lambda'-1)]^{V_3^-}.
\]

(4·25)

Since in the above expressions, the field quantities are given as the functions of \(\lambda\)'s, the shape of \(C^-\) in the \((x,t)\) space should be studied in order to obtain the spatial distribution of these values. In the following discussion we shall investigate the shape of \(C^-\) in this region; the knowledge about \(C^+\) may be obtained from that of \(C^-\) because of the symmetry. Noting the representation relations (4·9) between \(P^+\) and \(C^-\), we see that in the \((Y,X)\) space\(^*\), the directions of \(P^+\)'s and a \(C^-\) are symmetric with respect to \(Y\)-axis. If we draw a \(C^-\) in the \((Y,X)\) space, where \(t\) and \(x\) axis is put to coin-

\*) Coordinate space \((Y,X)\) is given by the \(\pi/4\) rotation of \((u,v)\) space.
cide with \(v\) and \(u\) axis respectively, we have the following equation;

\[
dX/dY = -dX'/dY,
\]

at a cross point between a \(I'\) and the \(C^-'\), in which \(dX/dY\) is the slope of the \(I'\) and \(X_1\) and \(Y_1\) is: \(X_1 = t - x\) and \(Y_1 = t + x\). Along \(I'\), we have

\[
Y' + Y' = C(\lambda')X'V^2 = 0,
\]

i.e.,

\[
dX/dY = - (\sqrt{3} + 1)X/(\sqrt{3} - 1)Y,
dX'/dY' = (\sqrt{3} + 1)X'/ (\sqrt{3} - 1)Y'.
\]

Hence we obtain the following equation for \(C^-'\);

\[
Y_1 = CX_1(Y' - 1)/(Y' + 1) \quad (\text{or } X_1 \approx Y_1^{3.75}); \quad (4.26)
\]

or

\[
(t + x)Y' + D(t - x)Y' = 0, \quad (4.26)'
\]

in which \(D\) is an integration constant and can uniquely be determined as the function of \(\lambda'\). Following the similar way we can obtain the equation for \(C^+\) in which the integration constant is expressed in terms of \(\lambda\). Then, representing \(\lambda\) and \(\lambda'\) in terms of \(x\) and \(t\), and inserting these expressions into (4.24.a, b), we have the solutions \(X\) and \(Y\) (or \(u\) and \(v\)) as functions of \(x\) and \(t\). However, as the final result obtained in this way is too complicated to evaluate, we rather prefer to use the following simple but strict graphical method with the aid of the analytical expression (4.26) for the curves \(C^\pm\).

From (4.26), we have Fig. (4.11) for the graph of \(C\) in which each curve is determined by each \(\lambda\) and \(\lambda'\). The values of \(X\) and \(Y\) are given by (4.24.a, b) in terms of \(\lambda\) and \(\lambda'\), hence the value at the cross points of a \(C^+\) and a \(C^-\), illustrated in Fig. 4.11, is to be uniquely determined.

(iv) Energy distribution

Let us now investigate the energy distribution. In the following discussion, we shall calculate the amount of energy contained in \(\beta\)-zone and \(\gamma\)-region after a sufficiently long time. From (4.2), the energy density \(H\) can be given by

\[
H = (v^2 - u^2) \left( (3/4)v^2 + \frac{3}{2}v^2 \right) \quad (4.27)
\]
At any instant, the energy contained in the whole \( \gamma \) region can be given as the total energy minus the energy contained in the whole \( \beta \)-region. Hence we try to calculate the energy contained in the \( \beta \)-zone.

From (4·22), the boundary curve \( \overline{PR} \) can be given by

\[
(t+x')^2 - V^+ = (2-\sqrt{3})^2(2a)^{-V^+}(t-x')^2 + V^+ ,
\]

(4·22)'
in which \( x' = x - a \), or

\[
\left[1 + \frac{1}{\lambda} \right] t^2 - V^+ = \rho \left[ \left(1 - \frac{1}{\lambda} \right) t^2 + V^+ \right] t^2 V^+ \]

(4·22)''
in terms of \( \lambda = t/x' \), in which \( \rho \) is defined by

\[
\rho = (2-\sqrt{3})^2(2a)^{-V^+} .
\]

(4·28)
The above equations give the value of \( x' \) on the \( \overline{PR} \) at a given instant, \( t = t_0 \), or the corresponding value of \( \lambda = \lambda_0 \); \( \lambda_0 \) tends to 1 as \( t_0 \to \infty \). Therefore we may assume the following expression for the above \( \lambda_0 \):

\[
\lambda_0 \equiv 1 + kt \quad (\nu < 0) \quad \text{for} \quad t \gg 1.
\]

