Magnetic and viscous coupling at the core-mantle boundary: inferences from observations of the Earth’s nutations

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SUMMARY
Dissipative core–mantle coupling is evident in observations of the Earth’s nutations, although the source of this coupling is uncertain. Magnetic coupling occurs when conducting materials on either side of the boundary move through a magnetic field. In order to explain the nutation observations with magnetic coupling, we must assume a high (metallic) conductivity on the mantle side of the boundary and a rms radial field of 0.69 mT. Much of this field occurs at short wavelengths, which cannot be observed directly at the surface. High levels of short-wavelength field impose demands on the power needed to regenerate the field through dynamo action in the core. We use a numerical dynamo model from the study of Christensen & Aubert (2006) to assess whether the required short-wavelength field is physically plausible. By scaling the numerical solution to a model with sufficient short-wavelength field, we obtain a total ohmic dissipation of 0.7–1 TW, which is within current uncertainties. Viscous coupling is another possible explanation for the nutation observations, although the effective viscosity required for this is 0.03 m² s⁻¹ or higher. Such high viscosities are commonly interpreted as an eddy viscosity. However, physical considerations and laboratory experiments limit the eddy viscosity to 10⁻⁴ m² s⁻¹, which suggests that viscous coupling can only explain a few percent of the dissipative torque between the core and the mantle.

Key words: core–mantle coupling, eddy viscosity, ohmic dissipation, magnetic field, turbulence.

1 INTRODUCTION
Tidal forces cause small periodic variations (nutations) in the Earth’s angular velocity (Mathews & Shapiro 1992). Precise measurements of nutations using radio interferometry (Herring et al. 1986, 1991, 2002) provide unique insights into the coupling between the core and the mantle. Sensitivity to core–mantle coupling arises because the angular velocities of the core and mantle become misaligned when the Earth is perturbed by tidal forces. The resulting relative motion across the core–mantle boundary alters the measured response at the Earth’s surface.

An important source of coupling in nutations is due to the effect of fluid pressure on the slightly elliptical core–mantle boundary. Motion of the fluid core relative to the mantle can be described to a first approximation as a rigid-body rotation. However, this flow is disturbed by the elliptical shape of the boundary. Fluid pressure acts on the boundary to produce a torque that limits the misalignment of the angular velocity vectors. Theoretical predictions for a hydrostatic earth model (Wahr 1981) show that the restoring force due pressure creates a mode of oscillation in which the angular velocities of the core and mantle precess about each other in a retrograde direction (e.g. opposite to the direction of rotation). Impulsive disturbances excite free oscillations of this mode, known as the retrograde free core nutation or RFCN (Sasao & Wahr 1981; Dehant et al. 1996). Periodic disturbances due to tidal forces are resonantly enhanced by the RFCN when the tidal frequency approaches the natural frequency of this mode.

Magnetic and viscous coupling alter both the frequency and damping of the RFCN. Changes in the forced response become most evident when the tidal frequencies are close to the resonance frequency. A shift in the frequency of the RFCN alters the in-phase response to tidal forces, whereas damping causes a small phase lag (e.g. an out-of-phase response). Herring et al. (1986) detected a small discrepancy of 1.8 milliarcseconds (mas) between the predicted and observed in-phase amplitude at the retrograde annual period. This discrepancy was attributed by Gwinn et al. (1986) to an excess flattening of the core–mantle boundary relative to the hydrostatic shape. This modification of the earth model increases the pressure torque and shifts the frequency of the RFCN towards the retrograde annual forcing, thereby increasing the resonant response at that tidal frequency. A comparable shift in the RFCN frequency has been detected in gravity measurements (Neuberg et al. 1987; Defraigne et al. 1994). Herring et al. (1986) also reported a 0.4-mas discrepancy in the out-of-phase amplitude of the retrograde annual nutation. This discrepancy was subsequently attributed by Buffett (1992) to magnetic coupling, although an effective viscosity
of $10^{-1} \text{ m}^2 \text{s}^{-1}$ for the liquid core was proposed as an alternative explanation.

