

## Optimal water supply zone size

Prabhata K. Swamee and Virendra Kumar

### ABSTRACT

Large water distribution networks are divided into several small sub-networks called water supply zones. Each sub-network contains a central input point and a distribution system. There is nominal connection between the zones for inter-zonal transfer of water under varying demand patterns. Splitting of an area into zones is not only easy to design, it is also economical. Using geometric programming the present investigation determines the optimal zone size of circular and rectangular geometry. It is hoped that the present work will provide a guideline for deciding the zone size of an arbitrarily shaped area.

**Key words** | circular networks, establishment cost, geometric programming, network decomposition, rectangular networks, service connections, water supply, zone size

**Prabhata K. Swamee** (corresponding author)  
**Virendra Kumar**<sup>1</sup>  
 Department of Civil Engineering,  
 Indian Institute of Technology Roorkee,  
 Roorkee 247 667,  
 India  
 E-mail: swamifce@iitr.ernet.in

### INTRODUCTION

Water distribution systems are generally designed with fixed configuration, but there must also be an optimal geometry to meet a particular water supply demand. For example, an area can be served by designing a single system, and also by dividing it into a number of small zones, each one having an individual pumping and network system. The choice is governed by economic and reliability criteria. The economic criterion pertains to minimizing the water supply cost per unit discharge. The optimum zone size depends upon the network geometry, population density, topographical features and the establishment cost  $E$  for a zonal unit. The establishment cost includes the capitalized cost of operational staff, capital expenditure on production and conveyance of water to the zonal reservoir site. The establishment cost occurs in many optimal studies. Swamee (2001) found the optimal number of pumping stages as a function of the establishment cost.

Given an input point configuration and the network geometry, Swamee & Sharma (1990) developed an algorithm to decompose the water supply network into the zones under the influence of each input point.

The present investigation was undertaken to find the optimal area of a water supply zone. The area of a water supply network can be divided into various zones of nearly

equal sizes. The pumping station (or input point) can be located as close to the center point as possible. It is easy to design these zones as separate entities and provide nominal linkage between the adjoining zones. Thus in this investigation locations of input points were not assumed in order to obtain the zone size.

### COST FUNCTION

#### Preliminaries

Consider a circular area of radius  $L$ . The area may be served by a radial distribution system having a pumping station located at the center and  $n$  equally spaced branches of length  $L$  (see Figure 1(a)). Assuming  $\sigma$  = peak water demand per unit area ( $\text{m}^3/(\text{s}\cdot\text{m}^2)$ ) the peak discharge pumped in each branch is  $\pi\sigma L^2/n$ . Further, considering continuous withdrawal, the discharge withdrawn in the length  $x$  is  $\pi\sigma x^2/n$ . Thus the discharge  $Q$  flowing at a distance  $x$  from the center is the difference between these two expressions. That is,

$$Q = \frac{\pi\sigma L^2}{n}(1 - \xi^2) \quad (1)$$

where  $\xi = x/L$ . For pipe resistance use of the Hazen-Williams equation is widespread among user agencies. However, a study conducted by Liou (1998) strongly

<sup>1</sup>Present address: 8/5 Raj Nagar, Ghaziabad 201 002, India

discouraged use of the Hazen–Williams equation. Instead, the study recommended use of the Darcy–Weisbach equation (with the Colebrook equation for the friction factor), which is theoretically sound and has an extensive database. Thus, considering a continuously varying diameter, and using the Darcy–Weisbach equation with constant friction factor, the pumping head  $h_0$  was obtained as

$$h_0 = \int_0^1 \frac{8fLQ^2}{\pi^2 g D^5} d\xi + z_L + H - z_0 \quad (2)$$

where  $f$  = friction factor,  $g$  = gravitational acceleration,  $D$  = pipe diameter,  $z_0$  and  $z_L$  = elevations of the pumping station and the terminal end of the radial branch respectively and  $H$  = terminal head. For optimality  $D$  should decrease with the increase in  $\xi$  and finally at  $\xi = 1$  the diameter should be zero. Such a variation of  $D$  is impractical, as  $D$  cannot be less than a minimum permissible diameter. Thus it is necessary that the diameter  $D$  should remain constant throughout the pipe length whereas the discharge  $Q$  will vary according to Eq. (1). Using Eq. (1), Eq. (2) was changed to

