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Optimal Scheduling of Water Supply Projects

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Some capacity expansion problems encountered in water resources development are studied. Models yielding least-cost shedules (i.e. optimal selection and sequencing of potential projects) in order to meet a stipulated increase in water demand are first discussed. However, recognizing uncertainties in the future demand, these ought to be taken into account. In order to do this, a new general dynamic programming algorithm has been developed. Depending on how detailed we are able to state the preferences against shortage, chance-constraints or penalty functions are included in the optimization. The use of the algorithm, as an aid in the current planning, is illustrated by solving an example under deterministic as well as stochastic assumptions about the future demand.

Introduction

The continually increasing water consumption in most communities is the reason for planners to investigate new water supply projects in order to meet future requirements. Often many independent projects are found to be potential new sources of water supply. Given the costs and capabilities of each project, the Water Authority is faced with the following decision problem of how to select a project and appoint its completion date in an optimal way, i.e. with regard to some stated objective. Considering the decision problem a static one and applying a minimum cost objective, the water authority is expected to select the project with the smallest unit cost. However, considering the problem a dynamic one i.e. a multistage decision problem, the first decision to be made becomes dependent on the remaining decisions in the planning period. Therefore, applying a minimum cost objective, the general decision problem is how to select some subset of the possible projects and to appoint their construction dates in order to minimize the present value of the total costs in the entire planning period.

Among others, Butcher et al. (1969); Morin and Esogbue (1971, 1974); Morin (1973); Haimes and Nainis (1974) have recently analyzed problems of this kind. The purpose of this paper is first to give a summary of the current progress in obtaining solutions to water supply planning problems, and secondly to present a new dynamic programming algorithm, which can take the stochastic nature of the future water demand into account.

In order to approach a more explicit formulation of the planning problems, we introduce the following notation:

Q_i the capacity of the i'th project.
c_i the construction cost of the i'th project.
b_i the per unit operation cost of the i'th project.
N the number of the possible projects.
r the discount rate
T the last period considered, i.e. the time horizon.
$D(t)$ water demand in time period t.

Note that only projects of fixed size will be considered. This assumption is made because current practice is oriented to investigations of discrete projects at specified locations. This implies that a project, if selected, will be constructed with exactly the specified capacity Q_i, whereas the yield can be variable, but cannot be greater than the capacity limit.

As indicated above, the planning problems will be formulated with time t as a discrete variable. Our planning period begins at time $t = 0$ and terminates T time periods (e.g. years) later at time $t = T$. Obviously, a project cannot be utilized before its completion date. Consequently, assuming the i'th project finished at time t, we are able to utilize the capacity Q_i in the following time period $t + 1$, $t + 2$, ..., T.

In Fig. 1, the difference between the discrete time and the corresponding time period is emphasized. Furthermore, an arbitrary demand function $D(t)$ is sketched. As a consequence of the discrete time formulation, $D(t)$ is given as a step function.
Our starting point in the presentation of the planning problems will be the case where the following rather restrictive assumptions are prevailing:

(a) The future demand is deterministic.
(b) Operation costs amount to a minor part of the total costs and/ or all the projects have about the same unit-operation cost.

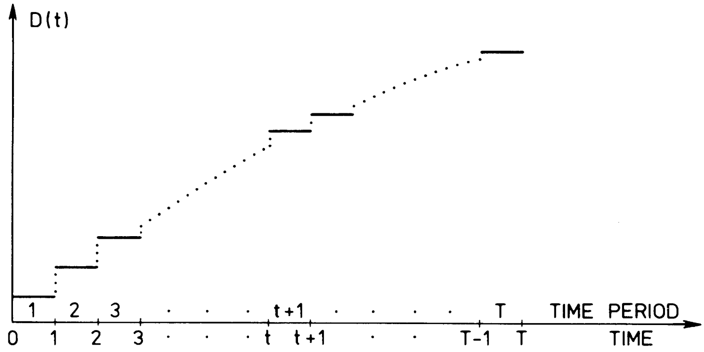

Fig. 1. The demand function *D(t)* and definition of time and the corresponding time period.

(c) The total capacity of the *N* projects equals the demand at the end of the planning period.

This is called the *sequencing problem,* as, obviously, all the projects have to be constructed and the only problem is to find the optimal sequence of the projects.

Subsequently, we will proceed by omitting the stated assumptions gradually, attaining more and more realistic planning conditions. Abandoning the assumption (c), we are dealing with the *scheduling problem.* The optimizing procedure now consists of both selection and sequencing of an appropriate number of projects. If the assumption (b) is omitted too, we have to include the minimization of the operation costs in each time period in the optimization. We denote this planning problem *the scheduling problem including operation costs.*

When at last the assumption (a) is omitted, the deterministic function *D(t)* is replaced by a series of stochastic variables $\tilde{D}(t)$, which of course will influence the optimal decisions to be taken. Depending on how detailed we are able to quotate a given shortage of water through a time period, we end up with two different models for *the optimal project scheduling under stochastic demand.* The first one, *the chance-constraint model* specifies that shortage of water in a given time period must only take place with a probability less than a specified value α, whereas the other one, *the penalty model,* quotates the shortage by introducing a penalty depending on the magnitude of the shortage.

As previously mentioned, a new dynamic programming algorithm has been developed. The application of this algorithm in connection with a stochastic demand function will be illustrated by solving a numerical example. Computational comments and experience will be stated. Finally, some conclusions will be given.

