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On Fitting Gamma Distrubution to Synthetic Runoff Hydrographs

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Twelve methods of fitting the two-parameter gamma distribution to unit hydrographs are evaluated using experimental data. Five of them are the methods of moments (MOM), cumulants (MOC), maximum likelihood estimation (MLE), least squares (MOLS), and principle of maximum entropy (POME). The remaining seven are based on specification of different point and planar boundary conditions. It is found that of the later seven methods, only one is satisfactory; the other six are inaccurate and should be abandoned for purposes of achieving a mathematical fit. The first five methods are reliable, easier and efficient, and should be preferred.

Introduction

The two-parameter gamma distribution is commonly employed for synthesis of instantaneous (IUH) or finite-period (UH) unit hydrographs (Dooge 1973). By making two hydrologic observations, Edson (1951) was perhaps the first to have derived it for describing a unit hydrograph. Nash (1958, 1959, 1960) showed, by using theory of linear systems, that the mathematical equation of the instantaneous unit hydrograph of a basin represented by a cascade of equal linear reservoirs would be a gamma distribution. This also results as a special case of the general unit hydrograph theory developed by Dooge (1959). On the other hand, using statistical and mathematical reasoning. Lienhard and associates (Lienhard 1964; Lienhard and Davis 1971; Lienhard and Meyer 1967) derived this distribution as a basis for describing the instantaneous unit hydrograph. Thus, these investigators laid the foundation of a hydrophysical basis underlying the use of this distribution in synthesizing the direct runoff. There has since been a plethora of studies employing this distribution in surface watershed hydrology (Gray 1962; Wu 1963; DeCoursey 1966; Dooge 1973; Gupta and Moin 1974; Gupta et al. 1974; Croley 1980; Aron and White 1982; Singh 1982).

Many synthetic unit hydrograph methods specify a few selected points on the unit hydrograph through which a curve must be fitted. Examples of such methods are the methods of Bernard (1935), Snyder (1938), Commons (1942), Taylor and Schwarz (1952), Espey, et al. (1965), to name but a few. In these methods graphs or equations are provided to determine values of such attributes as peak flow rate, the lag or rise time, the base time, and the hydrograph widths W_{50} and W_{75} at 50 and 75 percent of the peak. The fitting of a smooth curve over these points is subjective and inconvenient; it may sometimes be difficult to satisfy its volume condition. These reasons coupled with the fact that a UH can adequately be represented by a gamma distribution constitute the basis for its fitting. Consequently, many workers prefer to fit a gamma distribution to a unit hydrograph resulting from a synthetic method (Croley 1980; Aron and White 1982; Collins 1983).

There are several methods by which a fit of the gamma distribution to a synthetic unit hydrograph can be achieved. Since it has two parameters, these methods differ in the ways by which to estimate the parameters. Croley (1980) has discussed some of these methods, and proposed some others based on specification of different boundary conditions (Haan and Barfield 1978). This study compares twelve different methods of fitting a two-parameter gamma distribution to four experimentally observed runoff hydrographs. Five of these methods are based on statistical considerations, three on point boundary conditions and the remaining four on planar boundary conditions. It is shown that six of the methods based only on point or planar boundary conditions are not capable of producing an accurate fit; they, doubtless, reproduce the boundary conditions used to estimate the parameters. The statistical methods are equally easy and more accurate, and should, therefore, be preferred.

The Gamma Distribution

The two-parameter gamma distribution $f(t)$ can be written as

$$f(t) \equiv \frac{1}{k\,\Gamma(n)} \left(\frac{t}{k}\right)^{n-1} \exp\left(-\frac{t}{k}\right), \quad k > 0, \quad t > 0 \qquad (1)$$

in which t is time in any desired units, k is parameter with the dimensions of T, and n is dimensionless parameter. k is a scale parameter and n is a shape parameter. The quantity $\Gamma(n)$ is the gamma function tabulated in most mathematical handbooks, equals $(n\text{-}1)!$ and can be written as

