Inverse scattering of surface waves: imaging of near-surface heterogeneities

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SUMMARY
An efficient inverse scattering method is developed for imaging near-surface heterogeneities using scattered surface waves. Three dimensional elastodynamic wave propagation and scattering in a laterally invariant embedding medium is considered. The Born Approximation is used and the scattered wavefield is expressed as a domain type integral representation. The computation time of Green's tensor elements is reduced by considering the radial symmetry of the medium. The method is validated by numerical tests. Ultrasonic laboratory data obtained from a scale model experiment are used for imaging the near-surface inhomogeneities caused by an epoxy-filled hole in the surface of an aluminum block. Both synthetic and the scale model tests show that the location, the actual density contrast and the depth of the inhomogeneities are reasonably well estimated.

Key words: inverse problem, layered media, Rayleigh waves, scattering, seismic wave propagation, surface waves.

1 INTRODUCTION
Inverse scattering problems are of interest in many scientific and engineering problems and different methods of inverse scattering are widely used in medicine, material science and earth sciences to image the area of interest. The aim of imaging is to determine the shape, location and physical properties (such as density, velocity, electrical conductivity, refraction index, etc.) of a target object by using electromagnetic, acoustic or elastic waves (de Hoop 1995; Scales 2002; van den Berg 2002). If the scattering of the waves are due to a bounded object with a simple shape like square, circle or ellipse, integral equation methods are convenient and efficient to solve the scattering problem. Otherwise finite element or finite difference methods are appropriate which can handle any geometry, however they are compute expensive for similar accuracy. In this paper, an efficient inverse scattering method, based on an integral representation of the elastodynamic wave equation, is developed. By using the Born approximation, both forward and inverse scattering problems are solved. The main interest is to estimate the location and the density contrast of near-surface heterogeneities by using scattered surface waves.

Imaging shallow layers by body waves requires high resolution data acquired in a dense spatial array. Surface waves do not require the same dense sampling, since their wavelengths are longer when compared to body waves. Therefore, surface waves are more economic for this purpose. Surface waves are widely used in global, exploration and near-surface geophysics. A notable difference in the applications is the frequency content and the array aperture of the measurements which affect the investigation depth. The dispersive property of surface waves allows the estimation of the shear wave velocity structure and attenuation of shallow layers. In global seismology surface waves are used to investigate the crust and upper-mantle structure (Kovach 1978; Cong & Mitchell 1998; Chang & Baag 2005; Singh 2005) and the source properties of seismic events (Canitez & Toksoz 1971; Thiø et al. 1999; Ekström 2006). In geotechnical engineering shear wave velocity estimation from surface waves has become a popular tool and different techniques are applied to obtain the near-surface properties of the medium. This is relevant for the mitigation of the hazards that may be caused by earthquakes to constructions (Nazarian et al. 1983; Rix et al. 1998; Park et al. 1999; Zywicki 1999; Foti 2000; O’Neill 2003). They are also used in detecting buried objects, which are of great interest in civil and environmental engineering, archeological and land mine explorations. For example, Ganji et al. (1997) determined the location of buried objects by spectral analysis of surface waves while Leparoux et al. (2000) used common receiver gathers to locate tunnels and Gélis et al. (2005, 2006) proposed an elastic full waveform inversion method that can be used to detect cavities.

Complex structures near to the surface cause scattered body and surface waves. A general theory of elastodynamic wave scattering is given by Snieder (2002a, b). The scattered waves often have high amplitudes and mask the reflections from deeper parts of the Earth. In some
cases the information on the target reflectors is important, like in hydrocarbon exploration, therefore the elimination of the scattered surface waves become an important task. Blonk & Herman (1994, 1996), Ernst & Herman (1998) and Campman et al. (2005) considered the scattered surface waves as noise and studied the elimination of these waves to improve the reflections from the target reflectors. The scattered surface waves can also be used to image near-surface heterogeneities such as cavities, buried objects (e.g. archaeological ruins) and shallow water reservoirs. Sniider (1987) uses scattered surface waves to reconstruct lateral heterogeneities. Using scattered surface waves and elastodynamic wave theory Blonk et al. (1995) detected a large object while Herman et al. (2000) imaged small objects located in the first few metres of a weathered layer. These papers focused on detecting the object but not determining the actual contrast values of the inhomogeneities. Riyanti & Herman (2005) present a solution to the 3-D elastodynamic forward problem in a layered embedding medium including the effect of multiple scattering. It is an accurate solution of the wave equation, however it is quite compute expensive.

In this paper, a 3-D elastodynamic scattering by heterogeneities in a layered embedding medium is considered by using the Born approximation. Due to lateral invariance of embedding, the Green’s function is rotationally symmetric and they can be calculated in an efficient way. The method in this study allows to consider larger distances along observation locations, since it decreases the computational price. Numerical tests are performed and both the location and the actual density contrast of the scatterer are well estimated. Next, the method is applied to a scale model using the ultrasonic laboratory data collected at Colorado School of Mines, Physical Acoustics Lab. Two scatterers, formed by an epoxy-filled hole in the surface of an aluminum block, are imaged. The location of the scatterers, the actual density contrasts and their depths are reasonably well estimated.

The paper proceeds as follows. In Section 2, the formulation of the forward problem and the efficient calculation method of the Green’s tensor elements are explained. In Section 3, the inverse scattering problem is formulated and a conjugate gradient iterative inversion for its solution is outlined. The set-up and the material properties of the laboratory experiment, the numerical solution of the forward and inverse problems and the application of the method to the laboratory data are given in Section 4. The results are discussed in Section 5.