The Sequencing Problem

When the previously mentioned assumptions (a), (b) and (c) are valid, we are dealing with the sequencing problem. The assumption (c) can be stated mathematically as

$$\sum_{i=1}^{N} Q_i = D(T) \tag{1}$$

Defining a decision variable

$$Y_{it} \equiv \begin{cases} 1, & \text{if project No. } i \text{ is completed at time } t \\ 0, & \text{in all other cases} \end{cases}$$

then the problem can be formulated as an integer programming problem:

$$\min \sum_{t=0}^{T} \sum_{i=1}^{N} c_i (1+r)^{-t} Y_{it} \tag{2}$$

$$\sum_{i=1}^{N} Q_i \left(\sum_{\tau=0}^{t-1} Y_{i\tau} \right) \geq D(t) \qquad t = 1, \ldots, T \tag{3}$$

$$\sum_{t=0}^{T} Y_{it} \leq 1 \qquad i = 1, \ldots, N \tag{4}$$

$$Y_{it} \in \{0,1\} \tag{5}$$

The objective Eq. (2) is to minimize the present value of the total construction costs, under the restrictions that the total capability of the system in a given time period must not be less than the demand in this period, Eq. (3), and that no project can be constructed more than once, Eqs. (4) and (5). Considering the usually large number of 0-1 variables occurring in the problem, Eqs. (2) - (5), standard codes for solving the problem as an integer programming problem are not expected to be very efficient.

Butcher et al. (1969) has proposed a dynamic programming formulation which takes advantage of the special structure of the problem. Note that the only problem here is the order in which the projects should be constructed, because completion dates can be derived directly, given the optimal sequence. This is due to the fact that it is optimal to postpone a project until the water demand requires more capacity according to the present value criterion.

Regarding the problem as an N-stage sequential decision process and using Bellman's principle of optimality, Bellman (1962), Butcher et al. (1969) derived a recursive equation in forward dynamic programming form, using the supply capacity of the constructed projects as state variable.

They showed that the optimal sequence is determined by the shape of the water demand function, the interest rate, the relative costs and capacities of the facilities. Consequently, unit cost (c_i/Q_i) is not a satisfactory criterion sequencing the projects.

The Scheduling Problem

In general, the assumption (c) is not satisfied. When

$$\sum_{i=1}^{N} Q_i \geq D(T) \tag{6}$$

we are dealing with a scheduling problem, where only a subset of the N considered projects are expected to be selected for construction. Therefore, the sequencing problem just described is a special case of the scheduling (selecting and sequencing) problem.

Morin and Esogbue (1971) showed that the dynamic programming algorithm presented by Butcher et al. (1969) could be modified to solve the scheduling problem. Alternatively, a much more efficient dynamic programming algorithm using an imbedded state space approach was developed.

Furthermore, Morin (1973) identified two cases in which the algorithm of Butcher, Haimes and Hall may produce non-optimal solutions to the problem, Eqs. (2) - (5). The pathological behavior of the algorithm was illustrated by an example, and he proved that the algorithm using the imbedded state space approach always leads to the optimal solution.

The Scheduling Problem including Operation Costs

When the assumption (b) does not hold either, i.e. the operation costs amount to a substantial part of the total costs, we have to include them in the optimization.

When the yield of project No. i in time period t is denoted by w_{it}, the problem formulated as a mixed integer programming problem reads

$$\min \sum_{t=0}^{T} \sum_{i=1}^{N} [c_i(1+r)^{-t}\, Y_{it} + b_i(1+r)^{-t}\, w_{it}] \tag{7}$$

$$\sum_{i=1}^{N} w_{it} \geq D(t) \qquad t = 1, \ldots, T \tag{8}$$

$$Q_i \sum_{\tau=0}^{t-1} Y_{i\tau} - w_{it} \geq 0 \qquad \begin{cases} i = 1, \ldots, N \\ t = 1, \ldots, T \end{cases} \tag{9}$$

$$\sum_{t=0}^{T} Y_{it} \leq 1 \qquad i = 1, \ldots, N \tag{10}$$

$$Y_{it} \in \{0,1\}, \quad w_{it} \geq 0. \qquad \begin{cases} i = 1, \ldots, N \\ t = 0, \ldots, T \end{cases} \tag{11}$$

Here, the objective, Eq. (7), is to minimize the present value of the total construction costs plus the total operation costs. The minimization has to be performed under the restrictions that the total production of water in time period t must not be less than the demand in this period, Eq. (8), and that project No. i cannot deliver water before it has been constructed and then not more than its capacity limit, Eq. (9). The constraints, Eq. (10), are the same as Eq. (4), while Eq. (11) states that Y_{it} and w_{it} are a 0-1 variable and non-negative continuous variable, respectively.

In a study with a wider scope, Haimes and Nainis (1974) have also considered the model, Eqs. (7)-(11). Reformulating the problem as a dynamic programming problem, they proposed time in discrete years as the only state variable. However, this approach is only approximate and will not suffice in general.

In the present study, a dynamic programming algorithm equivalent to the mixed integer programming problem, Eqs. (7)-(11), will be derived. Before developing the new general recursive equation, it will be appropriate to determine the minimal operation costs as a function of the available projects.