Magnetic coupling occurs when conducting materials on either side of the boundary move through a magnetic field $\mathbf{B}$. Relative motion across the boundary generates electric currents $\mathbf{J}$, which produces a force that opposes the relative motion. The strength of the force is determined by the electrical conductivity $\sigma$ near the boundary because the high-frequency of the relative motion confines electric currents to a narrow skin depth. The value of $\sigma$ on the mantle side is particularly important because the conductivity affects electric currents to a narrow skin depth. The force is given by the equation $\sigma \mathbf{J} \times \mathbf{B}$.

The mantle side of the boundary move through a magnetic field $\mathbf{B}$ at a frequency $f$.

More recent work of Nakada (2006) has revisited the question of field distribution by highlighting the special role of non-axial components of the magnetic field in coupling axial and equatorial rotations. However, these calculations neglect the dominant influence of pressure coupling on nutations. More complete calculations, with different realizations of a randomly chosen non-dipole field (Deleplace & Cardin 2006), suggest that the spatial distribution of the field is much less important than other key parameters, such as the conductivity of the boundary region and the rms strength of the radial field.

Improvements in the nutation theory to allow for the influence of magnetic forces on the fluid motion (Buffett 2002) increase the radial field required to explain the nutation observations. The total rms radial field increases to 0.69 mT, which is partitioned into a dipole component (0.216 mT) and a non-dipole component (0.66 mT). (We use a smaller dipole component because Buffett et al. 2002 used the dipole value of $|\mathbf{B}|^\text{rms}$ rather than $B_0^\text{rms}$. However, the total rms radial field is still 0.69 mT.) The minimum field needed to fit the observations within the standard error is $B_0^\text{rms} = 0.65 \text{ mT}$. By comparison, downward continuation of the surface field in spherical harmonic degrees $l \leq 13$ yields $B_0^\text{rms} = 0.31 \text{ mT}$, based on the IGRF-10 model for 2005 (Macmillan & Maus 2005). The higher value inferred from observations implies substantial energy in the field at shorter wavelengths (e.g. $l > 13$), which cannot be measured at the surface because it is obscured by the crustal field. The goal of this paper is to assess whether the energy in the short-wavelength field is compatible with the results of recent geodynamo models (see Fig. 1). We also examine the consequences of the short-wavelength field on the power needed to drive the geodynamo. Finally, we use physical arguments and laboratory experiments to place limits on the size of eddy viscosities that can be used in calculations of viscous coupling for nutations (Palmer & Smylie 2005; Deleplace & Cardin 2006).

**2 Magnetic Spectrum at the Core-Mantle Boundary**

The internal magnetic field can be represented at the Earth’s surface by

$$\mathbf{B} = -\nabla V,$$  

(1)

where the scalar magnetic potential $V$ satisfies $\nabla^2 V = 0$. The solution in spherical coordinates $(r, \theta, \lambda)$ can be expanded in spherical harmonics as (Langely 1987)

$$V = a \sum_{l=1}^{\infty} \left( \frac{\alpha}{r} \right)^{l+1} \sum_{m=0}^{l} \left( g_l^m \cos m\lambda + h_l^m \sin m\lambda \right) P_l^m(\cos \theta),$$  

(2)

where $\alpha = 6371.2 \text{ km}$ is the average radius of the surface, $g_l^m$ and $h_l^m$ are the Gauss coefficients, and $P_l^m(\cos \theta)$ are the Schmidt normalized Legendre functions of degree $l$ and order $m$. The mean square field at the surface $r = a$ due to all harmonics of degree $l$ is given by (Lowes 1966)

$$R_l(a) = (l+1) \sum_{m=0}^{l} \left( g_l^m \right)^2 + \left( h_l^m \right)^2$$  

(3)

which can be extrapolated to other values of $r$ using

$$R_l(r) = R_l(a)(a/r)^{2l+4}$$  

(4)

when there are no sources between $a$ and the prescribed value of $r$. We also require the mean square radial field, which can be expressed in terms of $R_l(r)$ as

$$[B_l^\text{rms}(r)]^2 = \sum_{l=1}^{\infty} \frac{(l+1)}{(2l+1)} R_l(r).$$  

(5)