2 FORMULATION OF THE FORWARD PROBLEM

2.1 Domain integral representation of the wavefield

3-D elastodynamic wave propagation and scattering is considered in an isotropic, laterally homogeneous medium. The local form of the wave propagation is expressed by the equation of motion and the constitutive relation given in the frequency domain as

\[ \partial_t \tau_{ij}(x) + \omega^2 \rho u_{ij}(x) = -f_i(x), \]  

(1)

\[ \tau_{ij}(x) = \lambda \delta_{ij} \partial_k u_k(x) + \mu [\partial_i u_j(x) + \partial_j u_i(x)]. \]

(2)

In eq. (1) \( \omega \) is the angular frequency and for brevity the frequency dependence in the expressions are omitted. Unless mentioned the equations, throughout the text, are given in the frequency domain. The location of a point \( x \) in space is given by the right-hand side Cartesian coordinate system which is defined by the origin and three mutually perpendicular unit vectors \( \mathbf{i}_1, \mathbf{i}_2 \) and \( \mathbf{i}_3 \). In this case a position of \( x \) is denoted by \( x = x_1 \mathbf{i}_1 + x_2 \mathbf{i}_2 + x_3 \mathbf{i}_3 \), where \( x_1 = x \) and \( x_2 = y \) are used for the horizontal axes and \( x_3 = z \) for the vertical axis representing the depth direction. Throughout the text Einstein’s summation convention is used where the summation is over the repeated indices and the indices operate from 1 to 3. Differentiation with respect to \( x_i \) is denoted by \( \partial_i \). The material properties of the medium are described by its Lamé parameters, \( \lambda \) and \( \mu \), and mass density, \( \rho \). The displacement vector, the stress tensor and the body force are denoted by \( u_i \), \( \tau_{ij} \) and \( f_j \), respectively, and \( \delta_{ij} \) is the Kronecker delta.

To represent the elastodynamic displacement wavefield as a domain type integral, the medium is assumed linear and time invariant, and Betti–Rayleigh convolution type reciprocity is applied to eqs (1) and (2). By this reciprocity relation, two non-identical states occurring in the same domain are related to each other. Detailed derivations of the reciprocity relations for acoustic, elastodynamic and electromagnetic wavefields can be found in de Hoop (1995) and Fokkema & van den Berg (1993). Here the interaction of two states, the actual wave due to the presence of a scatterer embedded in the background medium. The domain occupied by the scatterer is denoted by \( D \) and the positions of the source, receiver and the scatterer are given by \( x', x' \) and \( x \), respectively. In this study only the mass

\[ u_{ij}(x', x) = W u_{ij}^D(x', x') + \omega^2 \int_{x' \in D} \sigma(x') u_{ij}(x', x) u_{ij}(x', x') dV(x'), \]

(3)

where

\[ W u_{ij}^D(x', x') = u_{ij}^{inc}(x', x') \]

(4)

is the incident wavefield (i.e. the wavefield in the absence of a scatterer) and

\[ \omega^2 \int_{x' \in D} \sigma(x') u_{ij}^D(x', x') u_{ij}(x', x') dV(x') = u_{ij}^{inc}(x', x') \]

(5)

is the scattered wavefield due to the presence of a scatterer embedded in the background medium. The domain occupied by the scatterer is denoted by \( D \) and the positions of the source, receiver and the scatterer are given by \( x', x' \) and \( x \), respectively. In this study only the mass
density contrast $\sigma(x')$, being the difference between the scatterer density $\rho_{sc}$ and the embedding (background) density $\rho_0$, is considered. Therefore, the contrast in Lamé parameters (or wave velocities) is explained by the mass density contrast. The response to a general impulsive point force is a tensor of $3 \times 3$ elements called as the Green’s tensor and it is denoted by $u_{ij}^G$. In eq. (3), $W$ is the source wavelet, $u_{ij}^G$ and $u_{ij}^W$ are the Green’s displacement tensors where the former one is generated by a vertical point force and expresses the direct propagation from the source $x'$ to the receiver $x$ and the latter from the scatterer to the receiver. $u_{3j}$ is the total displacement wavefield at $x'$ inside the scatterer. Both the scattered wavefield $u_{ij}^G$ and the total wavefield inside the scatterer $u_{ij}$ are unknown. For 3-D elastodynamic case this forward problem is solved by the method of moments by Ditzel et al. (2001) and Riyanti & Herman (2005) which gives an accurate solution of the wavefield, however it is quite compute intensive. To be able to consider the larger distances along the observation locations, which ensures the inclusion of the scatterer, the Born approximation is used to decrease the computational price. It is assumed that the density contrast value $\sigma(x')$ is much smaller than the background density, $\rho_0$, $\sigma(x') = \rho_{sc} - \rho_0 \ll \rho_0$. With this assumption the unknown wavefield inside the scatterer replaces the incident wavefield as

$$u_{ij}^3(x', x') \approx u_{ij}^{inc}(x', x')$$ (6)

and the scattered wavefield with Born approximation takes the following form

$$u_{ij}^G(x', x') \approx -\omega^2 W \int_{x \in D} \sigma(x') u_{ij}^G(x', x') u_{ij}^G(x', x') dV(x').$$ (7)

where all the variables under the integral sign are known. By using eqs (4) and (7) the incident and the scattered wavefields can be calculated and the total wavefield can be obtained as the sum of these wavefields as:

$$u_{ij}(x', x') = u_{ij}^{inc}(x', x') + u_{ij}^G(x', x').$$ (8)

By use of the Born approximation, an approximate solution of the wavefield is obtained resulting in an efficient calculation time.