Let $H^X(\tau,t)$ be the minimum present value of the operation costs satisfying the water demand from time period τ through time period t, using the projects in the set X, i.e.

$$H^X(\tau,t) = \min \sum_{p=\tau}^{t} \sum_{i \in X} b_i (1+r)^{-p} w_{ip} \qquad (12)$$

$$\sum_{i \in X} w_{ip} \geq D(p) \, , \quad p = \tau, \ldots, t \qquad (13)$$

$$0 \leq w_{ip} \leq Q_i \, , \quad i \in X, \quad p = \tau, \ldots, t \qquad (14)$$

Note that the problem, Eqs. (12)-(14), is separable in p, i.e.

$$H^X(\tau,t) = \sum_{p=\tau}^{t} h^X(p) \qquad (15)$$

where

$$h^X(p) = (1+r)^{-p} \min \sum_{i \in X} b_i w_{ip} \qquad (16)$$

$$\sum_{i \in X} w_{ip} \geq D(p) \qquad (17)$$

$$0 \leq w_{ip} \leq Q_i \, , \quad i \in X \qquad (18)$$

This is a very simple linear programming problem, which can easily be solved. The demand is met by introducing the projects in an order determined by their per unit operation cost (b_i, $i \in X$), arriving at the optimal solution when the total production equals the demand. If we cannot find a feasible solution to the problem, Eqs. (16)-(18), ie.

$$\sum_{i \in X} Q_i < D(p),$$

then we define $h^X(p) \equiv \infty$

In the scheduling problem, Eqs. (2)-(5), we were able to neglect the problem concerning the timing of the projects, because it was always optimal to postpone a project until more capacity was required. However, in the present problem, Eqs. (7)-(11), where an optimal production plan for each time period as well as an optimal schedule for the considered planning period have to be determined, the completion dates cannot be derived directly in general.

Although it is to be expected in most real-life problems that it will still be optimal to postpone a project as far into the future as possible, we will proceed in the following without this assumption. Therefore, it will be necessary to introduce time in discrete years as a state variable.

As Morin and Esogbue (1971) pointed out, the state space Ω, describing our possibilities of selecting combinations of projects for construction, can entirely be specified by the subspaces $\Omega_1, \ldots, \Omega_N$:

$$\Omega \equiv \Omega_1 \cup \ldots \cup \Omega_N$$

where Ω_n represents the $\binom{N}{n}$ possible sets of projects, each consisting of exactly n projects. Remembering that time in discrete years is also a state variable, our total state space therefore becomes $\Omega \times T$.

Exploiting the fact that the problem, Eqs. (7)-(11), can be regarded as a sequential decision process and recalling that our objective is to minimize the total costs, we define $G_n(X,t)$ as the minimum present value of the total costs, satisfying the demand up to time t by means of the n projects in the set X.

If

$$D(t) \quad \begin{cases} = 0\,, & t = 0 \\ > 0\,, & t = 1, \ldots, T \end{cases} \tag{20}$$

then, at the first stage where only one project is considered we formally state

$$G_1(X,t) = \min_{i \in X}\{c_i + H^X(0,t)\} \tag{21}$$

However, the minimization can be excluded, because at the first stage, X only consists of one distinct project. In Eq. (21) it has been utilized that we cannot satisfy the demand in the first period unless the project $i \in X$ is completed at time zero.

$G_1(X,t)$ must be computed for every combination of the state variables, i.e. $\forall X \in \Omega_1$ and $t = 0, \ldots, T$. For every combination of X and t, the corresponding completion date denoted by $\psi_1(X,t)$ is zero as explained above.

At the second stage where exactly two projects have to be constructed in the time interval 0-t, we compute:

$$G_2(X,t) = \min_{\substack{0 \le t_2 \le t \\ i \in X}} \{c_i(1+r)^{-t_2} + \eta^X(t_2,t) + G_1(X \setminus \{i\}, t_2)\} \qquad (22)$$

where η is defined by

$$\eta^X(t_2,t) \equiv \begin{cases} \sum_{\tau=t_2+1}^{t} h^X(\tau), & t_2 < t \\ 0, & t_2 = t \end{cases} \qquad (23)$$

$X \setminus \{i\}$ denotes the set of projects obtained by excluding project No. i from the set X. Equation (22) utilizes Bellman's principle of optimality, Bellman (1962). If a two-stage project is to be built satisfying the demand until time t, then *regardless of* the decision that might be made at t_2, the combined system cannot be optimal, unless the demand is satisfied until time t_2 by an optimal one-stage project. Eq. (22) consists of three terms, the first being the capital cost of completing project no. $i \in X$ at time t_2. The last term is the minimal cost of satisfying the demand up to time t_2 by the other project in the set X. When $t_2 < t$, the second term is the minimal operation costs in the time periods t_2 + 1,....,t. using both projects. For $t_2 = t$, project No. i will be built at the end of the considered interval of time and accordingly the operation costs in time periods 0-t will be included in the last term. Therefore, we have $\eta^X(t,t) = 0$.

Assume that the minimization for given values of X and t is obtained by adding the project No. $i \equiv i'$ at time $t_2 = t'$, then the corresponding set of completion dates $\psi_2(X,t) = \psi_1(X \setminus \{i'\}, t') \cup \{t'\}$. $G_2(X,t)$ must be computed for every value of the state variables, i.e. for $X \in \Omega_2$, $t = 0, \ldots, T$.

The general recursive relationship can now be formulated.

For every $n = 1, \ldots, N$, $X \in \Omega_n$, $t = 0, \ldots, T$ we have

$$G_n(X,t) = \min_{\substack{0 \le t_n \le t \\ i \in X}} \{c_i(1+r)^{-t_n} + \eta^X(t_n,t) + G_{n-1}(X \setminus \{i\}, t_n)\} \qquad (24)$$

where

$$\eta^X(t_n,t) \equiv \begin{cases} \sum_{\tau=t_n+1}^{t} h^X(\tau), & t_n < t \\ 0, & t_n = t \end{cases} \qquad (25)$$

with the boundary conditions

$$G_0(\emptyset,t) \equiv \begin{cases} 0 \text{ if } D(t) = 0 \\ \infty \text{ if } D(t) > 0 \end{cases} \tag{26}$$

where $\emptyset$ denotes the empty set.