$$h_0 = \frac{64fL^5 \sigma^2}{15gn^2 D^5} + z_L + H - z_0. \quad (3)$$

### Cost of distribution system

The pumping cost  $F_p$  was written as

$$F_p = \pi k_T \rho g \sigma L^2 h_0 \quad (4)$$

where  $\rho$  = mass density of water and  $k_T$  = pumping cost coefficient given by

$$k_T = \frac{(1 + s_b)k_p}{\eta} + \frac{8.76F_D F_A R_E}{\eta r} \quad (5)$$

where  $k_p$  = a proportionality constant,  $s_b$  = standby fraction, pumping head,  $\eta$  = combined efficiency of pump and prime mover,  $F_D$  = daily averaging factor,  $F_A$  = annual averaging factor,  $R_E$  = cost of energy per kWh and  $r$  = rate of interest in \$/\$/yr. Using Eq. (3), Eq. (4) was modified to

$$F_p = \frac{64\pi k_T \rho f \sigma^3 L^7}{15n^2 D^5} + \pi k_T \rho g \sigma L^2 (z_L + H - z_0). \quad (6)$$

The cost function  $F_m$  of the radial pipelines was written as

$$F_m = nk_m LD^m \quad (7)$$

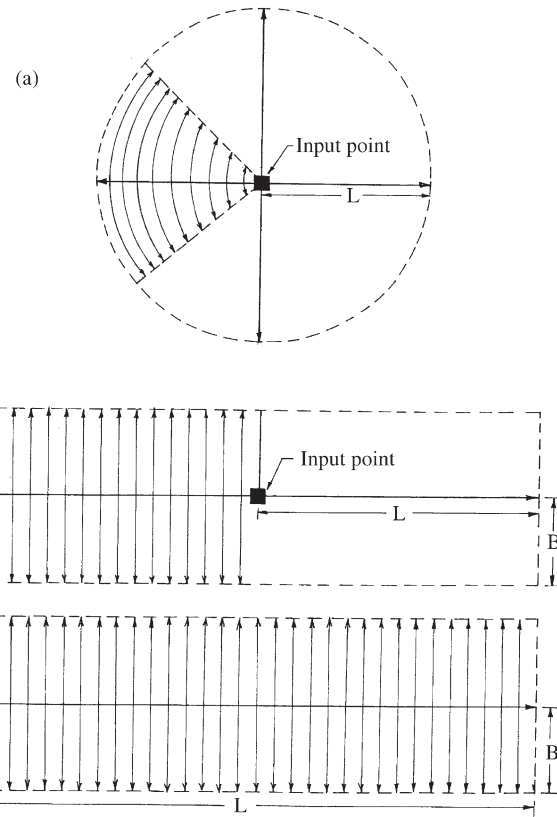


Figure 1 | Definition sketch. (a) Circular zone, (b) two branch strip zone, (c) single branch strip zone.

where  $k_m$  = proportionality constant and  $m$  = an exponent. Whereas  $k_m$  depends on the monetary unit employed,  $m$  remains fairly constant for a particular pipe material.

Adding Eqs. (6) and (7) the distribution system cost  $F_d$  was obtained as

$$F_d = nk_m LD^m + \frac{64\pi k_T \rho f \sigma^3 L^7}{15n^2 D^5} + \pi k_T \rho g \sigma L^2 (z_L + H - z_0). \quad (8)$$

For optimality, differentiating Eq. (8) with respect to  $D$  and equating it to zero and simplifying gives the following equation for the diameter:

$$D = \left( \frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m} \right)^{1/(m+5)}. \quad (9)$$

Using Eqs. (3) and (9) the pumping head works out to be

$$h_0 = \frac{64fL^5 \sigma^2}{15gn^2} \left( \frac{3mn^3 k_m}{64\pi k_T \rho f \sigma^3 L^6} \right)^{5/(m+5)} + z_L + H - z_0. \quad (10)$$

Using Eqs. (4) and (10), the pumping cost was obtained as

$$F_p = \frac{64\pi k_T \rho f L^7 \sigma^3}{15n^2} \left( \frac{3mn^3 k_m}{64\pi k_T \rho f \sigma^3 L^6} \right)^{5/(m+5)} + \pi k_T \rho g \sigma L^2 (z_L + H - z_0). \quad (11)$$

Similarly using Eqs. (7) and (9) the pipe cost was obtained as

$$F_m = nk_m L \left( \frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m} \right)^{m/(m+5)}. \quad (12)$$

