Comparison of quadratic and non-linear programming (QP and NLP) optimization models in groundwater management

Aris Psilovikos and Christos Tzimopoulos

ABSTRACT

This project is concerned with the comparison of two algorithms used in groundwater management models, based on Quadratic Programming (QP) and Non-linear Programming (NLP) models.

A quadratic objective function is used and solved in two different ways. The first one is the application of the Karush–Kuhn–Tücker (KKT) conditions and Wolfe's algorithm, which are used in solving QP models. The second one is the Conjugate Gradient Method (CGM), which is used in solving NLP models.

Two additional 'shell programs' are created to formulate the results of the management model. These results are organized in a Mathematical Programming System (MPS) file. This is the management model output and contains the response matrix coefficients and all the management model details in a coded format. The MPS data file is formatted via the two shell programs, constituting the import data file for the optimization procedure that takes place with the GINO model and spreadsheets.

An application took place in an aquifer in Northern Greece, just on the border with the Former Yugoslavian Republic of Macedonia (FYROM). The phreatic aquifer was divided into 271 small square areas, 200 m wide. The total area of the aquifer was 10.84 km^2. The time increment was equal to 1 month. Finally, the comparison of the two different optimization algorithms took place, concerning the pumping rates, the managed head distribution and the optimum pumping cost.

Key words | simulation, management and optimization groundwater models, quadratic and non-linear programming, KKT conditions, Wolfe's method, CG method, response matrix method

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INTRODUCTION

Our planet is experiencing a long period of increasing water demand and limited water resources. Water management, and specifically optimum water management in industrial, agricultural and urban use, are the only means we have in order to deal with this problem.

This study is concerned with minimizing pumping cost in a phreatic aquifer. A quadratic objective function is formulated and a solution is achieved, based on two different optimization algorithms. The quadratic function is used in order to avoid nonlinearity errors due to height loss, Δh, that occurred in the Linear Programming solution in previous research involving the same aquifer (Psilovikos 1996).

The research is focused on the comparison of the two different algorithms used, concerning the optimum pumping rates, the managed head distribution and the optimum pumping cost. The two different algorithms are:

1. The Karush–Kuhn–Tücker (KKT) conditions (Bazaraa *et al*. 1993) and Wolfe's algorithm (Wolfe

1959; Winston 1995). This algorithm is often used for Quadratic Programming Models.
2. The Conjugate Gradient Method (CGM) (Hestenes & Stiefel 1952; Fletcher & Reeves 1964; Hestenes 1980; Tzimopoulos *et al.* 1998; Psilovikos 1999*a*). This algorithm can be used for every non-linear objective function that has a range ≥ 2.

The optimum solution of the quadratic objective function is formulated in three steps:

1. Simulation model: application of a three-dimensional finite difference method. The solution is achieved using the backward differences method MODFLOW (McDonald & Harbagh 1988) and the Slice Successive Over-Relaxation (SSOR) (Moler 1969) procedure.
2. Management model: The response matrix coefficients are calculated (MODMAN) using space and time superposition. 'Influence surfaces' (Psilovikos & Tzimopoulos 1998) are produced in the field of wells due to unit pumping rates from individual pumping wells.
3. Optimization model:
   (a) The KKT conditions and Wolfe's algorithm are used so as to solve the model as a Quadratic Programming Model (QP).
   (b) The CGM is used so as to solve the model as a Non-Linear Programming Model (NLP).

The above model was used in the Eidomeni–Evzoni phreatic aquifer in Northern Greece. The collected data were based on 25 managed wells, over a period of 12 months. The quadratic terms in the objective function appeared only in 8 of the 25 managed wells that were used as control point wells. An extensive response matrix was obtained during the management model procedure. Two external shell programs were created in order to link the results of the management model (MODMAN), which were organized in an 'MPS' (Mathematical Programming System) format and the two optimization procedures based on KKT conditions, Wolfe's algorithm (QP) and the CG Method (NLP). Finally, from the comparison of the two optimization algorithms, interesting and quite useful conclusions were drawn which can be subjected to further research.

SIMULATION MODEL

The equation describing the groundwater flow is given by McDonald & Harbagh (1988):

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{zz}\frac{\partial h}{\partial z}\right)-W=S_s\frac{\partial h}{\partial t} \qquad (1)$$

where

K_{xx}, K_{yy}, K_{zz}	values of hydraulic conductivity along the x, y, z coordinate axes which are assumed to be parallel to the major axes of hydraulic conductivity [LT^{-1}],
W	volumetric flux per unit volume. It represents sources and sinks of water [T^{-1}],
t	time [T],
S_s	specific storage [L^{-1}],
$\left(\frac{\partial h}{\partial x}\right)$, $\left(\frac{\partial h}{\partial y}\right)$, $\left(\frac{\partial h}{\partial z}\right)$	space partial derivative with respect to x, y, z coordinate axes, respectively,
$\left(\frac{\partial h}{\partial t}\right)$	time partial derivative.