Fig. 2 shows $R_l(b)$ at the the core–mantle boundary $r = b$, calculated using the Gauss coefficients for $l \leq 10$ from the IGRF-10 model (open circles). An extrapolation of the IGRF-10 model to higher degrees is based on the theoretical spectrum of Vorhies et al. (2002). We also calculate the spectrum $R_l(b)$ using the Gauss coefficients from a geodynamo model (described below), which is scaled to match $R_l(b)$ in the IGRF-10 model. The non-dipole field from the dynamo model agrees reasonably well with the IGRF-10 model, where the spectra overlap in $l$, but the values of $R_l(b)$ from the dynamo model do not decay with $l$ as rapidly as the theoretical extrapolation of the IGRF-10 model. The third spectrum in Fig. 3 (called the geodetic model) is chosen to explain the nutation observations. This model is obtained by matching $R_l(b)$ in the IGRF-10 model and fitting the form of the function

$$R_l(b) = R_l \chi^l$$  

(6)

to the non-dipole field. The coefficient $\tilde{R} = 9.42 \times 10^8 \text{ nT}^2$ is fit to the average value of $R_l(b)$ from the IGRF-10 non-dipole field ($l = 2 - 10$), and the slope $\chi = 0.99$ is chosen to ensure that there is sufficient energy at short wavelengths to explain the nutation observations. The required model for the calculation of nutation observations is not altered if we replace the non-dipole field in $l = 2 - 10$ with the known spectrum from IGRF-10 because most of the power in the geodetic model is contained in scales beyond the resolution of the IGRF-10 model.

The spectrum $R_l(b)$ for the geodetic model is flatter than either the geodynamo model or the IGRF-10 extrapolation, particularly over intermediate scales ($l \approx 10 - 60$). Differences between the geodetic and geodynamo models are not imappable given the physical parameters currently used in geodynamo models. Geodynamo models must use unrealistically high viscosity $\nu$ and thermal diffusivity $\kappa$ to suppress small scales which cannot be resolved in numerical calculations. The magnetic spectrum at the smallest scales is clearly affected by the suppression of small-scale flow. However, it is less clear whether the spectrum at intermediate scales is also affected. Magnetic diffusion may reduce the magnetic spectrum at small scales to such a level that the omission of these scales is irrelevant for any of the larger scales. Christensen & Tilgner (2004) have...
Figure 1. (A) Radial magnetic field at the core–mantle boundary from a numerical model of Christensen & Aubert (2006). The control parameters that define the calculation include the Ekman number $E = 3 \times 10^{-5}$, the modified Rayleigh number $Ra = 0.18$, the Prandtl number $Pr = 1$, and the magnetic Prandtl number $Pm = 2.5$. Definitions are given in the text. (B) The same radial field filtered to the spatial resolution of models based on magnetic observations (e.g. $l \leq 12$).

argued that the characteristic dissipation length (and hence the nature of magnetic spectrum) depends only on the magnetic Reynolds number $Rm$ (defined below). This implies that the magnetic spectrum is independent of both $\nu$ and $\kappa$, and that reliable predictions are given by models with suitable values for $Rm$. On the other hand, we cannot rule out a dependence on $\nu$ or $\kappa$. Dynamo models with realistic values for $\nu$ and $\kappa$ could yield a flatter spectrum than that shown in Fig. 2. However, it is hard to imagine how the use of lower (and more realistic) values of $\nu$ and $\kappa$ in the dynamo model could make the magnetic spectrum decay faster with $l$. (Decreasing the convective vigor could reduce the magnetic energy at large $l$, but such models are less likely to match the expected value of $Rm$ for the Earth’s core.)

The spectrum in the dynamo model is much flatter than the one inferred from the IGRF-10 model, especially at large and intermediate scales. This difference implies much more ohmic dissipation in the dynamo model than would be inferred from an extrapolation of the IGRF-10 model. If the actual spectrum was flatter than the spectrum from the dynamo model, as is suggested by the geodetic model, then we need to ensure that the additional magnetic energy at small scales does not impose unreasonable demands on the power required by the Earth’s geodynamo.