The minimal present value of the total costs is obtained by executing the following minimizations: for $n = 1, \ldots, N$

$$Z_n(X_n^*) = G_n(X^*,T) = \min_{X \in \Omega_n} G_n(X,T) \tag{27}$$

and

$$Z^0(X^0) = \min_{n=1,\ldots,N} Z_n(X_n^*) \tag{28}$$

The optimal set of projects is given by X^0 and assuming that the minimum in Eq. (28) is occuring for $n \equiv n^0$ the corresponding set of completion dates is determined by $\psi_{n^0}(X^0,T)$. The optimal values of w_{it}, i.e. the production plan, can easily be derived if wanted.

Optimal Project Scheduling under Stochastic Demand

A common feature of the litterature mentioned in the introduction is that they all regard the demand function $D(t)$ as deterministic. In reality, $D(t)$ bust be regarded as a stochastic variable $\tilde{D}(t)$, because the future demand cannot be predicted with certainty. The extent to which the optimal decisions will be influenced depends on the magnitude of these uncertainties. The deterministic model could be solved using a number of probable demand functions, but this will in general not ensure an unambiquous solution.

Therefore, we generally define

$$P\{\tilde{D}(t) \leq c\} = F_t(c) \tag{29}$$

i.e. $F_t(c)$, $t = 0,\ldots,T$ is the probability distribution function for the demand in time period t. On this basis, we could try to set up a model for the future requirements.

One possibility would be to assume $\tilde{D}(t)$, $t = 0,\ldots,T$ independent, normally distributed random variables with mean μ_t and standard deviation σ_t, i.e. $\tilde{D}(t) \in N(\mu_t, \sigma_t^2)$, μ_t and σ_t in general being increasing functions of t. Letting $\phi(\cdot)$ be the probability density and $\Phi(\cdot)$ the cumulative distribution function of a standardized normal random variable, the following relationships are obtained.

$$F_t(c) = \Phi\left(\frac{c-\mu_t}{\sigma_t}\right) \tag{30}$$

$$f_t(c) = \frac{1}{\sigma_t} \phi \left(\frac{c-\mu_t}{\sigma_t} \right) \tag{31}$$

If we now return to our previous model Eqs. (7)-(11) and replace Eq. (8) with

$$\sum_{i=1}^{N} w_{it} \geq \tilde{D}(t) , \qquad t = 1, \ldots, T \tag{8a}$$

we have introduced the field of stochastic programming.

One way of classifying stochastic programming is to distinguish between the »wait and see« and »here and now« problems, Sengupta (1972), where the first group is characterized by the property that we can wait and observe the actual value of the random variable before making an optimal decision. However, before the value of the random variable comes to our knowledge, we could be interested in the experted output. The other, and most common situation, is the case where we are forced to make a »here and now« decision, i.e. we do not have the possibility of postponing the decision until the actual value of the random variable is known.

Examples of both the above-mentioned problems arise in the revised problem. The »wait and see« problem stems from the fact that we do not have to determine a production plan (i.e. w_{it}) ultimately at the beginning of the planning period ($t = 0$), i.e. we are in a »wait and see« situation concerning these variables. On the other hand, we have to make a »here and now« decision concerning the projects to be selected, their completion dates (i.e. Y_{it}), and accordingly the total available capacity in each time period. To depict this situation, we must calculate the expected minimum operation cost in each time period as a function of the available projects and relate the selection and timing of projects to our knowledge of $\tilde{D}(t)$, the aversion to shortage and the expected minimum operation costs.

Let $Q(t)$ denote the total capacity available in time period t, i.e.

$$Q(t) = \begin{cases} \sum_{i=1}^{N} Q_i \left(\sum_{\tau=0}^{t-1} Y_{i\tau} \right) & t > 0 \\ 0 & t = 0 \end{cases} \tag{32}$$

and let d_t denote the actual value of $\tilde{D}(t)$, then the operating policy problem for a given time period can be formally stated as

$$B_t(Q(t)) = \min \sum_{i=1}^{N} b_i (1+r)^{-t} w_{it} \tag{33}$$

$$\sum_{i=1}^{N} w_{it} = \begin{cases} d_t , & \text{if } d_t \leq Q(t) \\ Q(t), & \text{in all other cases} \end{cases} \tag{34}$$

$$0 \leq w_{it} \leq Q_i \left(\sum_{\tau=0}^{t-1} Y_{i\tau} \right) , \qquad i = 1, \ldots, N \tag{35}$$

where $B_t(Q(t))$ denotes the minimum present value of the operation costs in time period t. As seen from Eq. (32), the operation costs are a function of the available projects in the considered period of time.

In the problem formulation we have introduced the following assumptions. If shortage arises, all the available projects are operating up to their capacity limit in order to deliver the maximal possible amount of water. The opposite is the case if the actual demand is less than the total capacity of the available project, then only the part of the capacity corresponding to the demand will be delivered.

In the case of shortage, we then have

$$B_t^1(Q(t)) = \sum_{i=1}^{N} b_i(1+r)^{-t} Q_i\left(\sum_{\tau=0}^{t-1} Y_{i\tau}\right) \tag{36}$$

When d_t is less than $Q(t)$, we can easily obtain the minimal operation costs $B_t^2(d_t)$ as a function of d_t, because it will always be advantageous to use the available projects in an order determined by their per-unit operation cost, with the cheapest first and so on, until the demand is satisfied.