The partial derivative differential equation in three dimensions (Equation (1)) describes the groundwater flow and is used by the simulation model. This equation describes unsteady flow in heterogeneous and anisotropic media, considering that the major axes of hydraulic conductivity are identical to the Cartesian coordinate axes. S_s, K_{xx}, K_{yy} and K_{zz} can be space functions [$S_s=S_s(x,y,z)$, $K_{xx}=K_{xx}(x,y,z)$, $K_{yy}=K_{yy}(x,y,z)$, $K_{zz}=K_{zz}(x,y,z)$] and W a space and time function [$W=W(x,y,z,t)$].

NUMERICAL MODEL

Analyzing the first term of Equation (1), it can be expressed as follows:

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial h}{\partial x}\right) = \frac{K_{xx}\Big|_{i+1/2,j,k}\left(h^{n+1}_{i+1,j,k}-h^{n+1}_{i,j,k}\right)-K_{xx}\Big|_{i-1/2,j,k}\left(h^{n+1}_{i,j,k}-h^{n+1}_{i-1,j,k}\right)}{\Delta x^2} \qquad (2)$$

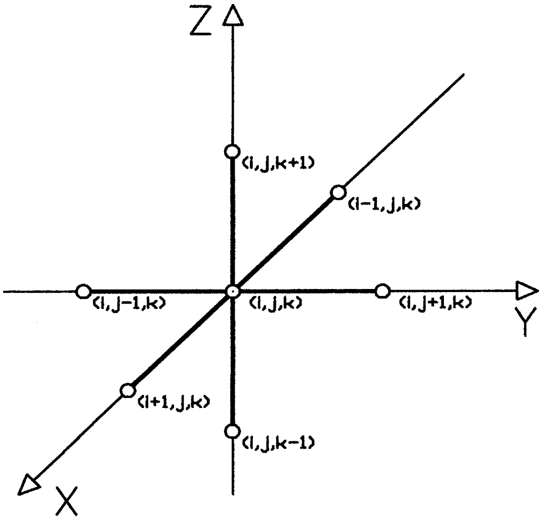

Figure 1 | Space discretization in *x*, *y* and *z* coordinate axes.

The same substitutions can be obtained for the next two terms of Equation (1). After space discretization in the x, y, z directions, the differential Equation (1) is transformed into a finite difference form in three dimensions (Figure 1) and the form of a fully implicit scheme (backward difference form) is obtained:

$$\frac{K_{xx}\Big|_{i+1/2,j,k}\left(h^{n+1}_{i+1,j,k}-h^{n+1}_{i,j,k}\right)-K_{xx}\Big|_{i-1/2,j,k}\left(h^{n+1}_{i,j,k}-h^{n+1}_{i-1,j,k}\right)}{\Delta x^2}$$
$$+\frac{K_{yy}\Big|_{i,j+1/2,k}\left(h^{n+1}_{i,j+1,k}-h^{n+1}_{i,j,k}\right)-K_{yy}\Big|_{i,j-1/2,k}\left(h^{n+1}_{i,j,k}-h^{n+1}_{i,j-1,k}\right)}{\Delta y^2}$$
$$+\frac{K_{zz}\Big|_{i,j,k+1/2}\left(h^{n+1}_{i,j,k+1}-h^{n+1}_{i,j,k}\right)-K_{zz}\Big|_{i,j,k-1/2}\left(h^{n+1}_{i,j,k}-h^{n+1}_{i,j,k-1}\right)}{\Delta z^2}$$
$$-W\Big|_{i,j,k}=S_s\Big|_{i,j,k}\frac{h^{n+1}_{i,j,k}-h^{n}_{i,j,k}}{t^{n+1}-t^{n}}. \tag{3}$$

The time partial derivative, which is the right part of Equation (3), is approximated with differences between a specific time t^{n+1}, where the head is unknown and equal to $h^{n+1}_{i,j,k}$, and just a previous time t^n, where the head is known and equal to $h^{n}_{i,j,k}$, as shown in the formula below:

$$\left(\frac{\partial h}{\partial t}\right)\Bigg|^{n+1}=\left(\frac{\Delta h_{i,j,k}}{\Delta t}\right)^{n+1}=\frac{h^{n+1}_{i,j,k}-h^{n}_{i,j,k}}{t^{n+1}-t^{n}}. \tag{4}$$

MANAGEMENT MODEL

The management model MODMAN (Greenwald 1994), works as a link between the three-dimensional groundwater simulation model and the various mathematical programming–operations research–optimization models.