The magnetic field must be continually regenerated by dynamo action to offset the persistent ohmic losses. The ohmic dissipation is given by

$$\Phi = \int_V \frac{J^2}{\sigma} \, dV,$$

where the volume $V$ includes both the core and conducting part of the mantle. Because the volume of a thin conducting layer at the base of the mantle is expected to be small, we confine our attention to the volume of the core. We also adopt a constant electrical conductivity of $\sigma = 5 \times 10^7$ S m$^{-1}$ for the core. A lower bound on the ohmic dissipation is given by the minimum current needed to sustain the potential field at the core–mantle boundary. Gubbins (1975) showed that

$$\int_V J^2 \, dV \geq \sum_{l} J_l^2,$$

where

$$J_l^2 = \frac{(2l + 1)(2l + 3)}{4\pi C} b R_l(b)$$

and $C = 10^{-14}$ converts the units of $b$ $R_l(b)$ (e.g. m-nT$^2$) to $A^2$ m$^{-1}$. Consequently, the minimum dissipation $\Phi_{\text{min}}$ is given by

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For the geodetic model, the minimum required magnetic spectrum with smaller energy in the truncated spectra. For example, we can define a new, smoothly varying spectrum that approximates the magnetic field vanishes for \( l \) above the truncation. It is more likely that the spectra decay more smoothly to zero with increasing \( l \). However, for any given truncation we can always define a new, smoothly varying spectrum that approximates the magnetic energy in the truncated spectra. For example, we can define a new geodetic spectrum with smaller \( \chi \) in eq. (6) which yields the same \( B_{r}^{\min} \) and nearly the same \( \Phi_{\min} \) as \( l \rightarrow \infty \). Assuming a particular degree of truncation in the geodetic spectrum is merely a convenient way of representing the amount of short-wavelength field that is required to explain the nutation observations. A truncation in the extrapolation of IGRF-10 might represent the scale where physical assumptions used by Voorhies et al. (2002) to construct the theoretical spectrum break down. A steeper theoretical spectrum at high \( l \) is necessary because the predicted \( \Phi_{\min} \) is unbounded as \( l \rightarrow \infty \). For the geodetic model, the minimum required field (e.g. \( B_{r}^{\min} = 0.65 \) mT) occurs at a truncation of \( l = 160 \). The corresponding value for the minimum ohmic dissipation of \( \Phi_{\min} = 0.010 \) TW. The best fit to the nutation observations is obtained when \( B_{r}^{\min} = 0.69 \) mT (Buffett et al. 2002), which occurs in the geodetic model at \( l = 250 \). The minimum dissipation in this case is \( \Phi_{\min} = 0.015 \) TW. For comparison, we also show the estimate inferred from the dynamo model, which has a truncation at \( l = 133 \).

The extrapolation of the IGRF-10 field model does not have sufficient magnetic energy at small scales to explain the nutation observations. Truncating this extrapolation at \( l = 300 \) yields \( B_{r}^{\min} = 0.407 \) mT, which explains only about 35 per cent (e.g. \( 0.407^2/0.69^2 \)) of the observed out-of-phase anomaly. In order to explain the entire signal we require a flatter magnetic spectrum at intermediate scales. This change in the spectrum implies a modification of the assumptions used by Voorhies et al. (2002). Rather than deal with differences in underlying assumptions, we attempt to determine whether the geodetic model is feasible on energetic grounds.

The actual dissipation in the core is liable to be much higher than \( \Phi_{\min} \) because of contributions from the toroidal part of the field, which is confined to the source region and cannot be directly observed at the surface. It is also possible that the poloidal field is stronger in the interior of the core than it is at the core–mantle boundary. Our approach is to calculate the total dissipation in the dynamo model and compare the result with the minimum bound \( \Phi_{\min} \), based on the potential field from the same model. We use the ratio \( \Phi/\Phi_{\min} \) to scale \( \Phi_{\min} \) for the geodetic model to obtain a total dissipation. Both \( \Phi \) and \( \Phi_{\min} \) are expected to have a similar dependence on the characteristic length scale of the magnetic field inside the core. For example, a decrease in the characteristic length
scale $L$ of the internal field field causes a predictable increase in $\Phi$ (e.g. $\Phi \propto L^{-2}$ when the field strength is constant). We assume that the change $L$ produces a change in the scale of the field at the core–mantle boundary, and a proportional change in $\Phi_{\text{min}}$.