$$\Gamma(n) = \int_0^\infty t^{n-1} \exp(-t) \quad , \qquad n > 0 \tag{2}$$

Eq. (1) has the following properties

(i) $$\int_0^\infty f(s)ds = 1 \tag{3}$$

(ii) The time to peak t_p of $f(t)$ is

$$t_p = (n-1)k \tag{4}$$

(iii) The temporal mean $\bar{t}$ is

$$\bar{t} = nk \tag{5}$$

(iv) The variance (temporal) S_t^2 is

$$S_t^2 = nk^2 \tag{6}$$

The function $f(t)$ rises and recedes like a typical unit hydrograph. To fit this distribution to a synthetic unit hydrograph involves estimation of its parameters n and k.

The Direct Runoff Hydrograph (DRH)

The direct runoff hydrograph $Q(t)$ due to an effective rainfall of constant intensity I, volume V and finite duration D has an appearance of a moderately skewed (to the right) distribution as shown in Fig. 1. It possesses the following attributes (Croley 1980)

$$\int_0^\infty Q(s)ds = V = I\,D \tag{7a}$$

$$\int_0^{t_a} Q(s)ds = aV = V(t=t_a) \, , \qquad 0 < a < 1 \tag{7b}$$

$$Q(t) = 0 \quad , \qquad t < 0 \tag{8a}$$

$$Q(t) > 0 \quad , \qquad t > 0 \tag{8b}$$

$$\lim_{t\to\infty} Q(t) = 0 \tag{9}$$

$$\int_0^\infty (s-\bar{t}) \quad Q(s)ds > 0 \tag{10}$$

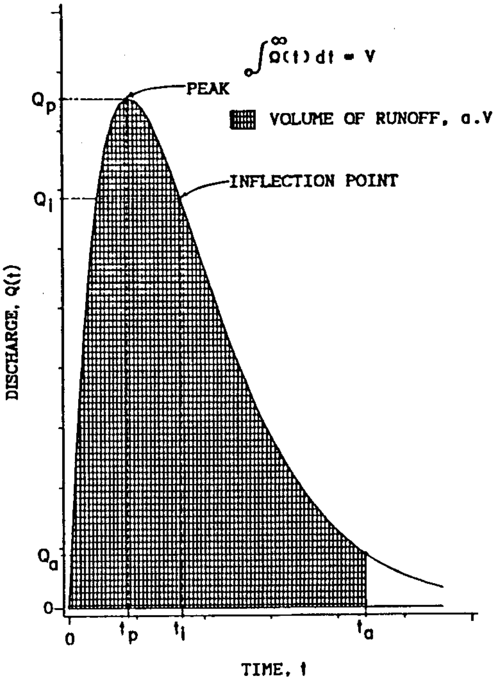

Fig. 1.
A typical direct runoff hydrograph.

Evidently, Eqs. (8) – (11) are satisfied by the gamma distribution. The D-hour unit hydrograph $h(D,t)$ is simply $Q(t)/V$. Thus, this distribution can be fitted to the unit hydrograph or equivalently to the direct runoff hydrograph. Therefore,

$$Q(t) = \frac{V}{k\,\Gamma(n)}\left(\frac{t}{k}\right)^{n-1}\exp\left(-\frac{t}{k}\right) \qquad (11)$$

Methods of Fitting

Twelve methods were selected for fitting a gamma distribution to synthetic unit hydrograph or direct runoff hydrograph. Five of these are methods of moments (MOM), cumulants (MOC), maximum likelihood estimation (MLE), principle of maximum entropy (POME) and least squares (MOLS); these are statistical in character. The methods of moments and cumulants have been used by Nash (1959) and Dooge (1973). The other three statistical methods have seldom been used for achieving a synthetic unit hydrograph fit. The mathematics of all these methods is straightforward and covered extensively elsewhere.