The major categories of groundwater management models are two. The first is the *embedding matrix* method (Remson & Gorelick 1980) and the second is the *response matrix* method (Gorelick 1983). The *response matrix* method is used in this project which has occurred from the assumption of space superposition for steady state problems and both space and time superposition for unsteady state–transient problems. According to the response matrix method, an external numerical simulation model is used to calculate coefficients. Each of the coefficients is associated with theoretical unit rates in each pumping well in the aquifer and unit drawdowns observed to the wells that are used as control points.

The mathematical problem is to formulate an algorithm according to which we will be able to compute the loss of head Δh_i that takes place at a determined position i, which is the control point well i, and is due to the pumping rates from N managed wells. This formula can be written as follows:

- steady state problems (space superposition):

$$H_i=U_i-\sum_{j=1}^{N}\alpha_{ij}Q_j \tag{5}$$

- unsteady state problems (space and time superposition) (Maddock 1971; Psilovikos 1996):

$$H^T_i=U^T_i-\sum_{k=1}^{T}\sum_{j=1}^{N}\alpha^{T-(k-1)}_{ij}Q^k_j \tag{6}$$

where:

i	control point well,
j	managed pumping well,
U^T_i	unmanaged head obtained in control point well i at the end of the last managing period T [m],
H^T_i	managed head obtained in control point well i at the end of the last managing period T [m],
$\alpha^{T-(k-1)}_{ij}$	average drawdown in each i control point well, at the end of the last managing period T, due to a

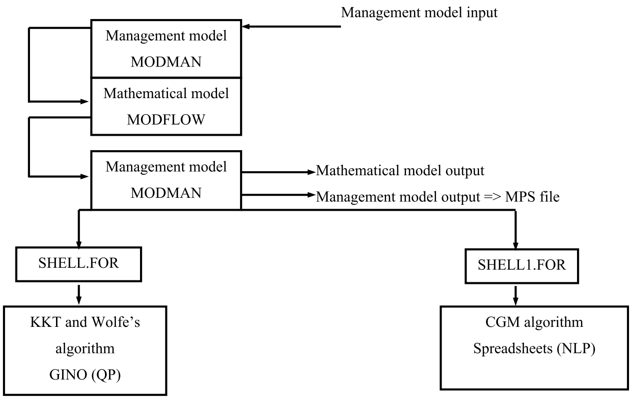

Figure 2 | Optimization procedure flowcharts in QP and NLP models.

unit pumping rate at the j managed well, applied throughout the k managing period,

Q_j^k pumping rate at well j during the k managing period [m^3/d].

The management model simulated the numerical model $(N+1)$ times, where N is the number of managed wells (Figure 2). During the first simulation the unmanaged heads U are computed with the assumption that no pumping takes place in the aquifer. During the next N simulations, the response matrix coefficients are computed, containing the unit head drawdowns referred to specific control point wells. In order to avoid scaling problems in calculating the response matrix coefficients, the theoretical unit pumping rates are chosen to be equal to 1,000 m^3/d because the real pumping rates have a range of 1,000–5,000 m^3/d.

The results of the $(N+1)$ simulations that occur during the management procedure are contained in the MPS file, which is an international code and can be read and processed directly as an input file from optimization models like LINDO (Schrage 1991). The LINDO model solves only Linear Programming (LP) and Mixed-Integer Programming (MIP) models and it was used in previous studies involving the same aquifer (Psilovikos 1996, 1999*b*; Psilovikos & Tzimopoulos 1998).

In the present study, two additional programs were used for the solution of the quadratic objective function, as is described in the optimization model below. These two programs (GINO and spreadsheets) cannot read the MODMAN MPS file results as an input file. The construction of two separate 'shell programs' in FORTRAN90 code (Psilovikos 1999*b*) gives us the opportunity to format the MPS data file into a compatible form, so it can be read as an input data file from the two programs (GINO and spreadsheets) used in the optimization procedure.

OPTIMIZATION MODEL

Quadratic programming model

The first solution is based on the KKT conditions and Wolfe's algorithm and is solved as a Quadratic Programming model.