3 OHMIC DISSIPATION IN GEODYNAMO MODEL

Geodynamo models are specified by the control parameters that appear in the governing equations. We use a geodynamo model from the study of Christensen & Aubert (2006), where models are described in terms of the Ekman number

$$E = \frac{\nu}{\Omega D^2}$$

the modified Rayleigh number

$$Ra' = \frac{\alpha g_c \Delta T}{\Omega^2 D}$$

the Prandtl number

$$Pr = \frac{\nu}{\kappa}$$

and the magnetic Prandtl number

$$Pm = \frac{\nu}{\lambda},$$

where $\nu$ is viscosity, $\kappa$ is thermal diffusivity, $\lambda$ is magnetic diffusivity, $\Omega$ is rotation rate, $D$ is the thickness of the spherical shell, $\Delta T$ is the temperature drop across the shell, $\alpha$ is thermal expansivity and $g_c$ is the value of gravity at the outer radius $r_o$ of the shell. The dynamo model used in this study is defined by $E = 3 \times 10^{-5}$, $Ra' = 0.18$, $Pr = 1$ and $Pm = 2.5$. This particular model was chosen because the magnetic Reynolds number $Rm = VD/\lambda \approx 900$, based on the calculated average velocity $V$, agrees well with the value inferred for the Earth (Christensen & Tilgner 2004). In addition, the ratio of power in the low-order non-dipole field to that of the dipole agrees roughly with that in the present geomagnetic field (see Fig. 1).

Christensen & Aubert (2006) choose $\Omega^{-1}$ as a timescale and define a magnetic scale using $(\rho \mu)^{1/2} \Omega D$, where $\rho$ is the fluid density and $\mu$ is the magnetic permeability. The magnetic energy $E_B$, scaled by $\rho \Omega^2 D^3$, is given by

$$E_B = \frac{1}{2} \int_V B^2 \, dV,$$

where the integral is taken over all space, although most of the energy is contained within the volume of the core. The corresponding rate of ohmic dissipation, scaled by $\rho \Omega^3 D^3$, is given by

$$D_\Omega = E_B \int_V \left( \nabla \times B \right)^2 \, dV,$$

where $E_B = E / Pm$. (We use $D_\Omega$ to distinguish the scaled dissipation from the physical dissipation $\Phi$ in units of W.) For the snapshot shown in Fig. 1, we have $E_B = 2.479 \times 10^{-3}$ and $D_\Omega = 1.108 \times 10^{-4}$. Neither of these values can be scaled directly to meaningful physical quantities because the dimensionless control parameters are very far from Earth-like values. However, we can still estimate $\Phi$ by assuming that the model gives a reasonable approximation for the spatial structure of the field. Equivalent approaches have been used previously by Buffett et al. (2002) and Christensen & Tilgner (2004).

We define a characteristic dissipation length scale $L$ by expressing $D_\Omega$ in the form

$$D_\Omega \approx \frac{E_B}{L^2} \int_V B^2 \, dV.$$

Both eqs (15) and (17) depend on the mean-square field, so the ratio $E_B / D_\Omega$ can be rearranged for the (non-dimensional) length scale

$$L \approx \sqrt{2 E_B / D_\Omega} = 0.02315.$$

Using $D = 2.26 \times 10^6$ m gives a dimensional dissipation length scale of 52 km. We use this length scale in eq. (7) to approximate $\Phi$ in physical units. Noting that

$$J = \frac{1}{\mu} \nabla \times B,$$

we can approximate the integral in eq. (7) using

$$\Phi \approx \frac{B^2 V_e}{\alpha \mu^2 L^2},$$

where $B = 1.8$ mT is the volume-averaged (rms) field from the dynamo model, after adjusting $R_1(b)$ to match the value from the IGRF-10 model, $V_e = 1.69 \times 10^2$ m is the volume of the outer core, and $\mu = 4 \pi \times 10^{-7}$ H m$^{-1}$. The resulting dissipation is $\Phi = 0.26$ TW, or about 70x larger than $\Phi_{\text{min}}$ for the scaled dynamo model (see Fig. 3).