The remaining seven methods can be divided into the methods based on point boundary conditions and those based on planar boundary conditions. In the former category are ($Q=Q_p$, $t=t_p$), ($Q=Q_i$, $t=t_i$), and ($Q=Q_a$, $t=t_a$); and in the latter category are ($Q=Q_p$, $t=t_i$), ($Q=Q_p$, $t=t_a$), ($t=t_p$, $t=t_i$), ($t=t_p$, $t=t_a$). The symbols are defined as follows: Q_p is peak discharge, t_p is time to peak discharge, t_i is time to the inflection point on the recession limb, t_a is time up to which volume of direct runoff equals aV, Q_i is discharge at $t=t_i$, and Q_a corresponds to the discharge at $t=t_a$. Five of these methods have been proposed by Croley (1980). For their discussion we will, therefore, rely on his work. For the sake of completeness of the treatment, the twelve methods employed for fitting are briefly summarized.

Method of Moments (MOM)

Since $f(t)$ in Eq. (1) has only two parameters, n and k, only the first two moments M_1 and M_2 about origin will suffice. Nash (1959) derived these as

$$M_1 = \int_0^\infty t\, f(t)dt = \int_0^\infty \frac{t}{k\,\Gamma(n)} \left(\frac{t}{k}\right)^{n-1} \exp\left(-\frac{t}{k}\right)dt = \bar{t} = nk \tag{12}$$

$$M_2 = \int_0^\infty t^2 f(t)dt = \int_0^\infty \frac{t^2}{k\,\Gamma(n)} \left(\frac{t}{k}\right)^{n-1} \exp\left(-\frac{t}{k}\right)dt = nk^2 + n^2k^2 \tag{13}$$

By knowing M_1 and M_2 from epirical data, the parameters n and k can be determined.

Method of Cumulants (MOC)

This involves finding the first two cumulants C_1 and C_2, and then solving for n and k. The r-th cumulant can be expressed as

$$C_r \equiv \frac{d^r}{d\theta^r} \ln G(\theta)\Big|_{\theta=0}\,; \qquad 1,2,\ \ldots$$

in which $G(\theta)$ is the moment generating function defined as

$$G(\theta) = \int_0^\infty \exp(\theta s)\, f(s)\, ds\,.$$

Therefore,

$$C_1 = nk \tag{12a}$$

$$C_2 = nk^2 \tag{13a}$$

Since cumulants and moments are uniquely related,

$$C_1 = M_1$$

$$C_2 = M_2 - (M_1)^2$$

C_1 and C_2 are obtained from M_1 and M_2, and then n and k are determined.

Method of Maximum Likelihood Estimation (MLE)

The maximum likelihood estimation (MLE) function L for Eq. (1) is

$$L = \prod_{i=1}^{N} \frac{1}{k\,\Gamma(n)} \left(\frac{t_i}{k}\right)^{n-1} \exp\left(-\frac{t_i}{k}\right) \tag{14}$$

in which N denotes the number of discrete data points. Taking $\log_e$ of L,

$$\ln L = \sum_{i=1}^{N} \ln \left[\frac{1}{k\,\Gamma(n)} \left(\frac{t_i}{k}\right)^{n-1} \exp\left(-\frac{t_i}{k}\right)\right]$$

$$= -Nn \ln k - N \ln \Gamma(n) + (n-1)\sum_{i=1}^{N} \ln t_i - \frac{1}{k}\sum_{i=1}^{N} t_i \tag{15}$$

Differentiating ln L once with respect to n and equating to zero, doing likewise with respect to k, and rearranging yields

$$nk = \frac{1}{N}\sum_{i=1}^{N} t_i = \bar{t} \tag{16}$$

$$\ln k + \psi(n) = \frac{1}{N}\sum_{i=1}^{N} \ln t_i \tag{17}$$

where $\psi(n) = d[\ln \Gamma(n)]/dn$, called the psi function. Eqs. (16) - (17) can be utilized to determine n and k.