The expected minimal operation cost in a given time period t can now formally be computed

$$E\{B_t(Q(t))\} = B_t^1(Q(t))\, P\{\tilde{D}(t) > Q(t)\} + \int_0^{Q(t)} B_t^2(z)\, f_t(z)dz$$

$$= B_t^1(Q(t))\, \left[1-\Phi\left(\frac{Q(t)-\mu_t}{\sigma_t}\right)\right] + \int_0^{Q(t)} B_t^2(z)\frac{1}{\sigma_t}\,\phi\left(\frac{z-\mu_t}{\sigma_t}\right) dz \tag{37}$$

This was the first element which had to be determined in the establishment of a rational basis on which optimal decisions have to be made. The other important element which we have to include is the aversion to future shortage.

Note that if we add the N constraints, Eq. (9), and make use of Eq. (8), we obtain

$$\sum_{i=1}^{N} Q_i\left(\sum_{\tau=0}^{t-1} Y_{i\tau}\right) = Q(t) \geq D(t), \qquad t = 1,\ldots,T \tag{38}$$

which means that the requirement of capacity being greater or equal to the demand is naturally inherent in the original deterministic model. If we replace $D(t)$ in Eq. (38) with its random analog $\tilde{D}(t)$

$$Q(t) = \sum_{i=1}^{N} Q_i\left(\sum_{\tau=0}^{t-1} Y_{i\tau}\right) \geq \tilde{D}(t), \qquad t = 1,\ldots,T \tag{39}$$

the restrictions do not make sense any more, because in general a given figure cannot always be greater than a random variable. Therefore, the interpretation of feasibility has to be revised according to the aversion to shortage. We can now proceed in two different ways which will be discussed in detail in the following.

The Chance Constraint Model

If the feasibility of the capacity restrictions, Eq. (39), is only desirable but not indispensable, and the consequences of shortage is difficult to evaluate in economic terms, the chance constraint linear programming techniques developed by Charnes and Cooper (1963) represent an alternative way of taking the stochastic future into account.

Suppose that the decision makers have expressed their preferences against shortage in the shape of a confidence level $1-\alpha$, defined by requiring that the probability of shortage in a given time period must not exceed α, i.e.,

$$P\{\tilde{D}(t) \leq \sum_{i=1}^{N} Q_i (\sum_{\tau=0}^{t-1} Y_{i\tau})\} \geq 1-\alpha, \quad t = 1,\ldots,T \tag{40}$$

which is equivalent to

$$\sum_{i=1}^{N} Q_i (\sum_{\tau=0}^{t-1} Y_{i\tau}) \geq \mu_t + \sigma_t \Phi^{-1}(1-\alpha), \quad t = 1,\ldots,T \tag{41}$$

where $\Phi^{-1}(\cdot)$ denotes the inverse normal standard distribution function. Eq. (41) being the deterministic constraints equivalent to the probabilistic constraints given by Eq. (40), the chance-constraint model becomes

$$\min \sum_{t=0}^{T} [\sum_{i=1}^{N} c_i (1+r)^{-t} Y_{it} + E\{B_t(Q(t))\}] \tag{42}$$

$$Q(t) = \sum_{i=1}^{N} Q_i (\sum_{\tau=0}^{t-1} Y_{i\tau}) \geq \mu_t + \sigma_t \Phi^{-1}(1-\alpha), \quad t = 1,\ldots,T \tag{43}$$

$$\sum_{t=0}^{T} Y_{it} \leq 1, \quad i = 1,\ldots,N \tag{44}$$

$$Y_{it} \in \{0,1\} \tag{45}$$

This problem can be solved efficiently by use of a slightly modified version of the dynamic programming formulation, Eqs. (24)-(26). With the purpose of ensuring that

Eq. (43) is satisfied, we define an indicator function

$$\delta_t(q) = \begin{cases} \infty , \text{ if } q < \mu_t+\sigma_t\Phi^{-1}(1-\alpha) \\ 0 , \text{ in all other cases} \end{cases} \tag{46}$$

Involving this function, the general recursive equation is still given by Eq. (24), but the term $\eta^X(t_n,t)$ now becomes:

$$\eta^X(t_n,t) = \begin{cases} \sum_{\tau=t_n+1}^{t} [E\{h^X(\tau)\}+\delta_\tau(q(X))], & t_n < t \\ 0 , & t_n = t \end{cases} \tag{47}$$

where $q(X)$ is the total capacity of the available projects,i.e. $q(X)=\sum_{i\in X} Q_i$ and $E\{h^X(\tau)\}$ is the expected minimum operation costs in time period τ, using the projects in the set X. These can be calculated in a way similar to Eqs. (33)-(37).

The boundary conditions are

$$G_0(\phi,t) = \begin{cases} 0 \text{ if } \mu_t+\sigma_t\Phi^{-1}(1-\alpha) \leq 0 \\ \infty \text{ in all other cases.} \end{cases} \tag{48}$$

The present value of the optimal solution is obtained by Eqs. (27)-(28) and the optimal set of projects and the corresponding completion dates are determined as previously mentioned.

Note that the optimal solution is a function of the predetermined confidence level, 1 - α. When, in general, α is determined by rules of thumb, it is important to establish a relationship between 1-α and the minimal costs in order to improve the basis on which the decisions are to be made. Performing these calculations, we have to remember that the model, Eqs. (42)-(45), does not take into account how much a possible shortage amounts to, but only how often shortage occurs.