### 2.2 Calculation of the Green’s displacement tensor in a layered medium

The considered medium is elastic, isotropic and constitutes from laterally invariant (homogeneous) stratified layers placed over a homogeneous, isotropic half-space. The layers are in welded contact, that is the traction and particle displacement across the interfaces are continuous. The top of the uppermost layer, the Earth’s surface, is assumed to be a stress-free boundary. For the wave propagation in layered medium the relations given in eqs (1) and (2) hold for each layer $l$, $(l = 1, \ldots, N)$, where the layers are represented by their own material properties, $\rho_j$, $\lambda_j$ and $\mu_j$. Dissipation is considered in the medium to avoid the singularities in the complex integration and it is implemented as the negative imaginary parts of compressional $\alpha_j$ and shear $\beta_j$ wave velocities (or Lamé constants $\lambda_j, \mu_j$) as $\text{Im} \{\alpha_j, \beta_j\} \leq 0$.

Since laterally invariant stratified layers are considered, the physical properties of the medium change only in vertical direction. Using this axial symmetry with respect to the vertical axis, the coupled differential equations of eqs (1) and (2) are expressed as a set of ordinary differential equations (Kennett 1983; Aki & Richards 2002) and the cylindrical coordinate system $(r, \phi, z)$ is employed, where $r$ is the radial distance in horizontal plane, $\phi$ is the azimuthal angle and $z$ represents the depth which is perpendicular to the stratification. By applying a Fourier–Hankel transform, and introducing the horizontal slowness $p = k(\omega)/\omega$, where $k$ is the horizontal wavenumber, the wavefield is decomposed into plane waves and the Green’s tensor elements are represented in terms of Bessel functions. The set of coupled first-order differential equations is solved for the displacement elements by using the recursive scattering matrix method. To obtain the displacements in the $(x, \omega)$ domain the transforms applied to solve the system of equations are inverted.

The Green’s tensor elements are calculated in an efficient way at the stage of inverse Fourier–Hankel transformations. Detailed formulations for the Green’s tensor elements in layered media can be found in Kennet (1983); Ditzel et al. (2001); Ditzel (2003) and Riyanti & Herman (2005). In the standard way of calculation the displacements $u_x$, $u_y$ and $u_z$ in the cylindrical coordinates are obtained by using the inverse Fourier–Hankel transform as follows (see Kennet 1983 pp. 34–35)

$$u_x(r, \phi, z) = \int_0^\infty dp d\phi \sum_m \left[ V(p, m, z) \frac{\partial J_m^{opr}(p)}{\partial \omega^{opr}} + W(p, m, z) \frac{i m}{\omega^{opr}} J_m^{opr}(p) \right] e^{im\phi},$$ (9)

$$u_y(r, \phi, z) = \int_0^\infty dp d\phi \sum_m \left[ V(p, m, z) \frac{i m}{\omega^{opr}} J_m^{opr}(p) - W(p, m, z) \frac{\partial J_m^{opr}(p)}{\partial \omega^{opr}} \right] e^{im\phi},$$ (10)

$$u_z(r, \phi, z) = \int_0^\infty dp d\phi \sum_m [U(p, m, z) J_{opr}(p)] e^{im\phi}.$$ (11)

Here $U$ is the vertical and $V$, $W$ are the horizontal elements of the stress–displacement vector. $J_n$ is the Bessel function of first kind with order $m$, where $m$ represents the azimuthal dependence caused by the source behaviour in case of an isotropic medium. Since a point source is considered the azimuthal order $m$ takes values up to $|m| < 2$. To perform the inverse Hankel transform, the above integrals need to be calculated numerically, since their analytical solutions do not exist. Therefore, the integrals are discretized by the midpoint rule where
the integration interval is divided into subdomains and the points in the middle of each subdomain is selected. Since the integration is over an infinite interval, when performing the numerical integration, a finite upper limit (cut-off value, $p_{\text{max}}$) and a sampling interval $\Delta p$ have to be determined. When determining these parameters the cut-off errors and the calculation time of the integral should be optimized and the wraparound and aliasing problems due to discrete sampling should be avoided (Jensen et al. 2000). When the source is close to the receiver and/or the frequency is low, the integrand vanishes quite slowly, thus greater $p_{\text{max}}$ values are required, while in the opposite conditions the integrand oscillates very fast and a dense $\Delta p$ sampling is required. To solve the former problem, a cosine taper is applied to the elements of the Green’s tensor by considering the value of $p_{\text{max}}$. Riyanti & Herman (2005) solved this problem by subtracting the free-space Green’s tensor elements from the Green’s tensor elements. When choosing the value of sampling the fastest oscillating case of the integrand is taken into account. The calculation of the displacements (eqs 9–11) can be given in the following discrete form,

$$u_{\alpha}(r, \phi, z) = \sum_{m} \sum_{p} p\Delta p A_{\alpha}(p, m)e^{im\phi},$$

$$u_{\phi}(r, \phi, z) = \sum_{m} \sum_{p} p\Delta p A_{\phi}(p, m)e^{im\phi},$$

$$u_{z}(r, \phi, z) = \sum_{m} \sum_{p} p\Delta p A_{z}(p, m)e^{im\phi},$$

where the expressions in the brackets of eqs (9)–(11) are replaced by the letters $A_{\alpha}$, $A_{\phi}$ and $A_{z}$, respectively. As it is seen from the equations, the summations are calculated for each frequency, $\omega$, distance, $r$, angle, $\phi$ and depth, $z$. Afterwards, a transformation from cylindrical to cartesian coordinates is applied and an inverse Fourier transform is employed to obtain the time domain counterparts of the displacements.