Adding Eqs. (11) and (12), the cost of distribution system was obtained as

$$F_d = nk_m L \left( 1 + \frac{m}{5} \right) \left( \frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m} \right)^{m/(m+5)} + \pi k_T \rho g \sigma L^2 (z_L + H - z_0). \quad (13)$$

### Cost of service connections

The frequency and length of the service connections will be less near the center, and more towards the outskirts. Considering  $q_s$  as the discharge per ferrule through a service main of diameter  $D_s$ , the number of connections per unit length  $n_s$  at a distance  $x$  from the center is

$$n_s = \frac{2\pi\sigma x}{nq_s}. \quad (14)$$

The average length  $L_s$  of the service main is

$$L_s = \frac{\pi x}{n}. \quad (15)$$

The cost of the service connections  $F_s$  was written as

$$F_s = 2n \int_0^L k_s n_s L_s D_s^{m_s} dx \quad (16)$$

where  $k_s$  and  $m_s$  = ferrule cost parameters. Using Eqs. (14) and (15), Eq. (16) was changed to

$$F_s = \frac{2\pi^2 k_s D_s^{m_s} \sigma L^3}{3nq_s}. \quad (17)$$

### Cost per unit discharge of the system

Adding Eqs. (13) and (17) and the establishment cost  $E$ , the overall cost function  $F_o$  was

$$F_o = nk_m L \left( 1 + \frac{m}{5} \right) \left( \frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m} \right)^{m/(m+5)} + \frac{2\pi^2 k_s D_s^{m_s} \sigma L^3}{3nq_s} + E + \pi k_T \rho g \sigma L^2 (z_L + H - z_0). \quad (18)$$

Dividing Eq. (18) by the discharge pumped  $Q_T = \pi\sigma L^2$ , the system cost per unit discharge  $F$  was

$$F = \left( 1 + \frac{m}{5} \right) \frac{nk_m}{\pi\sigma L} \left( \frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m} \right)^{m/(m+5)} + \frac{2\pi k_s D_s^{m_s} L}{3nq_s} + \frac{E}{\pi\sigma L^2} + k_T \rho g (z_L + H - z_0). \quad (19)$$

### OPTIMIZATION

As the last term of Eq. (19) is constant, it will not enter into the optimization process. Dropping this term, the objective function reduces to  $F_1$  given by

$$F_1 = \left( 1 + \frac{m}{5} \right) \frac{nk_m}{\pi\sigma L} \left( \frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m} \right)^{m/(m+5)} + \frac{2\pi k_s D_s^{m_s} L}{3nq_s} + \frac{E}{\pi\sigma L^2}. \quad (20)$$

The variable  $n \geq 3$  is an integer. Considering  $n$  to be fixed, Eq. (20) is in the form of a posynomial (positive polynomial) in the design variable  $L$ . Thus, minimization of Eq. (20) reduces to a geometric programming with a single degree of difficulty (Duffin *et al.* 1967). The contributions of various terms of Eq. (20) are described by the weights  $w_1$ ,  $w_2$  and  $w_3$  given by

$$w_1 = \left( 1 + \frac{m}{5} \right) \frac{nk_m}{\pi\sigma L F_1} \left( \frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m} \right)^{m/(m+5)} \quad (21)$$

$$w_2 = \frac{2\pi k_s D_s^{m_s} L}{3nq_s F_1} \quad (22)$$

$$w_3 = \frac{E}{\pi\sigma L^2 F_1}. \quad (23)$$

The dual objective function  $F_2$  of Eq. (20) was written as

$$F_2 = \left[ \left( 1 + \frac{m}{5} \right) \frac{nk_m}{\pi\sigma L w_1} \left( \frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m} \right)^{m/(m+5)} \right]^{w_1} \times \left( \frac{2\pi k_s D_s^{m_s} L}{3nq_s w_2} \right)^{w_2} \left( \frac{E}{\pi\sigma L^2 w_3} \right)^{w_3} \quad (24)$$