The KKT conditions are the necessary and sufficient conditions for $\overline{X}=(X_1,\ldots,X_n)$ to be an optimum solution for the following Quadratic Programming Model:

$$\max \text{ (or min) } f(X_1, X_2, \ldots, X_n) \tag{7}$$

subject to the constraints, which can be linear, non-linear or both:

$$g_1(X_1, X_2, \ldots, X_n) \nless b_1$$

$$g_2(X_1, X_2, \ldots, X_n) \nless b_2$$

. . . .

$$g_m(X_1, X_2, \ldots, X_n) \nless b_m. \tag{8}$$

If $\overline{X}=(X_1,\ldots,X_n)$ is an optimum solution to (7) then $\overline{X}=(X_1,\ldots,X_n)$ must satisfy the m constraints in (8) and there must exist multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ satisfying the KKT conditions given below:

$$\left.\begin{array}{l} \dfrac{\partial f(\overline{X})}{\partial X_j} \mp \displaystyle\sum_{i=1}^{i=m} \lambda_i \dfrac{\partial g_i(\overline{X})}{\partial X_j} = 0 \\ \lambda_i\left[b_i - g_i(\overline{X})\right] = 0 \\ \lambda_i \geq 0 \\ i=(1,2,\ldots,m), j=(1,2,\ldots,n) \end{array}\right\} \begin{array}{ll} \ll - \gg & \text{Maximization model} \\ \ll + \gg & \text{Minimization model} \end{array} \tag{9}$$

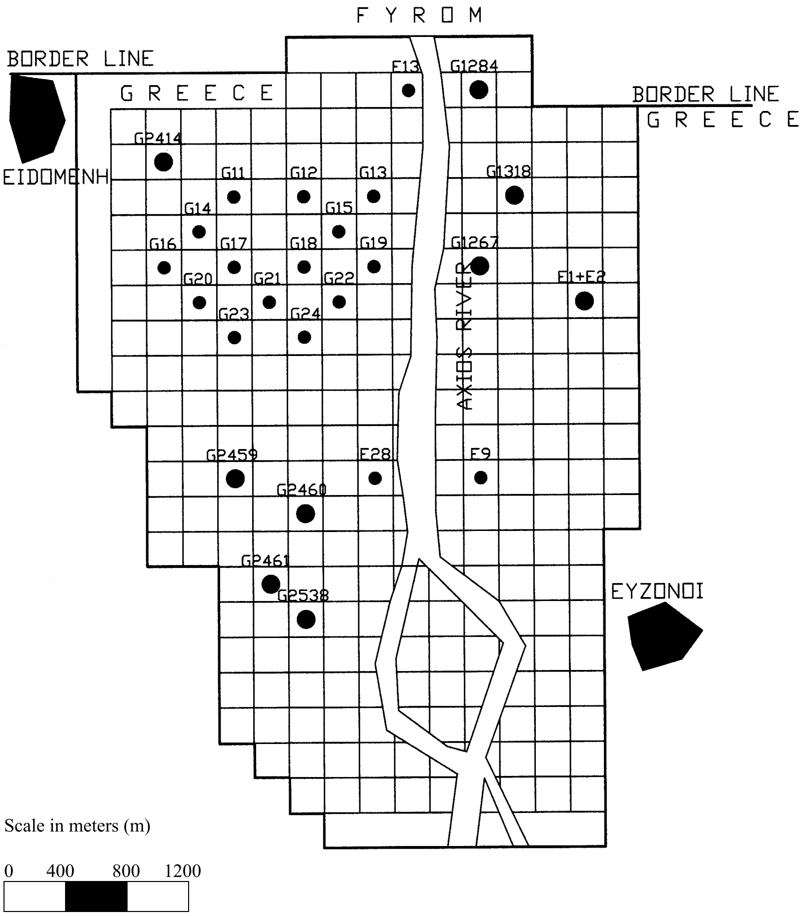

Figure 3 | The Eidomeni–Evzonoi phreatic aquifer. •: control point pumping wells, •: non-control point pumping wells.

where

$\lambda_1, \lambda_2, \ldots, \lambda_m$

Lagrange multipliers

$\overline{X} = (X_1, \ldots, X_n)$

solution vector matrix.

After application of the KKT conditions (Equation (9)), the quadratic programming model is transformed into a linear one and is solved according to Wolfe's algorithm. The GINO program is used (Liebman *et al.* 1986) to read the formatted data of the MODMAN MPS file via the shell program shell.for (Figure 2).

Non-linear programming model

The second algorithm is based on the *Conjugate Gradient Method*, proposed by Hesteness & Stiefel (1952) and transformed by Fletcher & Reeves (1964). This is an iterative algorithm.