Here the numerical integrations take the majority of the computation time. In this study, to take advantage of rotational symmetry of Green’s tensors in cylindrical coordinates, each term including the variable $m$ in eqs (12)–(14) is calculated separately by excluding the term $e^{im\phi}$ as

$$B_{m=2}^{\alpha}(r, z, sc) = \sum_{p} p\Delta p A_{\alpha}(p, m),$$

$$B_{m=2}^{\phi}(r, z, sc) = \sum_{p} p\Delta p A_{\phi}(p, m),$$

$$B_{m=2}^{z}(r, z, sc) = \sum_{p} p\Delta p A_{z}(p, m),$$

where $m = \{-1, 0, 1\}$ and $sc$ represents the source component. For each $\omega$, $r$, $z$, $sc$ and $m$ these calculated elements are stored in a table, which is called Green’s table henceforth. Although the integrations still take the majority of the computation time, once the table is obtained, the wavefield can easily be calculated for different combinations and numbers of source, receiver and scatterer in the considered model. This is especially important when an iterative inversion method is employed. The procedure is illustrated in Fig. 1 as a top view. The triangles represent the grid locations where the Green’s table elements are calculated. The source is denoted by a point at the centre and the surface projections of the scatterers are denoted by the squares. On each circle the Green’s table elements have the same values. For depth and horizontal locations of a scatterer, first the related depth and then the radial distance is found in the Green’s table. Since the table is calculated for particular radial distances, a linear interpolation is performed for intermediate distances. Then the exponential term including the angular

![Figure 1. Illustration for the calculation of the Green’s table elements (top view). The source, receivers and the scatterers are shown by ‘•’, ‘▼’ and ‘■’, respectively. $R$ is the radial distance and $\Phi$ is the azimuthal angle. The Green’s table elements are calculated for the ‘r’ distances. The elements have the same values over the circles. To obtain the Green’s tensor elements in the actual locations of the scatterers, the table elements are interpolated for intermediate distances and rotated.](https://academic.oup.com/gji/article-abstract/171/1/352/602824/fig1)
coordinate is supplemented. This implementation can be explained in the following matrix form as

\[
\begin{bmatrix}
  u_x \\
  u_y \\
  u_z \\
\end{bmatrix} =
\begin{bmatrix}
  B_1^x & B_2^x & B_3^x \\
  B_1^y & B_2^y & B_3^y \\
  B_1^z & B_2^z & B_3^z \\
\end{bmatrix}
\begin{bmatrix}
  e^{i(1-1)\phi} \\
  e^{i(0)\phi} \\
  e^{i(+1)\phi} \\
\end{bmatrix},
\]  
(18)

where \(B^x, B^y\) and \(B^z\) are given in eqs (15)–(17) and \(-1, 0, +1\) values at the exponential term corresponds to the values of \(m\).

Numerical tests have been performed to determine the interval for the grid locations of the Green’s table elements. Seismograms are calculated by using both exact and interpolated (from the Green’s table) values of Green’s elements. The normalized error of the interpolated seismograms with respect to the exact ones is calculated. The optimum mesh size that causes less than 1 per cent normalized error is chosen for the calculation of Green’s table elements. The tests have shown that a quarter of the minimum wavelength is an optimum mesh size for the Green’s table elements to make an accurate calculation of the wavefield. It should be noted that the elements of the Green’s table are calculated in one direction, such as from the source to the receiver. Therefore, to calculate a scattered wavefield by using the Green’s table a small change is required in eq. (7). Considering the reciprocity property of the Green’s tensor, the locations of the scatterer and the receiver are interchanged as

\[
u_{ij}^{(S)}(x', x) \equiv u_{ij}^{(S)}(x, x')
\]  

which is the appropriate form for using the Green’s table. In the following formulations eq. (19) is considered.

### 3 Formulation of the Inverse Problem

The inverse problem consists of the determination of the shape, location and the physical properties of the scatterer by using the measured scattered wavefield at the observation locations. Considering the measured scattered field as the data space (\(S\)), and the density contrast of the scatterer as model space (\(M\)), the scattered wavefield given in eq. (7) is calculated by mapping the model space (the given density contrast) to the data space (the measured scattered wavefield at the receivers). Also considering eq. (19), the scattered wavefield in eq. (7) can be represented in the following form

\[
u_{ij}^{(S)}(x', x) = \{G_3 \sigma\} = \omega^2 W \int \nu_{ij}^{(I)}(x', x) 2 \pi \nu_{ij}^{(I)}(x', x') dV(x') \quad \{x', x'\} \in S,
\]  

where the subscript \(i\) stands for the model data.

In the inverse problem, the unknown is the density contrast, \(\sigma\), under the integral sign in eq. (20). Because of the Born approximation, this is a linearized problem in the form of a Fredholm integral equation of the first kind. To solve the inverse problem, the conjugate gradient iterative inversion method is used which updates the contrast iteratively and generates a sequence of models that converge to an optimal solution (Tarantola 1984; Kleinman & van den Berg 1991). The method is based on minimizing the normalized cost function defined as

\[
F^{(n)} = \frac{\|e^{(n)}\|_S^2}{\|\nu_{ij}^{(S)}\|_S^2} + \eta \|\sigma^{(n)}\|_D^2,
\]  

where

\[
\|e^{(n)}\|_S^2 = \|\nu_{ij}^{(S)} - G_3 \sigma^{(n)}\|_S^2
\]  

is the residual function expressing the difference between the actual \(\nu_{ij}^{(S)}\) and the modelled data, and \(n\) is the iteration number. \(\|\cdot\|_2\) denotes the \(L_2\) norm and \(\eta\) is the Tikhonov regularization (damping) parameter that optimizes the trade-off between the best-fitting and the stability of the inversion (Hansen 1998; Aster et al. 2005). The value of the regularization parameter is determined by the \(\lambda\)-curve method as described in Hansen (1998). The minimization of the cost function is provided by finding a convenient \(\sigma\) value that approximates the model data to the actual one. This is done by iteratively updating the \(\sigma\) value until the residual function, \(\varepsilon\), becomes less than a given threshold value. The density contrast value is updated by computing an appropriate direction of decrease (update direction) \(w\) in the cost function \(F^{(n)}\), and an appropriate increment (step size) \(\xi\) in that direction given as

\[
\sigma^{(n)} = \sigma^{(n-1)} + \xi^{(n)} w^{(n)} (n \geq 1).
\]  