The Penalty Model

If the consequences of not being able to satisfy the demand can be estimated in economic terms, then it is suitable to introduce a penalty function, which accounts for the cost of a possible lack of capacity at certain time periods. The optimal decision can then be obtained by balancing the decreased construction cost as a result of later completion dates against the increased expected penalty cost. Compared with the chance constraint formulation, this represents an alternative procedure which requires more information but ought to be preferred, if the necessary data exist, because it gives a more realistic description.

Denoting by d_t the actual value of the random variable $\tilde{D}(t)$ and by z the total capacity in time period t, we assume that the penalty function for time period t is given by

$$P_t(z,d_t) \begin{cases} > 0, & \text{if } d_t > z \\ = 0, & \text{in all other cases} \end{cases} \qquad (49)$$

As a consequence of d_t, $t = 1,....,T$ not being known at the time when decisions concerning project scheduling have to be made, the prospective penalty cost cannot be determined exactly. However, the expected penalty cost for a given time period can be obtained as

$$E\{P_t(z,\tilde{D}(t))\} = \int_z^\infty P_t(z,x)\, f_t(x)\, dx \qquad (50)$$

which exclusively is a function of the available capacity z in the considered time period. Recalling that the total capacity in time period t is given by Eq. (32), our penalty model becomes

$$\min \sum_{t=0}^{T} [\sum_{i=1}^{N} c_i(1+r)^{-t} Y_{it} + E\{B_t(Q(t))\} + E\{P_t(Q(t), \tilde{D}(t))\}] \qquad (51)$$

$$Q(t) = \sum_{i=1}^{N} Q_i (\sum_{\tau=0}^{t-1} Y_{i\tau}) , \qquad t = 1,...,T \qquad (52)$$

$$\sum_{t=0}^{T} Y_{it} \leq 1 , \qquad i = 1,...,N \qquad (53)$$

$$Y_{it} \in \{0,1\} \qquad (54)$$

As an example of the determination of the expected penalty cost, let us assume the penalty being proportional to the magnitude of the shortage. Letting s be the cost per unit, then

$$P_t(Q(t),d_t) = \begin{cases} s(1+r)^{-t}(d_t-Q(t)), & d_t > Q(t) \\ 0, & \text{in all other cases} \end{cases} \qquad (55)$$

and accordingly the expected penalty cost in time period t becomes

$$E\{P_t(Q(t), \tilde{D}(t))\} = s(1+r)^{-t} \int_{Q(t)}^{\infty} [x-Q(t)]\, f_t(x)\, dx$$

$$= s(1+r)^{-t} \sigma_t[u_t(\Phi(u_t)-1) + \phi(u_t)] , \qquad (56)$$

$$u_t = \frac{Q(t)-\mu_t}{\sigma_t}$$

The assumption about linearly varying pentalty cost would probably not be very satisfactory in general. We would rather expect a quadratic function, e.g.

$$P_t^*(Q(t),d_t) \equiv \begin{cases} s(1+r)^{-t}(d_t-Q(t))^2, & d_t > Q(t) \\ 0 \text{ , in all other cases} \end{cases} \quad (57)$$

In this case, the expected pentalty cost in time period t becomes

$$E\{P_t^*(Q(t),\tilde{D}(t))\} \equiv s(1+r)^{-t}\,\sigma_t^2[(u_t^2+1)(1-\Phi(u_t)) - u_t\,\phi(u_t)],$$
$$u_t \equiv \frac{Q(t)-\mu_t}{\sigma_t} \quad (58)$$

Regardless of the shape of the penalty function, Eq. (49), the expected penalty cost for each time period can be computed. Accordingly, a dynamic programming formulation can be applied yielding the general recursive equation, Eq. (24), with the term $\eta^X(t_n,t)$ now being

$$\eta^X(t_n,t) = \begin{cases} \sum_{\tau=t_n+1}^{t} [E\{h^X(\tau)\} + E\{P_\tau(q(X),\tilde{D}(\tau))\}], & t_n < t \\ 0 \text{ ,} & t_n \equiv t \end{cases} \quad (59)$$

The boundary conditions become

$$G_0(\phi,t) = \sum_{\tau=1}^{t} E\{P_\tau(0,\tilde{D}(\tau))\} \quad (60)$$

The optimal set of projects, completion dates and the minimal present value of the total costs can be derived as previously described, Eqs. (27)-(28).

Illustrative Example and Discussion

In the previous sections, several different models for project scheduling have been discussed. Whatever the problem under consideration, the presented general dynamic programming algorithm can be applied. The algorithm is based on the following recursive equation

$$G_n(X,t) = \min_{\substack{0\le t_n\le t \\ i\in X}} \{c_i(1+r)^{-t_n} + \eta^X(t_n;t) + G_{n-1}(X\setminus\{i\},t_n)\} \quad (24)$$

where the boundary conditions and η are dependent on the actual problem.

The application of the algorithm determining optimal project scheduling will be illustrated by solving an example. Consider six potential projects with the data given in Table 1.

Table 1 - Costs and capacities of the potential projects.

Project no. i	Capital cost c_i	Per unit-cost b_i	Capacity Q_i
1	0.0	0.31	5.8
2	14.4	0.35	3.0
3	19.6	0.40	4.5
4	27.6	0.45	7.5
5	24.4	0.45	6.5
6	30.0	0.50	9.0

Let the deterministic excess demand $D(t)$, (demand at $t = 0$ is taken as the basis) be as given in the function represented in Fig. 2. Contrary to Fig. 1, the demand is depicted as a continous curve. This is done to improve clarity.