(4·29)

Inserting this into (4·22)'', we have

\[
\nu \approx -2 \frac{\sqrt{3}}{(2 + \sqrt{3})} \approx -0.93 ,
\]

\[
k \approx \left\{2(2 + \sqrt{3})aV^+ \right\}^{2(2-\sqrt{3})} \approx (7.46aV^+)\,0.84.
\]

(4·31)

On the other hand, inserting (4·21a, b) into (4·27) and integrating over the \( \beta^+ \) at an instant \( t = t_0 \), we have

\[
\bar{H}_\beta = (\alpha t/4) \int_{x_0}^{\alpha_0} \left\{Z^{(1+2\sqrt{3})} + Z^{(-1+2\sqrt{3})} + Z^{2\sqrt{3}} \right\} dx',
\]

(4·27)''
in which \( Z \) is given by

\[
Z = (2 + \sqrt{3})^{-1}(\lambda - 1)(\lambda + 1)^{-1} .
\]

The total energy is equal to \( \bar{H} \) at \( t = 0 \),

i.e.

\[
\bar{H} = 2 \times (\alpha t/4) 3a .
\]

(4·27)'''

Denoting

\[
\int_{x_0}^{\alpha_0} Z^t \, dx' = I_t ,
\]

and transforming \( x' \) into \( \lambda' \), we have

\[
I_t = 2t(2 + \sqrt{3})^{-1} \int_{0}^{\alpha_0} w^t(1+w)^{-2} \, dw ,
\]

in which

\[
w = (\lambda - 1)(\lambda + 1)^{-1} ;
\]
On the Wave Propagation in the Non-Linear Fields

and $0 \leq w \leq 1$, $(1 \leq \lambda \leq \infty)$ for $t_0 > 1$.

Using the above property of $w$, the integral of $(4 \cdot 32)$ may be estimated as follows:

$$
(4(\gamma+1))^{-1}w_0^{\gamma+1} \leq \int_0^{\infty} w^{\gamma+2}dw \leq (\gamma+1)^{-1}w_0^{\gamma+1}.
$$

(4 \cdot 33)

From $(4 \cdot 29), (4 \cdot 30), (4 \cdot 21)$, we have

$$
\psi_0^{\gamma+1} \approx \frac{(k/2)^{\gamma+1} t_0^{-2/V_2(\gamma+1)}(1+V_2)}{2 + V_2}.
$$

(4 \cdot 34)

Inserting $(4 \cdot 33), (4 \cdot 34)$ into $(4 \cdot 27)$, and neglecting the terms decreasing in the order of $t_0^{-1}$ or more, we have

$$
\frac{\alpha^4}{4} \leq \frac{8.66a(t_0/a)^{-0.072}}{2t_0} \leq \frac{2 \cdot 14a(t_0/a)^{-0.072}}{4}.
$$

(4 \cdot 35)

This indicates that the amount of energy contained in the $\beta$-zone decreases slowly with time, and is mainly determined by the factor $(t_0/a)^{-0.072}$. We have

$$
(t/a)^{-0.072} \approx 1/2.1 \quad \text{for } t \approx 2.2 \times 10^3,
$$

$$
\approx 1/11 \quad \text{for } t \approx 2.2 \times 10^{13};
$$

in the ordinary scale*, the above values of $t$ correspond to $10^{-20}$ sec. and $10^{-18}$ sec. respectively. In other words, the amount of energy contained in the $\gamma$-zone is one-half at $t \approx 10^{-20}$ sec and gradually increases with time and at $t \approx 10^{-18}$ sec. the major part of energy is contained in $\gamma$-region, or the energies concentrated near the wave fronts dissipate gradually with time.

§ 5. Concluding remarks

We have proved that under the specified initial condition, the Born type equation treated in this paper has not shown any feature characteristic to non-linear theory. However, it might be doubtful that this conclusion holds even for other equations of Born type. Similarly the nature of the smoothing away of the energy distribution shown in the extremely relativistic hydrodynamics might be changed due to the slight modification of the original equation. However, for the study of these kinds of problems, it is desirable to do somewhat more general discussion.

The author would like to express his sincere thanks to Professor Y. Tanikawa, Professor S. Tomotika and Professor Z. Koba for their advices and to Mr. N. Mugibayasi for his discussions.

*) The value of $a$ is taken to be $a = 0.07$ corresponding to the nuclear dimension (cf. § 3).
References

6) R. Courant and K. O. Friedlichs, loc. cit. See § 41, § 42 and § 48