In the conjugate gradient method the update direction is computed by considering the current gradient and the previous direction. The negative of the gradient, \(g^{(n)}\), is given by

\[
g^{(n)} = - \frac{G_3^{\eta} e^{(n-1)} + \eta \sigma^{(n-1)} }{\|\nu_{ij}^{(S)}\|_S^2}.
\]  

where

\[
G_3^{\eta} e^{(n)}(x') = \omega^2 W \nu_{ij}^{(I)}(x', x') \int \nu_{ij}^{(I)}(x', x') e(x') dV(x') \quad x' \in D.
\]  


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In eq. (25) the differential variable of the surface integral is denoted by $dA$, $\mathbf{U}$ represents the complex conjugate and $G'_{ns}$ is the adjoint operator which can be found from the following relation

$$
\{G'_{ns} \varepsilon, \sigma\}_D = \langle \varepsilon, G_S \sigma \rangle_S.
$$

Here $\langle \rangle$ represents the inner product. For the first iteration the update direction calculation starts with the negative gradient of $F^{(0)}$ with respect to $\sigma^{(0)}$ which is given by

$$
w^{(1)} = g^{(1)},
$$

and continue as

$$
w^{(n)} = g^{(n)} + \gamma^{(n)} w^{(n-1)} \quad (n \geq 2).
$$

In this paper for updating the directions, the method of Polak–Ribiére is considered (Polak 1971; Kleinman & van den Berg 1991) and $\gamma$ parameter is given by the following expression

$$
\gamma^{(n)} = \frac{\Re \{g^{(n)} - g^{(n-1)}, S^{(n)}\}_D}{\|g^{(n-1)}\|_D^2}.
$$

After the determination of the update direction, the step-size $\xi$ is determined by setting the directional derivative equal to zero. This requires the orthogonality of the residual $\varepsilon$ and the gradient $g$. With appropriate calculations the step-size can be obtained as

$$
\xi^{(n)} = \frac{\{\varepsilon^{(n-1)}, G_S w^{(n)}\}_S}{\|G_S w^{(n)}\|_S^2 + \eta \|w^{(n)}\|_D^2}.
$$

When $w^{(n)}$ and $\xi^{(n)}$ are calculated, the contrast $\sigma$ and hence the model data is updated until a satisfactory result is obtained.

## 4 SYNTHETIC AND SCALE MODEL TESTS

### 4.1 Scale model

To validate the method, the ultrasonic laboratory data collected at Physical Acoustics Laboratory of Colorado School of Mines (now at Boise State University) is simulated. In the laboratory experiment an aluminum block of dimensions $28 \times 23 \times 21.5$ cm is used as background medium. The scatterer is formed by drilling a hole on the surface of the aluminum block and filling it with an epoxy material. A picture of the aluminum block is given in Fig. 2(a). The epoxy is prepared at the laboratory of Applied Physics of Delft University of Technology. The aim was to have an epoxy which has the same elastic properties of aluminum but different density. This condition is not perfectly satisfied due to the absence of the proper materials. The experiment is performed by epoxy-filled holes having different depths. A laser point source is used to excite ultrasonic waves and the absolute particle velocity is measured via the Doppler shift by scanning the laser interferometer over the surface of the aluminum (Fig. 2b). The source is a pulsed (5ns) Nd:YAG (Yttrium aluminum garnet) laser with a wavelength of 1064 nm and an energy of 0.3J/pulse. To record the waveforms a laser Doppler vibrometer is used which has a beam diameter less than 1mm and a wavelength of 633 nm (red). The signal of the vibrometer is amplified with a low-noise preamplifier (SR560 with 12 db/octave 10 kHz high-pass filter) and digitized at 14 bit resolution using Gage digital oscilloscope card attached to a PC. When the beam hits a moving target its frequency is Doppler shifted. The absolute particle velocity is measured by decoding the beat-frequency of the output and the reflected signal in the hardware. This provides a non-contacting (free of mechanical disturbances) measurement of ultrasonic wave propagation (Scales & Malcolm 2003). To ensure high signal to noise ratio, thin reflective tape is applied to the model for a strong reflectivity of the interferometer beam (Fig. 2a). The sketch of acquisition geometry and the position of the scatter are shown in Fig. 2(c). The shaded area in the figure is the area where the measurement points representing the receivers are located. The wavefield is recorded by $51 \times 51$ measurement points, receivers, placed both in $x$ and $y$ directions with a spatial sampling of $\approx 0.6 mm$ where $\lambda_d$ is the dominant wavelength. $\lambda_d$ is calculated by considering the surface wave velocity (here Rayleigh wave) as $V_R = 2.85 \times 10^6$ mm s$^{-1}$ and the dominant frequency as 1 MHz. These quantities are obtained from the picked arrival times and from the calculated amplitude spectrums of the measurements, respectively. The background P-wave velocity is also obtained by time picking as $V_p = 5.7 \times 10^6$ mm s$^{-1}$. Using the relation $V_p/V_S \approx 0.92$ for the Poisson’s ratio of the aluminum ($\nu_{Al} = 0.35$) the S-wave velocity is obtained as $V_S = 3.00 \times 10^6$ mm s$^{-1}$. The densities of the background and the scatterer are $\rho_0 = 2.7 \times 10^{-6}$ and $\rho_{sc} = 7.0 \times 10^{-6}$ kg mm$^{-3}$, respectively, which gives a contrast value of $\sigma = 4.3 \times 10^{-6}$ kg mm$^{-3}$. A total wavefield for a scatterer with a dimensionless depth $d/\lambda_d = 0.7$ and diameter $\Phi/\lambda_d = 0.7$ is calculated by forward modelling and the results are compared with the measured data for the same configuration. Afterwards the calculated scattered wavefield is used to validate the inverse modelling. For the inversion of the scale model two scatterers with $d/\lambda_d = 0.5$ (relatively shallow) and $d/\lambda_d = 0.7$ (deep) are used to investigate the depth dependence of the method. Henceforth, the shallow and the deeper scatterers are named S1 and S2, respectively. In both cases the scatterer domain is discretized with a sampling interval of $\lambda_d/8$ in $x$, $y$ and $z$ directions.