It may be noted that $\bar{t}$, in actual practice, will be the weighted mean, not arithmetic mean. For example, it was computed in this study as

$$\bar{t} = \sum_{i=1}^{N} [t_i Q_i] \Big/ \sum_{i=1}^{N} Q_i$$

for N observations of discharge Q and its time t. Similarly, the right side of Eq. (17) was computed as

$$\left[\sum_{i=1}^{N} Q_i \ln t_i\right] \Big/ \sum_{i=1}^{N} Q_i$$

Method of Principle of Maximum Entropy (POME)

The entropy of Eq. (1) can be expressed (Shannon 1948) as

$$I[f] = -\int_0^\infty f(t) \ln f(t)dt = -\int_0^\infty \left[\frac{1}{k\,\Gamma(n)} \left(\frac{t}{k}\right)^{n-1} \exp\left(-\frac{t}{k}\right)\right.$$

$$\ln[\frac{1}{k\,\Gamma(n)}(\frac{t}{k})^{n-1}\exp(-\frac{t}{k})]]\,dt$$

$$= [\ln\Gamma(n) + n\ln k] - (n-1)E[\ln t] + \frac{E[t]}{k} \quad (18)$$

in which $E[\cdot]$ denotes the expectation of $[\cdot]$. For maximizing $I[\cdot]$, differentiate Eq. (18) with respect to n and k and equate each time to zero. After rearranging

$$nk = E[t] \quad (19)$$

$$\psi(n) + \ln k = E[\ln t] \quad (20)$$

The right sides of Eqs. (19) – (20) are known from empirical data. Hence, n and k can be computed. This method does not seem to have been used for fitting gamma distribution to a synthetic unit hydrograph.

Method of Least Squares (MOLS)

This minimizes the sum of squares of deviations between observed f_0 and computed values f_c of the discharge or the *UH* ordinates $f(t)$. To this end, it is more convenient to use 1n $f(t)$ in place of $f(t)$,

$$E = \sum_{i=1}^{N} [\ln f_0(i) - \ln f_c(i)]^2 = \sum_{i=1}^{N} [\ln f_0(i) + \ln\Gamma(n)$$

$$+ n\ln k - (n-1)\ln t_i + \frac{t_i}{k}]^2 \rightarrow \min \quad (21)$$

Differentiating Eq. (21) with respect to n and equating to zero, and doing likewise with respect to k result in two nonlinear equations which can be solved for n and k. However, the global minimum of E was more easily found by computing the surface of E in the n-k plane. This procedure is equally efficient and is more instructive as it pictures evolution of the error surface with variations in n and k.

Boundary Conditions: $Q = Q_p$ and t=t

These point boundary conditions lead to

$$Q_p = Q(t=t_p) \quad (22)$$

$$\frac{dQ}{dt}(t=t_p) = 0 \quad (23)$$

Since $t_p > 0$ and $k > 0$, $n > 1$ from Eq. (4). Inserting t_p for t in Eq. (11),

$$Q_p = V(\frac{t_p}{k})^{n-1}\exp(-\frac{t_p}{k}) / [k\,\Gamma(n)] \quad (24)$$

Coupling Eqs. (4) and (24),

$$\frac{Q_p t_p}{V} = (n-1)^n \exp(1-n) / \Gamma(n) \qquad \text{or} \tag{25}$$

$$\frac{Q_p t_p}{V} = [e / (\frac{t_p}{k})]^{-t_p/k} / \Gamma(\frac{t_p}{k}) \tag{26}$$

Eqs. (4) and (25) – (26) can be used to determine n and k (Croley 1980). Graphical solutions of Eq. (26) have been presented by Haan (1970), Haan and Barfield (1978), Aron and White (1982) among others.

Boundary Conditions: $t=t_p$ and $t=t_i$

These planar boundary conditions lead to

$$\frac{dQ}{dt}(t=t_p) = 0 \tag{22}$$

$$\frac{d^2Q}{dt^2}(t=t_i) = 0 \tag{27}$$

Eq. (27) indicates that the slope of the hydrograph stops decreasing and starts increasing at the inflection point. By employing Eqs. (11) and (27),

$$(\frac{t_i}{k})^2 - 2(n-1)(\frac{t_i}{k}) + (n-1)(n-2) = 0 \tag{28}$$

Eq. (28) has two roots. The smaller root corresponds to the inflection point prior to the peak and the larger root corresponds to the inflection point subsequent to the peak or t_i. Therefore,

$$t_i = k[n-1 + (n-1)^{0.5}] \tag{29}$$

The other equation is found by inserting Eq. (4) in Eq. (29) which is quadratic with two roots. The relevant root is

$$n-1 = (\frac{t_p}{t_i-t_p})^2 \tag{30}$$

Eqs. (29) – (30) can be used to determine n and k as done by Croley (1980).