At the beginning of the planning period (t=0), we assume that we have a certain amount of excess capacity, here represented by the capacity of project No. one, $Q_1 \equiv$ 5.8 units. Note that $c_1 = 0$ because that project has already been constructed. In the following cases, the annual rate of interest has been estimated at 7%

Case 1: The scheduling problem.

Neglecting the per unit-costs, i.e. $b_i = 0$, $i = 1,....,6$, the optimal solution turns out to be

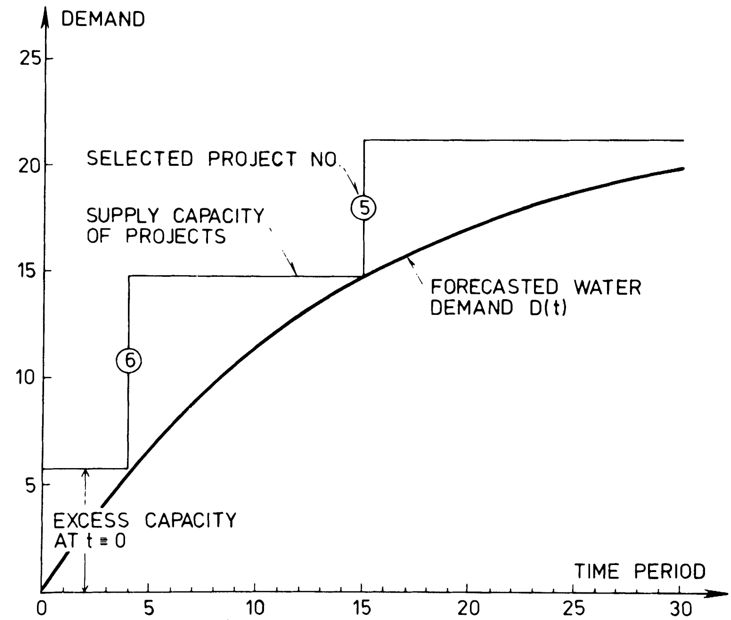

Fig. 2. Optimal project schedule, case 1.

Project no.	1	6	5
Completion date	-	4	15

with a minimal cost $Z_1^0 = 31.73$.

Case 2: The scheduling problem including operation cost.

Taking the operation cost into account, the optimal solution becomes

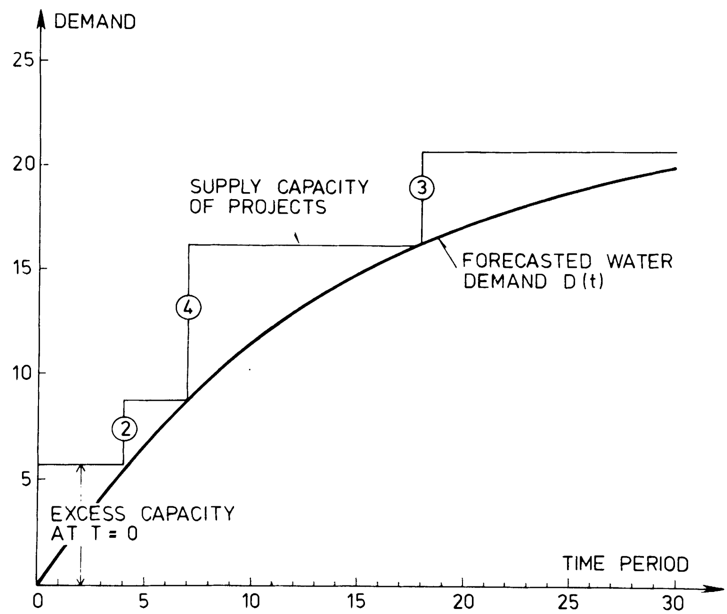

Fig. 3. Optimal project schedule, case 2.

Project no.	1	2	4	3
Completion date	-	4	7	18

with a minimal cost $Z_2^0 = 80.71$.

As expected, the optimal schedule as well as the minimal cost differs substantially in the two cases, because the operation costs account for at considerable part of the total costs.

In the following, we will assume that the demand function shown in Fig. 2 represents the *expected* water requirements, i.e. $\mu_t = D(t)$, t = 0,....,T and that the standard deviation of the demand at the end of the planning period has been estimated at $\sigma_T = 3$. If we, furthermore, assume that the variance is a linearly increasing function of the time t, i.e.:

$$\sigma_t^2 = \sigma_T^2 \frac{t}{T}, \qquad t = 0, \ldots, T \tag{62}$$

all the parameters of the random variables $\tilde{D}(t)$, $t = 0,....,T$ have been estimated.

Case 3: Optimal scheduling under chance-constraints.

If the Water Authority has accepted a confidence level of 90%, i.e. $\alpha = 10\%$, the optimal schedule becomes

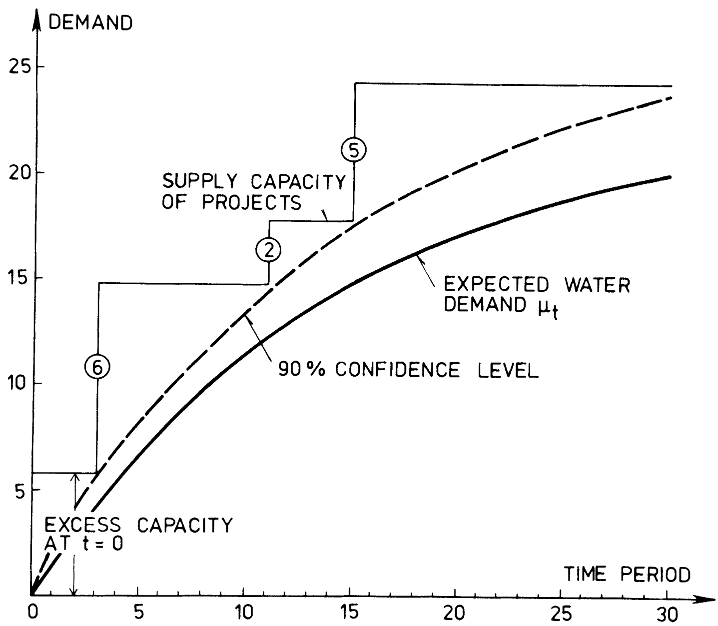

Fig. 4. Optimal project schedule, case 3.