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Figure 2. Picture of the aluminum block. The reflecting tape is the measurement area and the bright dot is the beam from the interferometer (a) The laser source (S) and the interferometer (R) mounted to a vibration-free optical bench (b). Schematic view of the aluminum block, the receivers (▼) are located at the grey shaded area.

4.2 Calculation and the use of the Green’s table

Determination of some parameters is necessary before calculating the Green’s table. These can be summarized as follows.

The medium parameters ($V_p$, $V_s$, $\rho$) have to be determined as explained in the previous subsection.

To calculate the Green’s table elements, the number of frequencies ($nf$) is required. The time domain seismograms are transformed to the frequency domain by using the Fourier transform. The amplitude spectrums of the traces are calculated and the value of the maximum frequency ($f_{max}$), where the amplitude spectrum dies out, is observed. Considering the frequency sampling ($\Delta f$), the number of frequencies are calculated as $nf = f_{max}/\Delta f = 37$.

The minimum wavelength of the Rayleigh wave is determined from the record. It is used to determine both the grid intervals for the discretization of the medium and the cut-of value, $p_{max}$. A quarter of the minimum wavelength is selected as the grid interval for all directions ($\Delta x = \Delta y = \Delta z = \lambda_{min}/4 = 0.4$mm). For the numerical calculation of eqs (9)–(11) a proper $\Delta p$ sampling is required (Jensen et al. 2000, pp. 224–247). The number of integration points is determined as $np = p_{max}/\Delta p = 8000$ by using $p_{max}$ and $\Delta p$ explained in Section 2.2.

The far grid location is determined as 50 mm by considering the greatest distance between the source and the receiver (the diagonal between the source and the far corner in the shaded area of Fig. 2c). This gives the number of grid locations as $nr = 126$ points. To determine the gridpoints in z direction, the scatterer which has the greatest depth in the experiment is considered and this results in $nz = 10$.

When calculating the Green’s table, besides these parameters the source ($ncs$) and receiver ($ncr$) components and the azimuthal dependence ($nm$) should be taken into account. This results in the number of $nf . nz . nr . np . ncs . ncr . nm = 37 . 10 . 126 . 8000 . 3 . 3 . 3 \geq 1 \times 10^{10}$ calculation steps. In a computer having an Intel Pentium 4 CPU of 3.0 GHz this calculation takes approximately 15 hr.

Once the Green’s table is obtained the forward and inverse calculations can be done for any shot, receiver and scatterer geometry. For example for one scatterer occupying one grid cell, one shot and one receiver, the standard calculation takes approximately 5 min while it takes less than 1 second if the Green’s table is used, or for one scatterer occupying four grid cells, one shot and one receiver, the standard calculation

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takes 18 min while it takes less than 2 s when using the Green’s table. This method allows a fast and efficient calculation even with a PC, without requiring any super computers or any parallel computing techniques.

4.3 Forward and inverse numerical modelling of the scale model

Considering the scale model parameters and the source–receiver line which is above the scatterer S2 (Fig. 2c), the noise free incident (Fig. 3a), scattered (Fig. 3b) and the total wavefields (Fig. 3c) are calculated by using eqs (4), (7) and (8), respectively. The source waveform used in numerical models is a zero-phase bandpass signal where its frequency content is determined from the amplitude spectrums of experimental traces. For the same shot-receiver line the record of the scale model is given in Fig. 3(d). In Figs 3(c) and (d) the compressional waves and the surface waves are clearly visible, although the waveform is somewhat different. In the scale model record (Fig. 3d), the scatterer causes a delay in the wavefield between 34.4 and 37.4 mm which is not visible in the modelled numerical data. This delay may be due to both low-wave velocity and the trapped (or multiply scattered) waves inside the epoxy. Since the wave velocity corresponding to epoxy is not considered and the Born approximation is used in the modelling, this delay is not precisely modelled by the method. On the other hand the scattered wavefield is visible in the numerical model while it is not clear in the scale model due to noise. For the same model the 3-D numerical and scale model wavefields are given in Figs 4(a) and (b), respectively for a snapshot of time 0.014 ms. In both figures the direct Rayleigh waves, represented by D, are clearly observed. The scattered Rayleigh wave, represented by S, is also visible in Fig. 4(a) while in Fig. 4(b) it is masked by the noise. For both 2-D and 3-D models it can be concluded that the results of numerical and scale models are quantitatively in agreement.

Considering the forward modelled scattered wavefield, calculated above the scatterer, and using the conjugate gradient iterative inversion method, the density contrast of the scatterer is estimated. For the inversion a grid size (number of cells) of 11 × 11 × 6 in x, y and z directions are used. The top view of the estimated density contrast is given in Fig. 5(a). The dotted rectangular box in Fig. 5(a) (and also in the following figures) shows the actual location of the scatterer. The shot-receiver line used for the inversion is along AA′ and hence the highest density contrast values, which are nearly the actual ones, are estimated along this line. The density contrast values decrease when the distance of the grids to the record line increases. This is due to lack of observation points. Numerical tests have shown that the number of the receiver lines affects the resolution both in horizontal and vertical direction. Estimations for both shape and the density contrast value of the scatterer become better with increasing receiver lines. Here the results are given only for one receiver line, which is above the scatterer. Many receiver lines are considered in the inversion of the experimental data. The cross-sections along AA′ and BB′ are shown in Figs 5(b) and (c). At shallower levels the highest density contrast values are estimated, however with the increasing depth the density contrast values are decreasing and the image is blurring. This is due to the exponential decay of Rayleigh waves with depth. The actual and the reconstructed traces of the scattered wavefield at distance 35.4 mm (see Fig. 3b for location) is plotted together in Fig. 6. There is a good agreement between the traces.