The orthogonality condition for Eq. (24) was written as

$$\frac{5(m-1)}{m+5} w_1^* + w_2^* - 2w_3^* = 0 \quad (25)$$

whereas the normality condition of Eq. (24) is

$$w_1^* + w_2^* + w_3^* = 1. \quad (26)$$

Solving Eqs. (25) and (26) in terms of  $w_1^*$ , the following equations were obtained:

$$w_2^* = \frac{2}{3} - \frac{7m+5}{3(m+5)} w_1^* \quad (27)$$

$$w_3^* = \frac{1}{3} - \frac{2(5-2m)}{3(m+5)} w_1^*. \quad (28)$$

Substituting Eqs. (27) and (28) in Eq. (24) and using  $F_1^* = F_2^*$ , the optimal cost per unit discharge is

$$F_1^* = \frac{2\pi(m+5)k_s D_s^{m_s}}{[2(m+5) - (7m+5)w_1^*]nq_s} \times \left[ \frac{2(m+5) - (7m+5)w_1^*}{m+5 - 2(5-2m)w_1^*} \frac{3nq_s E}{2\pi^2 \sigma k_s D_s^{m_s}} \right]^{1/3} \times \left\{ \frac{nk_m}{15E} \left( \frac{64\pi k_T \rho f \sigma^3}{3mn^3 k_m} \right)^{m/(m+5)} \frac{m+5 - 2(5-2m)w_1^*}{w_1^*} \times \left[ \frac{2(m+5) - (7m+5)w_1^*}{m+5 - 2(5-2m)w_1^*} \frac{3nq_s E}{2\pi^2 \sigma k_s D_s^{m_s}} \right]^{(7m+5)/(3(m+5))} \right\}^{w_1^*} \quad (29)$$

Following Swamee (1995), Eq. (32) is optimal when the part containing the exponent  $w_1^*$  is unity. Thus, denoting the parameter  $P$  by

$$P = \frac{15E}{nk_m} \left( \frac{3mn^3 k_m}{64\pi k_T \rho f \sigma^3} \right)^{m/(m+5)} \left( \frac{2\pi^2 \sigma k_s D_s^{m_s}}{3nq_s E} \right)^{(7m+5)/(3(m+5))} \quad (30)$$

the optimality condition was written as

$$P = \frac{m+5 - 2(5-2m)w_1^*}{w_1^*} \times \left[ \frac{2(m+5) - (7m+5)w_1^*}{m+5 - 2(5-2m)w_1^*} \right]^{(7m+5)/(3(m+5))} \quad (31)$$

For various  $w_1^*$  corresponding values of  $P$  were obtained by Eq. (31). Using the data so obtained the following equation was fitted:

$$w_1^* = \frac{2(m+5)}{7m+5} \left[ 1 + \left( \frac{P}{0.5 + 7m} \right)^{1.15} \right]^{-0.8} \quad (32)$$

The maximum error involved in the use of Eq. (32) is about 1.5%. Using Eqs. (29) and (31), the optimal objective function was obtained as

$$F_1^* = \frac{2\pi(m+5)k_s D_s^{m_s}}{[2(m+5) - (7m+5)w_1^*]nq_s} \times \left[ \frac{2(m+5) - (7m+5)w_1^*}{m+5 - 2(5-2m)w_1^*} \frac{3nq_s E}{2\pi^2 \sigma k_s D_s^{m_s}} \right]^{1/3} \quad (33)$$

where  $w_1^*$  is given by Eq. (32). Combining Eqs. (22), (27) and (33) the optimal zone size  $L^*$  was

$$L^* = \left[ \frac{2(m+5) - (7m+5)w_1^*}{m+5 - 2(5-2m)w_1^*} \frac{3nq_s E}{2\pi^2 \sigma k_s D_s^{m_s}} \right]^{1/3} \quad (34)$$

Equation (34) reveals that the size  $L^*$  is a decreasing function of  $\sigma$  (which is proportional to the population density). Thus a larger population density will result in a smaller zone size.

## STRIP ZONE

Equations (14) and (15) are not applicable for  $n = 2$  and 1, as for both these cases the water supply zone degenerates to a strip. Following Figures 1(b, c) the pipe discharge was given by

$$Q = 2\sigma BL(1 - \xi) \quad (35)$$

where  $B$  = half the zone width and  $L$  = length of the zone for  $n = 1$  and half the zone length for  $n = 2$ . Using the

Darcy–Weisbach equation the pumping head was given by

$$h_0 = \frac{32f\sigma^2 B^2 L^3}{3\pi^2 g D^5} + z_L + H - z_0. \quad (36)$$

For  $n = 2$  the pumping discharge  $Q_T = 4BL\sigma$ . Thus the pumping cost  $F_p$  is

$$F_p = 4k_T \rho g \sigma B L h_0. \quad (37)$$

Combining Eqs. (36) and (37), the following equation was obtained:

$$F_p = \frac{128k_T \rho f \sigma^3 B^3 L^4}{3\pi^2 n^2 D^5} + 4k_T \rho g \sigma B L (z_L + H - z_0). \quad (38)$$