Boundary Conditions: $Q = Q_p$ and $t=t_i$

These are also planar boundary conditions and lead to

$$Q(t=t_p) = Q_p \tag{22}$$

$$\frac{d^2Q}{dt}(t=t_i) = 0 \tag{27}$$

Therefore, Eqs. (4), (24) and (29) hold. Combining Eqs. (4) and (24),

$$Q_p = V(n-1)^{n-1} \exp[-(n-1)] / [k\,\Gamma(n)] \quad (31)$$

Eliminating k between Eqs. (29) and (31),

$$\frac{Q_p t_i}{V} = [\{n-1+(n-1)^{0.5}\}(n-1)^{n-1} \exp(1-n)] / \Gamma(n) \quad (32)$$

Eqs. (29) and (32) can be used to compute n and k as done by Croley (1980).

Boundary Conditions: $t=t_p$, $t=t_a$ and $V(t=t_a) = aV$

These, again, are planar boundary conditions and yield

$$\frac{dQ}{dt}(t=t_p) \equiv 0 \quad (23)$$

$$\int_0^{t_a} Q(t)dt = aV = V(t=t_a) = \int_0^{t_a} V(\frac{t}{k})^{n-1} \exp(-\frac{t}{k}) / [k\,\Gamma(n)]dt \quad (33)$$

If $x = t/k$ then Eq. (33) reduces to an incomplete gamma function,

$$\int_0^{t_a/k} \frac{x^{n-1}}{\Gamma(n)} \exp(-x)dx = a \quad (34)$$

Eqs. (4) and (34) can be used to compute n and k as done by Croley (1980).

Boundary Conditions: $Q=Q_p$, $t=t_a$ and $V(t=t_a) = aV$

These planar boundary conditions yield

$$\frac{dQ}{dt}(t=t_p) = 0 \quad (23)$$

$$\int_0^{t_a} Q(t)dt = aV \quad (33)$$

Therefore, Eqs. (32) and (34) can be used to compute n and k as done by Croley (1980).

Boundary Conditions: $Q=Q_i$, $t=t_i$

These point boundary conditions yield

$$\frac{d^2Q}{dt^2}(t=t_i) \equiv 0 \quad (27)$$

$$Q(t=t_i) = Q_i \quad (35)$$

Therefore, Eqs. (11) and (29) can be used to compute n and k.

Boundary Conditions: $Q=Q_a$, $t=t_a$

These point boundary conditions lead to

$$\int_0^{t_a} Q(t)dt = aV \tag{33}$$

$$Q(t=t_a) = Q_a \tag{36}$$

Therefore, Eqs. (11) and (34) can be used to compute n and k.

Experimental Data

Four rainfall-runoff events were selected for evaluating the goodness of fit of each method, and to minimize the degree of fortuity in fitting. These events cover a portion of a large outdoor rainfall-runoff experimental facility located at Colorado State University. This portion has an area of approximately 296 m^2. Since the surface of this area is impervious, virtually the entire rainfall becomes surface runoff. The rainfall intensity was uniform for each event. Some pertinent characteristics of these events are given in Table 1.