Figure 3. The forward modelled incident (a), scattered (b) and total (c) wavefields for a shot-receiver line crossing over the scatterer (the source–receiver line at Fig. 2(c), where beginning of the receivers are represented by ▼). The total wavefield of the scale model data for the same configuration is shown in (d). The compressional, surface and scattered waves are represented by C, S and Sc, respectively.
Consequently, it is seen that it is possible to locate the scatterer and estimate its actual density contrast by the method. After verifying the method by the numerical calculations, the scale model is used for testing the method.

### 4.4 Inversion for the scale model data

For the inversion of the scale model data some pre-processing steps are necessary. A priori information required in the inversion is the background velocities, background density, scattered wavefield and the source waveform. The success of the estimation of the location and the density contrast of the scatterer is related to the accurate estimation of this a priori information. The first step is the estimation of the background velocity. For calculating the Green's table and performing the numerical modelling, the wave velocities ($P$, $S$ and Rayleigh) of the background were obtained from the record of the ultrasonic data (Fig. 7a) as explained in Section 4.1. The background density of the aluminum is a well-known quantity. Next step is the estimation of the scattered wavefield. In the inversion of the numerical model, the scattered wavefield of the forward model was used as the input data. This time the scattered wavefield is estimated from the total wavefield by using a frequency–wavenumber ($f$–$k$) dip filter (Yilmaz 1987). The filter allows separating the direct wavefield according to its dip (Fig. 7b) and the scattered wavefield (Fig. 7c) can be estimated as the difference of the total (Fig. 7a) and the direct wavefields (Fig. 7b). At last, the source waveform is estimated. For the estimation of the source a deterministic deconvolution in the frequency domain is applied by assuming that the medium parameters (velocity, density and attenuation) are exactly known. From the forward modelled and estimated incident (direct) wavefields two suitable traces are selected. A cosine window is employed to obtain the Rayleigh waveforms, since the scattering of Rayleigh waves are of interest. Then the source waveform is estimated by the following expression

$$W(\omega) = R(\omega) \frac{\bar{M}(\omega)}{M(\omega) \bar{M}(\omega) + \zeta^2}.$$  

(31)

The Rayleigh waveforms of forward modelled and the estimated incident wavefields are denoted by $M(\omega)$ and $R(\omega)$, respectively. $\bar{M}(\omega)$ is the complex conjugate of $M(\omega)$ and $\zeta$ is a parameter to stabilize the division. The estimated source waveforms for the scatterers S1 and S2 are given in Fig. 8(a). To see how well the estimated waveforms resemble the real trace, the forward modelling is performed by these estimated waveforms and an example for S1 is illustrated in Fig. 8(b). The measure of their similarities is given by the semblance which is calculated as 0.93 for this example. It is seen that the waveform is well estimated and can be used for the inversion.

To ensure the inclusion of the scatterer, an inversion grid of size $9 \times 9 \times 5$ cell is selected. For the inversion, the records around the scattered wavefield are used. The rectangular box in Fig. 7(c) is an example for a receiver line of S1 above the scatterer. Different than for the numerical modelling, for the inversion of the real data, four shot-receiver lines crossing over and near to the scatterer are used. The triangles in Figs 9(a) and (b) show the beginning of the receiver lines which are aligned along $x$ direction.
Figure 5. The top view of the density contrast image of the scatterer (a) and the cross-sections along AA' (b) and BB' (c). The real location of the scatterer is shown by the dashed line.

Figure 6. The comparison of the actual and the reconstructed traces at position 35 mm.

The top view of the estimated density contrast images for the scatterers S1 and S2 are given in Figs 9(a) and (b). The dashed circles represent the actual locations of the scatterers. The highest density contrast values for S1 are estimated in the middle of the circle while for S2 nearly the whole circle is represented by high-density contrast values. For the first depth level, the location and the actual density contrast values of S1 and S2 are reasonably well estimated. To see the variation of the density contrast values at the other depth levels, Figs 10(a) and (b)
Figure 7. The total wave field of the scale model data above the scatterer (a), estimated incident wave field (b) and the scattered wave field (c) by \( f-k \) dip filter. The rectangular box shows the traces that are used in the inversion.

Figure 8. The estimated source waveforms for the shallow (S1) and the deep (S2) scatterers (a) and the comparison of the scale model (real) trace by the forward modelled one (b) which is obtained by using waveform of S1.
are given. It is seen that S1 is clearly observed at the first two levels, while S2 is visible up to the fourth depth level. The cross-sections for S1 and S2 along AA’ and BB’ are given in Figs 11(a) and (b) with the dashed boxes showing the location of the scatterers. As it is seen from the figures, the images are getting blurred with depth as it is also observed in Figs 10(a) and (b). In Figs 10 and 11 it is seen that the method can distinguish the depth of the scatterers. For the shot record which is above the scatterer S1, the estimated scattered wave field, which is the input for the inversion, and the related reconstructed scattered wave field are given in Figs 12(a) and (b), respectively as an example. The difference of them is depicted in Fig. 12(c). The residuals indicate a reasonably good fit of data. During the calculations, iterations are terminated when the objective function approximately reaches to its plateau. For the scale model inversion, 20 iterations were sufficient.