The pipe cost function  $F_m$  of the two diametrically opposite radial pipelines is

$$F_m = 2k_m L D^m. \quad (39)$$

Summing up Eqs. (38) and (39) the distribution system cost  $F_d$  was given by

$$F_d = 2k_m L D^m + \frac{128k_T \rho f \sigma^3 B^3 L^4}{3\pi^2 n^2 D^5} + 4k_T \rho g \sigma B L (z_L + H - z_0). \quad (40)$$

Differentiating Eq. (40) with respect to  $D$ , equating it to zero and simplifying:

$$D = \left( \frac{320k_T \rho f \sigma^3 B^3 L^3}{3\pi^2 m k_m} \right)^{1/(m+5)}. \quad (41)$$

Combining Eqs. (36), (37) and (41), the pumping cost was

$$F_p = \frac{128k_T \rho f \sigma^3 B^3 L^4}{3\pi^2} \left( \frac{3\pi^2 m k_m}{320k_T \rho f \sigma^3 B^3 L^3} \right)^{5/(m+5)} + 4k_T \rho g \sigma B L (z_L + H - z_0). \quad (42)$$

Using Eqs. (39) and (41) the pipe cost was obtained as

$$F_m = 2k_m L \left( \frac{320k_T \rho f \sigma^3 B^3 L^3}{3\pi^2 m k_m} \right)^{m/(m+5)}. \quad (43)$$

Adding Eqs. (42) and (43), the cost of the distribution system is

$$F_d = 2k_m L \left( 1 + \frac{m}{5} \right) \left( \frac{320k_T \rho f \sigma^3 B^3 L^3}{3\pi^2 m k_m} \right)^{m/(m+5)} + 4k_T \rho g \sigma B L (z_L + H - z_0). \quad (44)$$

The number of ferrule connections  $N_s = 4BL\sigma/q_s$ . Thus, the cost of the service connection is

$$F_s = \frac{4\sigma B^2 L k_s D_s^{m_s}}{q_s}. \quad (45)$$

Adding Eqs. (44), (45) and  $E$ , the overall cost function was obtained as

$$F_d = 2k_m L \left( 1 + \frac{m}{5} \right) \left( \frac{320k_T \rho f \sigma^3 B^3 L^3}{3\pi^2 m k_m} \right)^{m/(m+5)} + \frac{4\sigma B^2 L k_s D_s^{m_s}}{q_s} + E + 4k_T \rho g \sigma B L (z_L + H - z_0). \quad (46)$$

Dividing Eq. (46) by  $4\sigma B L$ , the system cost per unit discharge  $F$  was obtained as

$$F = \frac{k_m}{2\sigma B} \left( 1 + \frac{m}{5} \right) \left( \frac{320k_T \rho f \sigma^3 B^3 L^3}{3\pi^2 m k_m} \right)^{m/(m+5)} + \frac{k_s D_s^{m_s} B}{q_s} + \frac{E}{4\sigma B L} + k_T \rho g (z_L + H - z_0). \quad (47)$$

Following the procedure described for  $n = 2$ , it was found that for  $n = 1$  Eq. (41) remained unchanged while Eqs. (46) and (47) respectively change to

$$F_d = k_m L \left( 1 + \frac{m}{5} \right) \left( \frac{320k_T \rho f \sigma^3 B^3 L^3}{3\pi^2 m k_m} \right)^{m/(m+5)} + \frac{2\sigma B^2 L k_s D_s^{m_s}}{q_s} + E + 2k_T \rho g \sigma B L (z_L + H - z_0) \quad (48)$$

$$F = \frac{k_m}{2\sigma B} \left( 1 + \frac{m}{5} \right) \left( \frac{320k_T \rho f \sigma^3 B^3 L^3}{3\pi^2 m k_m} \right)^{m/(m+5)} + \frac{k_s D_s^{m_s} B}{q_s} + \frac{E}{2\sigma B L} + k_T \rho g (z_L + H - z_0). \quad (49)$$