Discussion of Results

The gamma distribution was fitted to the observed runoff hydrographs with the parameters estimated by each method which are given in Table 2. The MOM and MOC are identical, and so are MLE and POME. For the purposes of discussion we first discuss statistical methods, then point boundary-condition methods and finally planar boundary-condition methods. Fig. 2 provides a typical comparison of observed hydrographs with the hydrographs computed by MOM, MOC, MLE, POME and MOLS for the four events. Clearly MOLS provided a somewhat better overall fit for each event. It overestimated the bulk of rising limb as well as the recession limb, but underestimated peak discharge and overestimated the time to peak. This was consistently seen for each event. MLE and MOM provided comparable fits with the former having a slight advantage over the latter. Both methods had a tendency to overestimate the rising limb and underestimate the recession limb, and were superior to MOLS in estimating peak characteristics.

The relative error in peak discharge and its time was computed for each method as given in Table 3. Also given there is the mean of squared deviations between computed and observed discharges. Table 4 provides values of errors averaged over the four events for each method. The MLE method was slightly better than

A comparison of observed and computed runoff hydrographs for event 4.

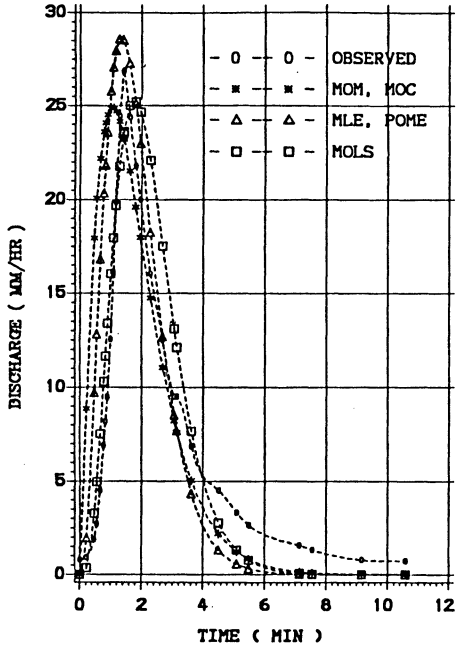

Fig. 2.
The methods for computation are MOM, MOC, MLE, POME, and MOLS.

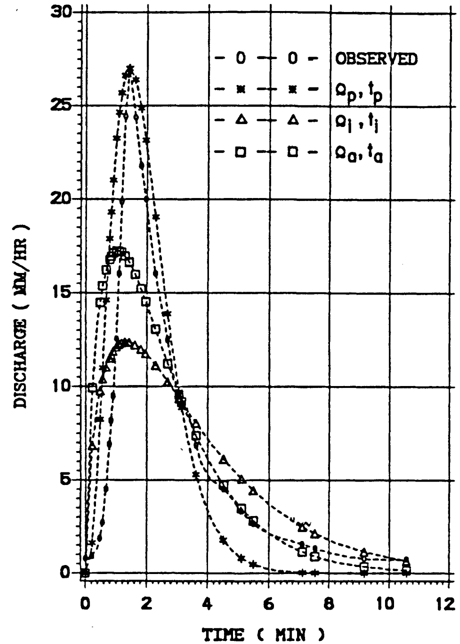

Fig. 3.
The methods of computation employ point boundary conditions (Q_p, t_p), (Q_i, t_i) and (Q_a, t_a).

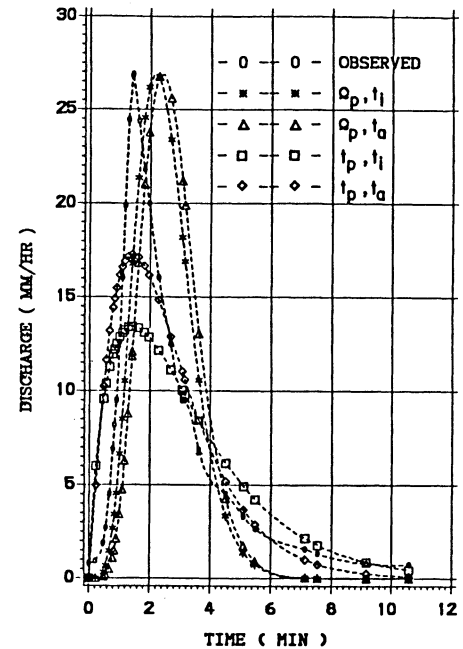

Fig. 4.
The methods of computation employ planar boundary conditions (Q_p, t_i), (Q_p, t_a), (t_p, t_i) and (t_p, t_a).

