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Randomizing quantum states to Shatten p-norm for all $p \geq 1$

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Abstract. We formularize a method for randomizing quantum states with respect to the Shatten p-norms ($p \geq 1$) in trace class. In particular, this work includes the operator norm, $p = \infty$, and the trace norm, $p = 1$, simultaneously in a single statement via McDiarmid's inequality and a net construction.

Keywords: ε-randomizing channel, Shatten p-norm, McDiarmid's inequality, η-net
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INTRODUCTION

Randomizing quantum states in quantum information theory has many applications in quantum communications such as super-dense coding [1], data hiding [2] and proof of the additivity violation for the classical capacity on quantum channels [3]. Following Hayden, Leung, Shor and Winter's result [2] and Dickinson and Nayak's [4], here we make a general formula for the randomization of all quantum states. In this study, actually we formularize a method for randomizing quantum states with respect to the Shatten p-norms in trace class for all $p \geq 1$ [5].

Let $\mathscr{B}(\mathbf{C}^d)$ be the space of (bounded) linear operators and $\mathscr{U}(d) \subset \mathscr{B}(\mathbf{C}^d)$ the unitary group on the d dimensional Hilbert space $\mathbf{C}^d$, and $\mathbf{I}$ stands for the $d \times d$ identity operator on the space. Let $\mathscr{P}(\mathbf{C}^d)$ denote the set of all pure states i.e., unit vectors on $\mathbf{C}^d$. For all $p \geq 1$, the Shatten p-norm can be described in trace class by $\|A\|_p = \left(Tr(A^\dagger A)^{p/2}\right)^{1/p}$ for any matrix A [1].

Now, let's define an ε-randomizing maps with respect to the Shatten p-norm: A completely positive and trace-preserving map $\mathscr{R} : \mathscr{B}(\mathbf{C}^d) \to \mathscr{B}(\mathbf{C}^d)$ is *ε-randomizing* with respect to the Shatten p-norm $\|\cdot\|_p$ if, for all states $\rho \in \mathscr{B}(\mathbf{C}^d)$,

$$\left\| \mathscr{R}(\rho) - \frac{\mathbf{I}}{d} \right\|_p \leq \frac{\varepsilon}{\sqrt[p]{d^{p-1}}}. \tag{1}$$

If ε is equal to zero, the map $\mathscr{R}$ is called by completely randomizing map. Above definition of ε-randomizing map is well defined for some special cases p. Since, for the

[1] The Shatten p-norm of a matrix A is defined by $\|A\|_p = \left(\sum_{i=1}^d |s_i|^p\right)^{1/p}$, where s_i denotes the singular values of A.

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map $\mathscr{R}$ with respect to the trace norm, the ε-randomizing map is defined by the condition $\|\mathscr{R}(\rho) - \mathbf{I}/d\|_1 \leq \varepsilon$. Similarly, for $p = \infty$ case, the condition is naturally defined as $\|\mathscr{R}(\rho) - \mathbf{I}/d\|_\infty \leq \varepsilon/d$.

MAIN RESULTS

We are interesting to approximating the randomizing map $\mathscr{R}$ by mapping with small cardinality of unitary operators, and reproducing the known two results [2, 4] exactly. Next statement is our *main theorem*.

Let φ be a pure state in $\mathscr{P}(\mathbf{C}^d)$, and μ be the Haar measure on the unitary group $\mathscr{U}(d)$. For all $\varepsilon \geq 0$ and sufficiently large d, there exists a choice of unitaries in $\mathscr{U}(d)$, $\{U_i | 1 \leq i \leq m\}$ with $m \geq \frac{c_p \cdot d}{\varepsilon^2} \log\left(\frac{10 d^{(p-1)/p}}{\varepsilon}\right)$, which is independent μ-distributed random matrices, such that the map

$$\mathscr{R}(\varphi) = \frac{1}{m} \sum_{i=1}^{m} U_i \varphi U_i^\dagger \tag{2}$$

on $\mathscr{B}(\mathbf{C}^d)$ is ε-randomizing with respect to the Shatten p-norms for all $p \geq 1$ with probability at least $1 - e^{-m}$, and c_p is an absolute constant.

As mentioned above, c_1 and c_∞ are absolute constants. Notice that if $p = 1$, the map $\mathscr{R}$ is ε-randomizing with respect to the trace norm with the cardinality $m = O\left(\frac{c_1 \cdot d}{\varepsilon^2} \log\left(\frac{10}{\varepsilon}\right)\right)$ in Ref. [4]. If $p = \infty$, then $m = O\left(\frac{c_\infty \cdot d}{\varepsilon^2} \log\left(\frac{10d}{\varepsilon}\right)\right)$ in Ref. [2]. So, the theorem well represents the operator norm and the trace norm cases simultaneously in a single formula. See the proof details of above theorem in Ref. [5]. The scheme of main proof is similar to the References [2, 4]. For the proof, we make use of two key-lemmas known as McDiarmid's inequality [6] and η-net argument. The first one is a large deviation estimates and the second is a general method for discretization of all pure quantum states.

Including above mentioned two lemmas [2, 6] in the proof, we also make use of another two technical lemmas on the Shatten p-norms [5]: (i) For all $r > p \geq 1$ and for any *density* matrix $A \in \mathscr{B}(\mathbf{C}^d)$,

$$\left\| A - \frac{\mathbf{I}}{d} \right\|_p^r \leq d^{\frac{r-p}{p}} \|A\|_r^r - \frac{d^{(r-p)/p}}{d^p}. \tag{3}$$

(ii) For a fixed pure state $\varphi \in \mathscr{B}(\mathbf{C}^d)$, define $Y_{[\varphi]} = \left\| \mathscr{R}(\varphi) - \frac{\mathbf{I}}{d} \right\|_p$. Then. for all $r > p \geq 1$,

$$\mathbf{E} Y_{[\varphi]} \leq \left(\frac{\sqrt[p]{d}}{m^p} + \frac{r}{m^{p-1} \cdot \sqrt[p]{d}} \right)^{1/r}, \tag{4}$$

where $\mathbf{E}$ stands for the expectation value.

By exploiting above lemmas and several concatenated inequalities, we can bound the following probability by 1 with high probability. That is, for any pure state φ, next bound implies the proof of the main result

$$\mathbf{P}_{\forall\varphi}\left[Y_{[\varphi]} := \left\| \frac{1}{m}\sum_{i=1}^{m} U_i\varphi U_i^{\dagger} - \frac{\mathbf{I}}{d} \right\|_p \geq \frac{\varepsilon}{\sqrt[p]{d^{p-1}}}\right] < 1. \tag{5}$$

CONCLUSION

In conclusion, we can obtain a formula for randomizing quantum states with respect to the Shatten p-norms on d dimensional Hilbert space. That is, there exists a choice of unitary operators in $\mathscr{U}(d)$ selected according to the Haar measure, $\{U_i\}_{i=1}^m$ with $m = O(d\log(d^{(p-1)/p}/\varepsilon)/\varepsilon^2)$ such that the completely positive and trace-preserving map $\mathscr{R}(\varphi) = \frac{1}{m}\sum_{i=1}^m U_i\varphi U_i^{\dagger}$ on $\mathscr{B}(\mathbf{C}^d)$ is ε-randomizing with respect to the p-norm with high probability. We hope to contribute this work to quantum communication.

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