According to this algorithm, at the kth step, a direction $\vec{s}_k$ is derived as a linear combination of $\nabla f(\vec{x}^k)$ and the previous directions $\vec{s}^a$, $(a = 1, 2, \ldots, k-1)$, where f is the objective function (Tzimopoulos *et al.* 1998).

The coefficients of linear combinations are chosen in such a way that the derived directions must be conjugate with respect to the Hessian matrix of f (Hesteness 1980):

$$\vec{s}_i^{\,T} \cdot \boldsymbol{H} \cdot \vec{s}_j = 0. \quad (10)$$

Only the current gradient $\nabla f(\vec{x}^k)$ and the previous $\nabla f(\vec{x}^{k-1})$ are used for the estimation of these coefficients.

This method started with an initial approach $\vec{x}^0$ and the steepest descent direction is chosen to be the first direction of minimization:

$$\vec{s}^{\,0} = -\nabla f\left(\vec{x}^{\,0}\right). \tag{11}$$

For the next point the following equation is chosen:

$$\vec{x}^{\,1} = \vec{x}^{\,0} + \lambda_0 \vec{s}^{\,0} \tag{12}$$

where λ_0 is a parameter, obtained by the Optimum Step Size Procedure (OSSP). The next direction $\vec{s}^{\,1}$ is obtained by the formula below:

$$\vec{s}^{\,1} = -\nabla f\left(\vec{x}^{\,1}\right) + \omega_1 \vec{s}^{\,0} \tag{13}$$

which is a linear combination of $\nabla f(\vec{x}^{\,1})$ and the previous direction $\nabla f(\vec{x}^{\,0})$. The parameter ω_1 is chosen using the property of conjugate directions (Equations (10) and (14)) and is given in Equation (15):

$$\vec{s}_1^{\,T} \cdot \boldsymbol{H} \cdot \vec{s}_0 = 0 \tag{14}$$

$$\omega_1 = \left\{ \frac{\|\nabla \vec{x}^{\,1}\|}{\|\nabla \vec{x}^{\,0}\|} \right\}^2. \tag{15}$$

A convergence criterion is examined at every iteration step and the general algorithm appears in the three formulae below:

$$\vec{s}^{\,k} = -\nabla f\left(\vec{x}^{\,k}\right) + \omega_k \vec{s}^{\,k-1} \tag{16}$$

$$\vec{x}^{\,k+1} = \vec{x}^{\,k} + \lambda_k \vec{s}^{\,k} \tag{17}$$

$$\omega_k = \left\{ \frac{\|\nabla \vec{x}^{\,k}\|}{\|\nabla \vec{x}^{\,k-1}\|} \right\}^2. \tag{18}$$

Spreadsheets are used for the solution of the algorithm. There is an embedded NLP solver routine in the Excel package that uses the CGM method, as described above, and reads the formatted data of MODMAN MPS files via the shell program shell1.for (Figure 2).

RESULTS AND DISCUSSION

The application of the combined simulation–management–optimization model was carried out in the Eidomeni–Evzoni phreatic aquifer in northern Greece (Psilovikos *et al.* 1996), just on the border with the Former Yugoslavian Republic Of Macedonia (FYROM).

The Axios River flows through this area with a direction from north to south. In Figure 3 the map of the phreatic aquifer is shown divided into 271 small areas, 200 m square. The total area of the aquifer is 10.84 km^2. The labelled filled circles represent the pumping wells and the largest of them represent the control point pumping wells. The white areas in the northwest and south represent boundary conditions of constant charge and all the other boundary square areas are conditions of vertical impermeable limits.

The data used for construction of the simulation–optimization model were precipitations, specific irrigation recharge, injections of 26 wells for 12 months, average head in the river, specific storage, porosity, hydraulic conductivity, ground levels and well base level. The average depth of the phreatic aquifer is 18 m. The time increment of Δt is equal to 1 month. The management period involves a period of 12 months (March 1996–March 1997), during which the 25 wells operate only during the four months of the irrigation period ($k = 3^{rd}$ month (June 1996), . . ., 6^{th} month (September 1996) and the water supply well G_{2414} is operating continuously.

The 8 control point wells are G_{1284}, G_{1318}, G_{1267}, $E_1 + E_2$, G_{2459}, G_{2460}, G_{2461} and G_{2538} (Figure 3). They are chosen as control point wells because of their extended pumping rates (Table 1), which are responsible for the extended loss of head Δh_i^T in these wells.