Table 1 – Some pertinent characteristics of four experimental rainfall-runoff events.

	Rainfall			Runoff							
Event	Intensity mm/hr	Duration sec	Depth mm	Q_p mm/hr	t_p sec	Q_i mm/hr	t_i sec	a	Q_a mm/hr	t_a sec	Duration sec
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1	11.6	111	.36	4.2	205	4.1	265	.68	2.3	349	796
2	26.4	79	.57	10.6	154	8.5	189	.72	4.5	272	700
3	60.7	71	1.19	33.2	102	20.1	147	.66	12.5	184	545
4	32.7	112	1.02	26.8	86	9.5	183	.68	9.5	183	635

Table 2 – Parameters n and k of two-parameter gamma distribution estimated by various methods of four experimental rainfall-runoff events. Note: $V(t = t_a) = aV$

Method	Event 1		Event 2		Event 3		Event 4	
	n	k (sec)	n	k (sec)	n	k (sec)	n	k (sec)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Method of Moments	3.24	89.4	4.33	39.8	3.63	33.5	2.31	47.8
Method of Cumulants	3.24	89.4	4.33	39.8	3.63	33.5	2.31	47.8
Method of Maximum Likelihood of Estimation	3.30	87.8	5.60	30.8	5.40	22.5	3.70	29.9
Principle of Maximum Entropy	3.30	87.8	5.60	30.8	5.40	22.5	3.70	29.9
Method of Least Squares	4.25	70.0	5.00	40.0	5.00	30.0	4.50	30.0
Given: $Q = Q_p$, $t = t_p$	3.93	70.0	5.27	36.0	5.00	25.5	3.69	31.8
Given: $Q = Q_i$, $t = t_i$	5.40	40.0	4.60	34.0	3.30	38.0	1.60	133.0
Given: $Q = Q_a$, $t = t_a$	2.90	102.0	2.70	82.0	2.00	81.5	1.70	89.8

cont.

Table 2 – cont.

Given: $Q \equiv Q_p$, $t \equiv t_i$	2.60	92.0	3.40	47.8	4.70	26.2	7.00	21.7
Given: $Q \equiv Q_p$, $t \equiv t_a$	5.20	58.6	8.30	28.0	9.50	17.5	8.20	19.9
Given: $t = t_p$, $t = t_i$	12.67	17.6	19.61	8.3	6.08	20.1	1.77	111.1
Given: $t = t_p$, $t = t_a$	3.18	94.0	3.33	66.0	2.72	59.5	2.19	71.8

Table 3 – Errors in fitting gamma distribution by different methods.

Method	Relative Error %								Mean Deviation Squared (Discharge)			
	Event 1		Event 2		Event 3		Event 4		Event 1	Event 2	Event 3	Event 4
	Q_p	t_p	Q_p	t_p	Q_p	t_p	Q_p	t_p				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
MOM, MOC	11.86	0	– 2.66	16.86	8.16	13.62	7.49	23.83	0.181	6.687	76.32	61.659
MLE, POME	11.31	0	–14.02	8.13	– 6.91	4.21	– 6.36	9.70	0.168	5.549	55.212	46.278
MOLS	5.45	– 8.85	6.45	– 7.88	16.10	–22.33	6.21	–26.84	0.053	0.737	11.761	6.072
Given: Q_p,t_p	0.69	0	– 1.06	0	1.02	0	– 0.17	0	0.143	2.223	40.055	30.037
Given: Q_i,t_i	– 43.12	12.71	–15.85	17.90	13.70	13.62	54.11	9.70	1.197	12.283	72.128	42.489
Given: Q_a,t_a	16.87	0	31.73	8.13	41.65	20.08	35.98	23.83	0.256	4.607	68.934	42.109
Given: Q_p,t_i	0.20	31.91	0.03	26.43	0.01	4.21	0.76	–59.81	1.030	11.218	56.105	35.318
Given: Q_p,t_a	0.04	–18.31	0.10	–30.99	0.04	–44.37	0.67	–59.81	0.145	4.723	75.722	65.303
Given: t_p,t_i	–102.32	0	114.21	0	–11.98	0	50.18	0.70	2.344	21.264	50.427	36.394
Given: t_p,t_a	15.12	0	26.57	0	36.95	0	35.81	0.71	0.171	2.929	43.015	25.538