In comparison with the dynamo model, the geodetic model requires more magnetic energy in short wavelengths at the core–mantle boundary. This presumably implies a shorter dissipation length scale, which is consistent with higher values of $\Phi_{\text{min}}$ for the geodetic model. A radial field of 0.65–0.69 mT for the geodetic model implies a minimum dissipation of $\Phi_{\text{min}} = 0.01 – 0.015$ TW. Assuming that the ratio $\Phi / \Phi_{\text{min}}$ remains constant as $L$ varies (e.g. both $\Phi$ and $\Phi_{\text{min}}$ have a similar dependence on $L$), we obtain a total dissipation for the geodetic model of 0.7–1.05 TW. These values are somewhat higher than recent estimates (e.g. Christensen & Tilgner 2004), but still within current uncertainties. For example, an ohmic dissipation of 1 TW can be sustained at the present time with a core heat flow of 4–9 TW, depending on the choice of parameter values (e.g. Buffett 2002; Gubbins et al. 2004). Somewhat higher heat flows are required if radioactive heat sources are present in the core because the rate of inner-core growth is reduced. We conclude that the radial field needed to explain the nutation observations with magnetic coupling is physically plausible. Indeed, the nutation observations may provide an important constraint on the magnetic field if the source of high electrical conductivity at the base of the mantle can be adequately resolved.

These results are in conflict with the study of Deleplace & Cardin (2006), which concludes that magnetic coupling cannot explain the nutation observations, even if a conducting layer is present at the base of the mantle. This disagreement can be attributed to the truncation of the spherical harmonic expansion for the radial field at the core–mantle boundary. Deleplace & Cardin (2006) truncated their field model at $l = 40$, whereas we truncate the field model at $l = 160$ or higher. The severe truncation used by Deleplace & Cardin (2006) is not consistent with the dynamo model of Christensen & Aubert (2006), which has plenty of energy beyond $l = 40$. However, the field from the dynamo model is still insufficient to explain the nutation observations. The geodetic model puts more magnetic energy at small scales, but this short-wavelength field does not produce unrealistic power demands for the geodynamo.

4 TURBULENT DISSIPATION AT THE CORE-MANTLE BOUNDARY

Several recent studies (Mathews & Guo 2005; Palmer & Smylie 2005; Deleplace & Cardin 2006) have investigated the effects of
viscous coupling as an alternative explanation of the nutation observations. The estimates of viscosity inferred from these studies range from 0.03 to 0.06 m$^2$ s$^{-1}$. Because these values exceed laboratory and theoretical estimates by four or five orders of magnitude (Gans 1972; de Wijs et al. 1998; Dobson et al. 2000; Vočadlo et al.2000; Rutter et al. 2002; Terasaki et al. 2006), it is often assumed that the viscosity recovered from a viscous coupling calculation represents a turbulent or eddy viscosity (e.g. Brito et al. 2004; Deleplace & Cardin 2006). In dealing with turbulent viscosities it is important to remember that the effective viscosity is a property of the fluid rather than a property of the fluid. This connection to physical process limits the magnitude of the eddy viscosity for a given flow.

Dimensional arguments show that the eddy viscosity $\nu_e$ can be approximated by (Tennekes & Lumley 1972)

$$\nu_e \approx v l,$$

where $v$ and $l$ are the velocity and size of the eddies. An upper bound on $v$ is defined by the large-scale flow in the core because we expect the convective velocity to decrease when the length scale $l$ becomes small. Taking $v = 10^{-4}$ m s$^{-1}$ for the large-scale velocity, we obtain diffusivities within the range of recent estimates when $l = 300 - 600$ m. The overturn time for these eddies is $\tau = l/v$, which is in excess of 35 d. A more realistic extrapolation of convective velocities to small scales would inevitably give a smaller $v$ and require a larger $l$ to explain the inferred viscosity. The resulting overturn time is probably much longer than 35 d. These scales should be compared with those for nutations, which involve circular motion of fluid elements in the core relative to the mantle with amplitude of roughly 1 m and a period of 1 d. Over the short period of a nutation, the relatively large eddies would appear to be nearly stationary. More importantly, the eddies have little opportunity to transport horizontal momentum away from the boundary over one nutation cycle. This is equivalent to saying that the Reynolds shear stress is small. Thus the incompatibility between the period of the nutations and the overturn time of the eddies means that the Reynolds stress is potentially too small to explain the out-of-phase nutation amplitude.