The application of the optimization model consists of the steps given below:

1. formulation of a quadratic objective function,
2. constraints in piezometric level in control point wells,
3. balance constraints,
4. constraints in minimum and maximum pumping rates.

Each one of the steps can be analytically written as follows:

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Table 1 | Optimum pumping rates obtained through the two models (m^3/d)

	Congugate gradient method—non-linear programming solution					**KKT conditions and Wolfe's algorithm—quadratic programming solution**				
Wells	**Mar–May, Oct–Feb**	**June**	**July**	**Aug**	**Sept**	**Mar–May, Oct–Feb**	**June**	**July**	**Aug**	**Sept**
G_{2414}–fixed	1,800	1,800	1,800	1,800	1,800	1,800	1,800	1,800	1,800	1,800
Control point wells ($i = 1, \ldots, 8$)										
G_{1284}, G_{1318}	0	4,500	4,500	4,500	4,500	0	4,500	4,500	4,500	4,500
G_{1267}	0	1,600	1,600	1,600	1,600	0	1,600	1,600	1,600	1,600
$E_1 + E_2$	0	2,100	2,100	2,100	2,100	0	2,100	2,100	2,100	2,100
G_{2459}	**0**	**1,120**	**1,120**	**1,120**	**1,120**	**0**	**1,600**	**1,600**	**1,600**	**183**
G_{2460}	**0**	**1,600**	**1,600**	**1,600**	**1,600**	**0**	**1,600**	**1,600**	**1,600**	**1,441**
G_{2461}	**0**	**1,006**	**1,006**	**1,006**	**1,006**	**0**	**1,600**	**1,600**	**1,600**	**1,096**
G_{2538}	**0**	**3,500**	**3,500**	**3,500**	**3,500**	**0**	**3,500**	**3,500**	**3,500**	**3,032**
Non control point wells ($j = 9, \ldots, 25$)										
E_{13}, G_{13}	0	1,100	1,100	1,100	1,100	0	1,100	1,100	1,100	1,100
G_{14}	**0**	**726**	**726**	**726**	**726**	**0**	**0**	**0**	**0**	**1,100**
G_{15}, G_{16}	0	1,100	1,100	1,100	1,100	0	1,100	1,100	1,100	1,100
G_{17}	**0**	**0**	**0**	**0**	**0**	**0**	**1,100**	**1,100**	**1,100**	**1,100**
G_{18},G_{19},G_{20},G_{21},E_9,E_{28}	0	1,100	1,100	1,100	1,100	0	1,100	1,100	1,100	1,100
G_{22}	**0**	**1,100**	**1,100**	**1,100**	**1,100**	**0**	**0**	**0**	**0**	**1,100**
G_{23}	**0**	**1,100**	**1,100**	**1,100**	**1,100**	**0**	**752**	**752**	**752**	**1,100**
G_{24}, E_9, E_{28}	0	1,100	1,100	1,100	1,100	0	1,100	1,100	1,100	1,100

1. Formulation of a quadratic objective function, considering negative rates for pumping. The aim is for the pumping cost to be minimized. The objective function is given below:

$$\mathbf{max}\, f(X) = \sum_{k=1}^{12} C_1^k Q_1^k + \sum_{i=2}^{26} \sum_{k=4}^{7} C_i^k Q_i^k . \tag{19}$$

The water supply well G_{2414}, which is operated continuously, is added as a fixed cost $\Sigma_{k=1}^{12} C_1^k Q_1^k$ in the objective function and represents the first term of Equation (19).

2. Constraints in piezometric level in the 8 control point wells of the aquifer in the end of the last managing period T. These constraints can be written as

$$\Delta h_i^T = U_i^T - H_i^T = \sum_{k=1}^{T} \sum_{j=1}^{N} \alpha_{ij}^{T-(k-1)} Q_j^k \le b_i^T = U_i^T - H_{i,\,\mathbf{min}}^T \tag{20}$$

where
$H^T_{i\,min}$
the imposed minimum head in the end of the last managing period T [m]. All the other terms have been explained above.

3. Balance constraints, concerning the total amount of pumping water:

$$\sum_{k=1}^{T}\sum_{j=1}^{N} Q_j^k = B \tag{21}$$

where $\boldsymbol{B}$ is a constant quantity.

These constraints represent the total amount of pumping water per month, which has to be equal to irrigation and water supply demands in every pumping period month.

4. Constraints in minimum and maximum pumping rates (negative values for pumping) in each managing period:

$$0 > Q_j^k > Q_{j\,\max}^k \tag{22}$$

where
$i = 1,2, \ldots, 8$
control point wells,
$j = 1,2, \ldots, 25$
all the pumping wells,
$k = 1,2, \ldots, T$
pumping periods: months,
$T = 6$
end of irrigation period,
$T = 12$
end of management period.