Thus for  $n \leq 2$ , Eqs. (47) and (49) were generalized as

$$F = \frac{k_m}{2\sigma B} \left( 1 + \frac{m}{5} \right) \left( \frac{320k_T \rho f \sigma^3 B^3 L^3}{3\pi^2 m k_m} \right)^{m/(m+5)} + \frac{k_s D_s^{m_s} B}{q_s} + \frac{E}{2n\sigma\sigma B} + k_T \rho g (z_L + H - z_0). \quad (50)$$

The last term of Eq. (50) is constant. Dropping this term, Eq. (50) reduces to

$$F_1 = \frac{k_m}{2\sigma B} \left(1 + \frac{m}{5}\right) \left(\frac{320k_T \rho f \sigma^5 B^3 L^3}{3\pi^2 m k_m}\right)^{m/(m+5)} + \frac{k_s D_s^{m_s} B}{q_s} + \frac{E}{2n\sigma\sigma B} \quad (51)$$

Considering  $B$  and  $L$  as design variables, the minimization of Eq. (51) boils down to a geometric programming with zero degree of difficulty (Wilde & Beightler 1967). The weights  $w_1$ ,  $w_2$  and  $w_3$  pertaining to Eq. (51) were given by

$$w_1 = \frac{k_m}{2\sigma B F_1} \left(1 + \frac{m}{5}\right) \left(\frac{320k_T \rho f \sigma^5 B^3 L^3}{3\pi^2 m k_m}\right)^{m/(m+5)} \quad (52)$$

$$w_2 = \frac{k_s D_s^{m_s} B}{q_s F_1} \quad (53)$$

$$w_3 = \frac{E}{2n\sigma B L F_1} \quad (54)$$

The dual objective function  $F_2$  of Eq. (51) was written as

$$F_2 = \left[ \frac{k_m}{2\sigma B w_1} \left(1 + \frac{m}{5}\right) \left(\frac{320k_T \rho f \sigma^5 B^3 L^3}{3\pi^2 m k_m}\right)^{m/(m+5)} \right]^{w_1} \times \left(\frac{k_s D_s^{m_s} B}{q_s w_2}\right)^{w_2} \left(\frac{E}{2n\sigma B L w_3}\right)^{w_3} \quad (55)$$

The orthogonality conditions for Eq. (55) were written as

$$B: -\frac{5-2m}{m+5} w_1^* + w_2^* - w_3^* = 0 \quad (56)$$

$$L: \frac{3m}{m+5} w_1^* - w_3^* = 0. \quad (57)$$

On the other hand, the normality condition for Eq. (55) is  $w_1^* + w_2^* + w_3^* = 1$ . (58)

Solving (56)–(58), the following optimal weights were obtained:

$$w_1^* = \frac{m+5}{5(m+2)} \quad (59)$$

$$w_2^* = \frac{m+5}{5(m+2)} \quad (60)$$

$$w_3^* = \frac{3m}{5(m+2)}. \quad (61)$$

Equations (59) and (60) indicate that, in a strip zone, the optimal contribution of a water distribution network and

service connections are equal. Thus, for  $m = 1$  the optimal weights are in the proportion of 2:2:1. With the increase in  $m$  the optimal weights even out. Thus, for maximum  $m = 1.75$  the proportion of weights becomes 1.286:1.286:1. Further, a similar procedure gives the following equation for the optimal objective function for a strip zone:

$$F_1^* = (m+2)k_m \left[ \frac{5k_s D_s^{m_s}}{2(m+5)k_m \sigma q_s} \right]^{(m+5)/(5(m+2))} \times \left( \frac{5000k_T \rho f E^3}{81\pi^2 n^3 m^4 k_m^4} \right)^{m/[5(m+2)]} \quad (62)$$

Using (62) for  $n = 1$  and 2, the ratio of optimal objective functions was obtained as

$$\frac{F_{1,n=1}^*}{F_{1,n=2}^*} = 2^{2m/[5(m+2)]} \quad (63)$$

Thus, for the practical range  $1 \leq m \leq 1.75$ , it is 15–21% costlier to locate the input point at the end of a strip zone. Using Eqs. (53), (60) and (62), the optimum strip width  $B^*$  was found to be

$$B^* = \frac{(m+5)q_s k_m}{5k_s D_s^{m_s}} \left[ \frac{5k_s D_s^{m_s}}{2(m+5)k_m \sigma q_s} \right]^{(m+5)/(5(m+2))} \times \left( \frac{5000k_T \rho f E^3}{81\pi^2 n^3 m^4 k_m^4} \right)^{m/[5(m+2)]} \quad (64)$$