There is also an incompatibility in length scales. The thickness of the viscous boundary layer is nominally

$$\delta \approx \sqrt{\frac{2 \nu_v}{\Omega}},$$

although the effect of rotation causes modest variations with latitude (e.g. Mathews & Guo 2005). For a turbulent viscosity $\nu_v = 0.03$ m$^2$ s$^{-1}$ and $\Omega = 0.73 \times 10^{-4}$ s$^{-1}$, we obtain a boundary layer thickness of 30 m. By comparison, the eddies required to obtain $\nu_v = 0.03$ m$^2$ s$^{-1}$ are at least 300 m, and probably larger. Consequently, the eddy size is not consistent with the boundary of the core, rather than the boundary thickness. Consequently, the eddy size is not consistent with the boundary thickness when the influence of the eddies is represented using an effective viscosity. This inconsistency suggests that an eddy viscosity may not be the appropriate description for the nutations, particularly when the eddies are relatively large. In order to assess the magnitude of the Reynolds stress, we note that the effective stress in an eddy-viscosity model is defined by

$$\tau = \rho \nu_e \frac{\partial v_f}{\partial r} \approx \rho \nu_e \frac{v_f}{\delta},$$

where $v_f$ is the velocity of the core relative to the mantle due to nutations (specifically for the retrograde annual term). For the values required to explain the nutation observations, $v_f \approx 10^{-4}$ m s$^{-1}$, $\nu_e = 0.03$ m$^2$ s$^{-1}$ and $\delta = 30$ m, we obtain the necessary value of shear stress $\tau = 10^{-3}$ Pa. However, the inconsistencies in both length and timescales raise questions about the validity of this estimate.

More standard treatments of turbulent boundaries typically assume that the length scale of eddies is limited by the distance from the wall (e.g. Tennekes & Lumley 1972).

A more likely source of dissipation is due to the effects of flow over a rough core–mantle boundary. Roughness on a scale of approximately 1 m would promote flow separation and turbulence, enhancing dissipation in the fluid. The velocity of the eddies might be comparable to the flow velocity and the size of the eddies might be comparable to the boundary roughness (e.g. $l \approx 1$ m). Combining these results gives $\nu_v \approx 10^{-4}$ m$^2$ s$^{-1}$, which is about two orders of magnitude smaller than the viscosity required to explain the nutations. The boundary-layer thickness is $\delta \approx 1.7$ m and the resulting shear stress is $\tau \approx 6 \times 10^{-5}$ Pa, or about 20 times smaller than the required stress.

Le Mouël et al. (2006) take a different approach to the problem of flow over a rough boundary, but they obtain a very similar result. They consider a solid boundary moving over a stationary fluid, and assume that turbulent flow in the wake of the topography is accelerated (on average) to the velocity of the boundary (see Fig. 4). For the special case of boundary topography comprised of regular pyramids with a basal area of $A^2$ and height $h$, they obtain an estimate for the effective stress on the boundary

$$\tau = \frac{1}{2} \frac{h}{A} \rho v_f^2.$$

Taking $h/A = 1$ gives $\tau = 5 \times 10^{-5}$ Pa, which is comparable to the estimate based on an eddy viscosity with $l \approx 1$ m.

Both of the preceding estimates for $\tau$ are probably too large, based on comparisons with laboratory experiments of turbulent oscillatory flow over a rough boundary (e.g. Sleath 1987). Experimental studies

\[\text{Figure 4. Schematic illustration of flow at a rough boundary. (A) When the solid boundary moves relative to a stationary fluid, turbulent flow is generated behind the crests of small-scale (≈1 m) topography. The turbulent flow moves (on average) at the velocity of the solid. (B) A buoyant fluid at the top of the core is trapped by the boundary topography. Pressure forces from the boundary accelerate the buoyant fluid to the velocity of the solid provided the buoyancy is sufficient to prevent the fluid from spilling over the crests of the topography. Magnetic coupling to the underlying fluid transfers momentum to the bulk of the core and induces a cavity flow in the buoyant fluid. Topography of roughly 200 m or more is needed to ensure a sufficient magnetic stress on the fluid core.}\]
often report \( \tau \) in terms of a friction coefficient

\[
f_w = \frac{\tau}{2 \rho v^2_{\infty}},
\]