The cost coefficient is given by the following formula:

$$C = \frac{K}{Q_{unit}} Hman \tag{23}$$

where

Q_{unit} unit pumping rates equal to 1,000 m^3/d, applied in the management model to avoid scaling problems in calculating the response matrix coefficients, because the real pumping rates have a range from 1,000–5,000 m^3/d,

K coefficient that contains the cost of electric energy for the operation of the pumps and the cost of maintenance.

The total head is equal to

$$Hman_i^T = L_i^T + \Delta h_i^T + ct \tag{24}$$

where the above terms are

L_i^T the beginning head difference between the surface ground level and the unmanaged heads U [m],

Δh_i^T the height loss—drawdown—between the unmanaged heads U and the heads H [m] obtained after the application of the simulation model, equal to

$$\Delta h_i^T = U_i^T - H_i^T = \sum_{k=1}^{T}\sum_{j=1}^{N} \alpha_{ij}^{T-(k-1)} Q_j^k. \tag{25}$$

This expression (Equation (25)) is responsible for the quadratic terms of Q appearing in the objective function. These terms are expressed only for the control point wells ($i = 1, \ldots, 8$). For all the other pumping wells that are not considered to be control points ($j = 9, \ldots, 25$) we cannot express the loss of head Δh as a linear combination function of the response matrix coefficients and the pumping recharges (Equation (25)), so we consider Δh to be equal to an average $\overline{\Delta h}$. So $Hman$ remains a constant quantity for the non-control point wells.

ct the additional manometric head (62 m) for operation of the irrigation network [m].

So the cost coefficients used for the control point wells ($i = 1, \ldots, 8$) in the optimization model are

$$C_i^T = \frac{K}{Q_{unit}} \left(L_i^T + \sum_{k=1}^{T}\sum_{j=1}^{N} \alpha_{ij}^{T-(k-1)} Q_j^k + ct \right) \tag{26}$$

and the cost coefficients used for the non-control point wells ($j = 9, \ldots, 25$) are

$$C_j^T = \frac{K}{Q_{unit}} \left(L_j^T + \overline{\Delta h}_j^T + ct \right). \tag{27}$$

The assumptions considered for the construction of the model are:

1. The G_{2414} well is used only for water supply demands. The cost coefficients and the pumping rates are considered constant throughout the management periods, so the terms are added as a fixed cost in the objective function.
2. After the above assumption, the management periods are identical to the four months of the irrigation period ($k = 3^{rd}$ month (June 1996), . . ., 6^{th} month (September 1996), where all the pumping rates take place.
3. Quadratic terms are included in the objective function, due to the application of the response matrix method, referred to control point wells ($i = 1, \ldots, I_{control}$).
4. The other managed wells that are non–control points ($j = I_{control+1}, \ldots, J_{max}$), do not contain quadratic terms and the Δh terms are considered constant, as they are obtained from the simulation model.

Finally the objective function has the form below:

Quadratic terms referred to control points

$$\max f(X) = \frac{K}{Q_{unit}} \left\{ \sum_{i=1}^{I_{control}} \sum_{k=1}^{T} \left(L_i^k + \sum_{k=1}^{T} \sum_{j=1}^{J_{max}} \alpha_{i,j}^{T-(k-1)} Q_j^k + ct \right) Q_i^k \right\} + \sum_{j=I_{control}+1}^{J_{max}} \sum_{k=1}^{T} \left(L_j^k + \overline{\Delta h}_j^T + ct \right) Q_j^k + \sum_{fixed=1}^{12} C_{fixed} Q_{fixed} \tag{28}$$

Linear Terms referred in non-control points **Fixed cost due to the water supply well G_{2414}**

where $I_{control} = 8$, $J_{max} = 25$, $T = 4$ and fixed $= 12$.

The results obtained from both models are demonstrated in the tables and diagrams and are analytically described below.

The pumping rates obtained by solution of the two specific models are shown in Table 1. The two solutions give different optimum pumping rates for the wells G_{2459}, G_{2460}, G_{2461}, G_{2538}, G_{14}, G_{17}, G_{22} and G_{23}, comparing the two optimization models mentioned. Specifically, using the CG method and being solved as a NLP model, the optimum pumping rates for each one of the wells are the same during the four management periods (months). On the other hand, using KKT conditions and Wolfe's algorithm—the QP model—the optimum pumping rates are minimized only during the fourth month for the control point wells G_{2459}, G_{2460}, G_{2461} and G_{2538}. This fact leads the control point wells to a higher piezometric level for the QP model than the NLP one.