5 CONCLUSIONS

An efficient inverse scattering method is introduced for estimating the location, actual density contrast and the depth of inhomogeneities. The method is verified both by numerical and scale model data. The use of the Born Approximation and forming a Green’s table to use in forward and inverse modelling accelerates the calculation and allows including larger distances in the inversion. Both numerical and scale model studies have shown that the location, the actual density contrast as well as the depth of the scatterers are reasonably well estimated. Essentially the method is developed for imaging the near-surface inhomogeneities in seismic exploration and geotechnical scale; however it may also be possible to apply the method to the global scale data. The application of the method to a field seismic data, for imaging the near-surface inhomogeneities in geotechnical scale, is currently in progress. Up to now the purpose was to see whether it is possible to estimate the location, depth and the density of the scatterer with the proposed method. Therefore, controlled experiments are used where, at least, the location of the scatterer is well known. After the verification of the method by the seismic field data, the method will be applied.

Figure 9. The top view of the density contrast image of the scatterers S1 (a) and S2 (b). The dashed circles show the real location of the scatterers and V shows the starting locations of the receiver lines that are used in the inversion.
to passive seismic data where irregular source and receiver geometries will take place. In this case, numerical tests of the method should be performed to understand the effect of the distance of acquisition line to the scatterer to be able to detect it. For example, the relation between the measurement line and location of the scatterer is investigated by Herman et al. (2000) and Snieder (1987) with different methods. Besides the medium properties, acquisition geometry (line, circular, orthogonal, etc.), size, density and depth (or height) of the scatterer will play a role on the estimated results and each should be examined separately.

Although a controlled experiment is used to verify the method, there is a difference between the estimated results and the experimental data. Some reasons for this difference can be summarized as follows.

The use of the Born approximation provides an efficient calculation, however it doesn’t consider the multiple scattering of waves. This may effect the accuracy of the estimated results. Here, by multiple scattering both the interaction between the grid cells of the discretized scatterer domain and also the interaction between different scatterers are meant. In this experiment the scatterers (S1 and S2) were far enough from each other, therefore, they can be considered as separate scatterers embedded in separate background mediums (no interaction between the scatterers). In case of several embedded inhomogeneities, it is believed that the method still can be used if the multiple scattering effects are not dominant. For example single scattering is a frequently used assumption in local coda wave generation (Herraiz & Espinoza 1987). Certainly this expectation should be examined with numerical tests. A method which considers multiple scattering is proposed by Riyanti (2005) and Riyanti & Herman (2005). Their method can be considered as the extended version of the presented method. This method gives an accurate solution of the wavefield, however it is quite compute expensive.

In the method only the mass density contrast is considered. Density and wave velocities (or Lamé parameters) are the fundamental parameters for the elastic wave propagation. It is known that variation in density causes variation in velocities. Here the density contrast is chosen as a starting point to develop the method since it simplifies the model and calculations. Both density and Lamé parameters are important to understand the subsurface properties and the negligence of the possible contrasts in the Lamé parameters can cause some uncertainties in the estimation of density contrast. Future work will include the contrasts in Lamé parameters besides the density contrast.

In the method it is also assumed that the density contrast is much smaller than the background density (Born approximation). Although this assumption is not fulfilled, it is seen that the method can still predict some reasonable results for both location and the density contrast value of the scatterer.
Figure 11. The cross-sections along AA' and BB' of S1 (a) and S2 (b) (see Fig. 9a and b for the locations of AA' and BB').

Figure 12. The estimated (a), the reconstructed scattered wavefields (b) and their residuals (c) for S1.

It is assumed that the elastic parameters of the epoxy (scatterer) are the same those of the aluminum background and the only difference is in densities. This condition is not perfectly satisfied due to the absence of proper materials. Besides, during the filling process of the holes with epoxy there may be possible occurrences of micro-air bubbles which may change the elastic properties and cause some deviations from this assumption.

The method used in this study is based on elastic wave propagation and inelastic attenuation or dissipation is not considered. In elastic wave propagation a slight dissipation can be modelled using complex velocities or complex elastic parameters according to correspondence...
principle (Toksoz and Johnston 1981). Here dissipation is used only as a mathematical tool to avoid the singularities when performing the numerical integration. Since the intrinsic attenuation causes decrease in the wave amplitude, negligence of it may affect the accuracy of the estimation for density contrast values due to improper modelling of the wave amplitudes. Inelastic attenuation is dominant especially in lossy materials or in fluid or gas saturated materials. However, in the experiment, aluminum is used as background medium and inelastic attenuation in aluminum can be neglected for the dominant wavelength used in the experiment. For the scatterers, epoxy filled holes, both inelastic attenuation and multiple scattering may play a role in damping of the wave amplitudes. Since the method does not consider inelastic attenuation and multiple scattering, computational results may differ than the experimental one. On the other hand, it may be expected that inelastic attenuation does not have a considerable effect on the estimated results since the sizes of the scatterers are not larger than the dominant wavelength.

In the density contrast images it is observed that the images are getting blurred with the increasing depth and imaging the deeper levels become difficult. This is due to the exponential decay of the Rayleigh waves. It can be concluded that the estimations are reliable up to a depth of about one wavelength.

Here the scatterers of cylindrical shape are discretized by cubic cells. If the selected grid cell sizes are small enough than the scatterer size (or in general the mean curvature, $\frac{\partial V}{\partial A}$, where $V$ is the volume $A$ is the surface area of the scatterer), then the difference in shapes of the scatterer and grid cells does not affect the results.

The estimation of the scattered wavefield and the source waveform also comprises uncertainties that affect the accuracy of the results.

In spite of these uncertainties the method works well and the estimated results are reasonable and the method is promising for the field data applications.

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