Project no.	1	6	2	5
Completion date	-	3	11	15

with a minimal cost $Z_3^0 = 90.03$.

Case 4: The scheduling problem including penalty cost.

Assuming the penalty to be a linear function of the shortage, and the per unit penalty cost to be estimated at $s = 10$, the optimal capacity expansion plan becomes

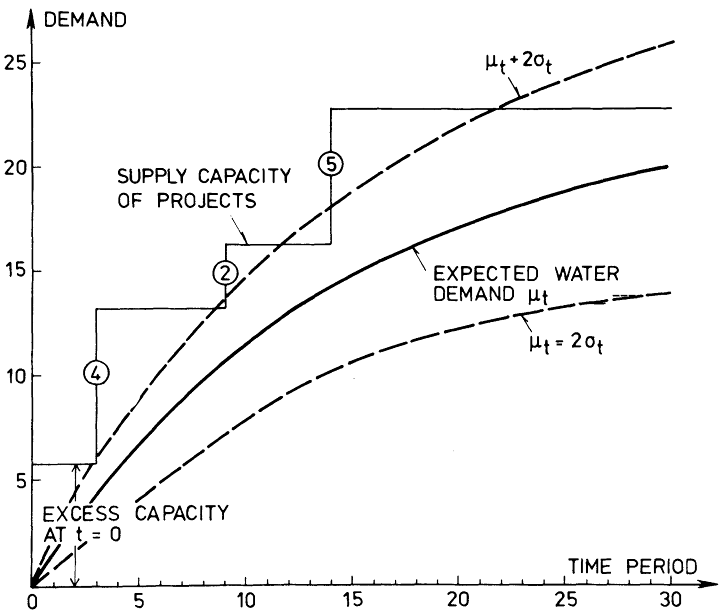

Fig. 5. Optimal project schedule, case 4.

Project No.	1	4	2	5
Completion date	-	3	9	14

with a minimal cost, $Z_4^0 = 91.21$.

The main purpose of carrying out the analysis is to assess the first decision to be taken. In the presented example, the first proper decision consists of selecting the project to succeed project no. one and appoint its completion date.

In comparison with case 2, the results from cases 3 and 4 demonstrate that taking the uncertainties in the future demand into account affect the first decisions to be taken considerably. Furthermore, we may realize that different solutions are obtained depending on the way the uncertainties are taken into account.

Recognizing uncertainties in the future demand, it is therefore important to assess the aversion to future shortage and include it in the analysis. Using the suggested algorithm we can decide on which project to construct first. By the time a new decision is needed, additional data have been accumulated. A new analysis can now be carried out, using the collected information to determine the next project to be constructed. In this way, a current planning can be performed, revising the data of the problem when more information is available.

One might ask if it is worth determining the optimal solution. How much is expected to be saved applying the optimal schedule instead of projects selected on the basis of rational rules-of-thumb? Of course, we cannot give answers which hold in general. We may confine ourselves to state that our calculations show that about 5-10% can be saved by introducing the optimal schedule instead of solutions obtained by rules of thumb.

Finally, a few computational comments. When the number of possible projects (N) is relatively small, the proposed algorithm is very efficient. In the presented example ($N = 6$, $T = 30$), it did only take between 2.2. - 3.3 sec to solve one of the problems on an IBM 370/165. However, when N increases the computational requirements increase rapidly. This is due to the fact that the number of states

$$(T+1)\sum_{n=0}^{N}\binom{N}{n} = (T+1)2^{N}$$

we had to examine increases dramatically when N increases. However, when N becomes larger than 15, a heuristic algorithm can be applied. Assume that the decisions which have been made in order to satisfy the demand until a certain time t with minimal costs, also represent the optimal basis for further capacity expansions. Exploiting this assumption - which not always had to be true - a heruristic algorithm can readily be constructed, where the only state variable is time in discrete years. Experience with this heuristic algorithm has been encouraging. The solutions have in almost all cases been the same as the exact optimal solution. In the remaining cases, the heuristic solution has been near-optimal.

Conclusion

Because of the dimension encountered in complex real life problems, it is highly desirable to develop efficient algorithms that can assist in water planning. Algorithms bases on dynamic programming principles have proved to be an excellent tool when searching for optimal solutions of capacity expansion problems.

In this paper, a general dynamic programming algorithm has been presented. When dealing with the most simple case, i.e. neglecting the operation costs and taking the future demand as deterministic, the model can be simplified, taking advantage of the fact that completion dates can be directly computed for a given sequence of projects. Applying this modification, the presented algorithm becomes identical to the DP2 algorithm of Morin and Esogbue (1971). However, when uncertainties in the future demand are recognized, the problem of project-timing has to be included in the problem formulation. If the aversion to future shortage in water supply can be described by chance-constraints or by a certain penalty function, the presented algorithm becomes an efficient tool in determining the optimal project schedule, provided the number of potential projects is not too large.

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