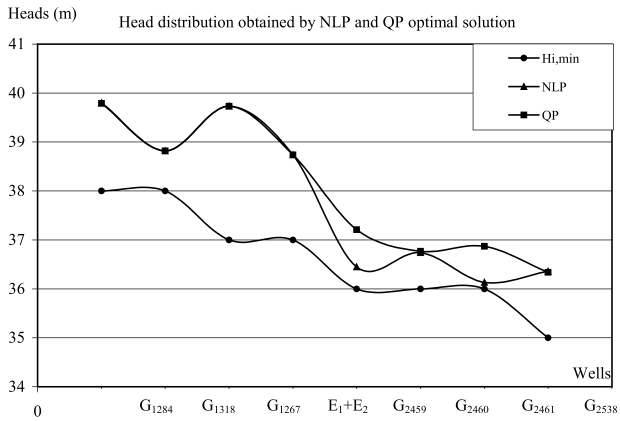

Figure 4 | Comparison of head levels distribution obtained by the two models.

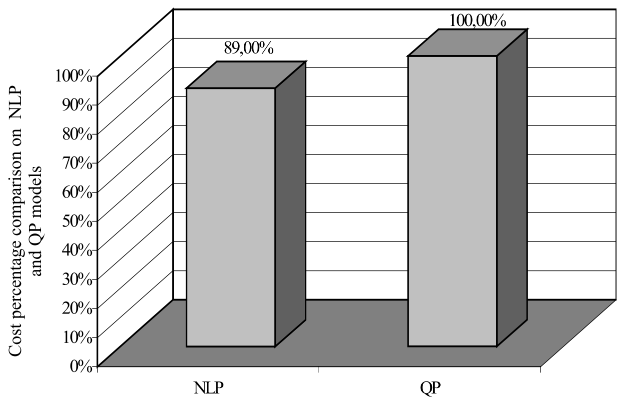

Figure 5 | Cost percentage comparison between NLP and QP models.

As we can see in Figure 4, three curves are depicted. The first one refers to the minimum imposed heads $H_{i,min}$ that appeared in the constraints in the piezometric level (Equation (20)) for the 8 control point wells. The second and third curves represent the solution obtained through the CG method (NLP) and Wolfe's algorithm (QP), respectively. The heads in the QP solution are kept at higher piezometric levels because the pumping rates from the 8 control point wells in the last managing period (fourth month) are much smaller than the rates obtained from the NLP solution. In order for the balance constraint to be satisfied for the fourth month in the QP solution, the non-control point wells G_{14}, G_{17}, G_{22} and G_{23} are pumped at rates equal to 1,100 m^3/d, which are the maximum imposed pumping rates that are given in the constraints in Equation (22).

The comparison cost between the NLP and QP model is referring to the percentage cost rate (Figure 5). The optimization cost does not refer to the total but to the non-constant pumping cost that is due to the terms $L + \Delta h$ (Equations (25)–(28)). The constant term *ct* is due to the additional manometric head, which is necessary for the irrigation network operation. It is the same for both of the models used in this study and it is equal to 62 m. So the non-constant pumping cost obtained from the CG method—the NLP model—is reduced to 89% of the cost obtained through Wolfe's method—the QP model—giving an 11% better optimization cost (Figure 5).

CONCLUSIONS

A quadratic objective function of minimizing the pumping cost in a phreatic aquifer is solved with two separate algorithms. The first one is the application of the KKT conditions and Wolfe's algorithm and the model is solved as a QP optimization model. The second one is the application of the CG method and the model is solved as a NLP optimization model.

Finally in the present study, the NLP model, which is solved with the CG method, is more preferable than the QP one which is solved with Wolfe's algorithm.

The optimum piezometric heads obtained through the NLP model are kept in lower levels for G_{2459} and G_{2461} control point wells than the solution obtained through the QP–Wolfe algorithm (Figure 4).

The constraints concerning the objective function are satisfied. These constraints are the balance constraints (Equation (21)), the minimum and maximum imposed pumping rates (Equation (22)) and the minimum imposed piezometric levels (Equations (20) and (25)).

Furthermore, from the solution of the NLP model a better cost optimization was obtained, which was 11% reduced from the non-constant optimum pumping cost obtained from the QP model.

Further research can be carried out for other confined or phreatic aquifers and groundwater catchments areas, contributing to optimum water management policy.

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