(25)

where \( v_{\infty} \) represents the amplitude of the far-field velocity relative to the boundary. (This is equivalent to \( v_f \) in nutations.) The value of \( \tau \) in the experiments is computed from the acceleration of the fluid in the boundary region using

\[
\tau = \rho \int_{0}^{\infty} \frac{\partial [v_w - v(y)]}{\partial t} \, dy,
\]

(26)

where the velocity \( v(y) \) is measured as a function of distance \( y \) from the mean level of the topography on the boundary. The friction coefficients reported by Sleath (1987) vary from \( 10^{-2} \) to \( 10^{-1} \). The largest values are obtained when the orbital amplitude of fluid parcels outside the boundary layer is comparable to the scale of the roughness. Taking \( f_w = 0.1 \) gives \( \tau = 5 \times 10^{-6} \) Pa, which is hardly bigger than the stress expected with a molecular viscosity. The reason is simply that the inertia associated with \( v_f \) is small. The Reynolds number, based on molecular viscosity \( \nu \approx 10^{-6} \) m² s⁻¹ and the roughness scale \( l \approx 1 \) m is \( Re = v_f l / \nu \approx 100 \). (By comparison the experiments with the largest \( f_w \) have \( Re = 10^3 \).) At \( Re \approx 100 \), it is not obvious that flow associated with nutations is turbulent. Even if turbulence develops we do not expect the eddy viscosity to exceed \( 10^{-4} \) m² s⁻¹. The associated stress is probably less than \( 5 \times 10^{-5} \) Pa. To put this into context, we note that a comparable magnetic stress arises based on molecular viscosity.

\[\tau \approx \frac{B_{rms}^2}{\rho c} = 0.18 \text{ mT}^2.\]

5 DISCUSSION

Magnetic coupling provides a viable explanation for the dissipative coupling observed in nutations, but only if a thin conducting layer is present at the base of the mantle. A lower bound on the radial field of \( B_{rms} = 0.65 \) mT implies substantial energy at small wavelengths. However, the total ohmic dissipation associated with this field is estimated to be 0.7 TW. Dissipation increases to 1 TW if the radial field increases to 0.69 mT. These estimates of dissipation are within current uncertainties, but even modest increases in the radial field could raise the dissipation to implausibly high levels. Consequently, the viability of magnetic coupling to explain the nutation observations depends critically on the conductivity of the layer at the base of the mantle. A number of mechanisms have been proposed to incorporate liquid iron into the base of the mantle (e.g. Porier & LeMouël 1992; Buffett et al. 2000; Petford et al. 2005; Kanda & Stevenson 2006), but the bulk conductivity of the mixture is limited by the volume fraction of metallic iron. Bulk conductivities below \( 5 \times 10^4 \) S m⁻¹ require stronger radial fields to match the observations. The attendant ohmic dissipation is potentially a problem.

An alternative source of conductivity might develop if a buoyant fluid collects at the top of the core (see Fig. 4). The fluid would be isolated by topography on the boundary, and a sufficient density deficit relative to the bulk of the core (possibly due to an excess concentration of light elements) would prevent this fluid from being swept over the crests of the topography. To give a rough idea of the required density contrast, we note that \( \Delta \rho / \rho \approx 10^{-7} \) is probably enough to suppress Kelvin-Helmholtz instabilities when a velocity difference of \( 10^{-4} \) m s⁻¹ between the buoyant fluid and the underlying core is separated by a viscous boundary layer with thickness \( \delta \approx 0.1 \) m. A thicker shear layer promotes stability (see Chandrasekhar 1981) for a simple stability criterion based on the Richardson number). Greater stability would also be expected if the buoyant fluid was immiscible in the bulk of the liquid core because the density difference could potentially be quite large (Eaton & Kendall 2006).

Buoyant fluid near the boundary would move with the mantle because of the pressure exerted by the topography. Magnetic coupling would transfer momentum from the buoyant fluid to the bulk of the core, potentially dissipating the energy needed to explain the nutation observations. Clearly, the dynamics of the buoyant fluid needs to be considered in calculating the consequences for nutations, but this example illustrates that there are many ways to magnetically coupling the core to the mantle. The same may also be true for viscous coupling, although the effects of convective turbulence and flow over a rough boundary are insufficient to explain more than a few percent of the dissipative torque inferred from the retrograde annual nutation.

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