Relative Error ≡ (Observed Quantity – Computed Quantity) * 100/Observed Quantity
Mean Squared Deviation (Discharge) ≡ Σ (Observed Discharge – Computed Discharge)^2/N
N ≡ Number of Observed Discharge Values over a Hydrograph

Table 4 – Average errors (ignoring algebraic sign) of different methods in fitting two-parameter-gamma distribution.

Method	Average Relative Error Q_p %	Average Relative Error t_p %	Average Absolute Error Real Time to Peak (sec)
(1)	(2)	(3)	(4)
MOM, MOC	7.54	13.58	11
MLE, POME	9.65	5.51	2
MOLS	8.55	16.48	19
Given: Q_p, t_p	0.74	0	0
Given: Q_i, t_i	31.70	13.48	19
Given: Q_a, t_a	31.56	13.01	13
Given: Q_p, t_i	0.25	30.49	40
Given: Q_p, t_a	0.21	38.37	45
Given: t_p; t_i	69.67	0.18	0
Given: t_p; t_a	28.61	0.18	0

MOM and MOLS which did not produce t_p as accurately as Q_p. The average relative error in Q_p was below 10 percent for all statistical methods; and that in t_p was below 6 percent for MLE, below 14 percent for MOM and below 17 percent for MOLS. The MOM and MOLS were comparable for fitting Q_p and t_p: The mean squared deviation in discharge was the least for MOLS. Thus, MOLS may be preferred for fitting the entire hydrograph, whereas MLE may be preferred if time to the peak discharge was desired. All these methods were comparable for reproducing the peak discharge. It may, however, be noted that these methods deviated from one another more as the hydrograph rose and receded more steeply.

Of the three methods based on point boundary conditions, the (Q_p, t_p) method is the most accurate as evident from a sample Fig. 3. This is also seen from Tables 3-4. Even for this method, the mean squared deviation became quite large for events 3 and 4; however, it was less than that of MOM and MLE for all 4 events. This then says that its performance may be fortuitous. The methods employing (Q_i, t_i) and (Q_a, t_a) were unsatisfactory as illustrated by Fig. 3.

All four methods based on planar boundary conditions – (Q_p, t_i), (Q_p, t_a), (t_p, t_i) and (t_p, t_a) – were unsatisfactory as seen from sample Fig. 4 as well as Tables 3-4. These methods were unsuitable for fitting the gamma distribution to synthetic unit hydrographs. This was consistently seen for all four events. This, however, was not surprising since these methods did not take into account the entire runoff hydrograph and were based on partial information on two or three points.

Conclusions

The following conclusions are drawn from this study: 1) The method of moments, cumulants, maximum likelihood estimation, principle of maximum entropy and least squares were satisfactory for fitting a gamma distribution to a synthetic unit hydrograph or direct runoff hydrograph. The methods of maximum likelihood estimation and principle of maximum entropy were, however, more accurate than the other three for fitting peak discharge and its time. 2) The method of least squares provided a better fit to the overall hydrograph. 3) The method based on the boundary condition $Q=Q_p$ and $t=t_p$ also produced a reasonable fit. 4) The methods based on other point and planar boundary conditions: $(V(t_a)=aV, t=t_a)$, $(Q=Q_p, t=t_i)$, $(t=t_p, t=t_i)$, $(Q=Q_p, t=t_a, V(t_a)=aV)$, $(Q=Q_i, t=t_i)$, $(Q=Q_a, t=t_a)$ were unsatisfactory and should not be used for achieving a mathematical fit.

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