According to Eq. (64) for  $n = 1$  and 2, the optimal strip width ratio is same as the cost ratio. Thus the optimal strip width is 15–21% larger if the input point is at one end of the strip. Similarly, using Eqs. (54), (61), (62) and (64) the optimum length  $L^*$  was obtained as

$$L^* = \frac{25k_s D_s^{m_s} E}{6mn(m+5)k_m^2 \sigma q_s} \left[ \frac{2(m+5)k_m \sigma q_s}{5k_s D_s^{m_s}} \right]^{[2(m+5)]/[5(m+2)]} \times \left( \frac{81\pi^2 n^3 m^4 k_m^4}{5000k_T \rho f E^3} \right)^{2m/[5(m+2)]} \quad (65)$$

Thus, using Eq. (65) for  $n = 1$  and 2 gives the ratio of optimal strip lengths as

$$\frac{L_1^*}{L_2^*} = 2^{(110-m)/[5(m+2)]} \quad (66)$$

For the practical range  $1 \leq m \leq 1.75$ , the zone length is 23–36% longer if the input point is located at the end of the strip zone. Equations (64) and (66) reveal that both  $B^*$  and  $L^*$  are inverse functions of  $\sigma$ . On the other hand, the use of smoother pipes will reduce the zone width and increase its length.

## DESIGN EXAMPLE

The objective of the design example is to illustrate the use of equations presented in this paper. It is intended to find the optimal zone size for the following data:  $m = 1.62$ ,  $k_T/k_m = 0.06$ ,  $k_s/k_m = 3.0$ ,  $E/k_m = 7500$  (ratios in SI units),  $\sigma = 10^{-7}$  m/s,  $q_s = 0.001$  m<sup>3</sup>/s,  $D_s = 0.025$  m,  $m_s = 1.0$  and  $f = 0.02$ .

### Design steps

Considering a strip zone and using Eqs. (59)–(61) the optimal weights are  $w_1^* = w_2^* = 0.3657$ ; and  $w_3^* = 0.2685$ . Adopting  $n = 1$  for the input point at one end, and using Eq. (62),  $F_1^* = 49416k_m$ . Using Eqs. (53) and (60),  $B^* = 241$  m. Furthermore, using Eqs. (54) and (61),  $L^* = 11727$  m covering an area  $A^*$  of 5.65 km<sup>2</sup>. Similarly, adopting  $n = 2$  for a centrally placed input point, the design variables are  $B^* = 200$  m,  $L^* = 8508$  m and  $A^* = 6.81$  km<sup>2</sup>, yielding  $F_1^* = 41024k_m$ .

On the other hand, considering a circular zone with  $n = 3$  and using Eq. (31)  $P = 378.759$ . Furthermore, using Eq. (32),  $w_1^* = 0.0329$ ; using Eqs. (27) and (28),  $w_2^* = 0.6396$  and  $w_3^* = 0.3275$ . Using Eq. (33),  $F_1^* = 78760k_m$ ; and using Eq. (34)  $L^* = 962$  m. The corresponding area  $A^* = 2.908$  km<sup>2</sup>. Similar calculations for  $n > 3$  can be made. The calculations for different  $n$  values are depicted in Table 1, providing a guideline to the designer.

A perusal of Table 1 shows that for rectangular geometry with  $n = 1$  and 2 in the share of the total cost, the contribution of the main pipes is about one-third ( $w_1^* = 0.3657$ ). On the other hand, for circular geometry with  $n = 3$ , the contribution of radial pipes to the total cost is insignificant ( $w_1^* = 0.0369$ ) and this ratio increases slowly with the number of radial lines. Thus from the consumer's point of view the rectangular zone is superior as they have to bear about one-third of the total cost ( $w_2^* = 0.3657$ ) in

comparison to the case of the radial zone, in which their share increases to about two-thirds. Thus, for the circular zone, the major part of the cost is shared by the service connections. If this cost has to be passed to consumers then the problem reduces considerably. Dropping the service connection cost, for a circular zone, Eq. (20) reduces to

$$F_1 = \left(1 + \frac{m}{5}\right) \frac{nk_m}{\pi\sigma L} \left(\frac{64\pi k_T \rho f \sigma^3 L^6}{3mn^3 k_m}\right)^{m/(m+5)} + \frac{E}{\pi\sigma L^2}. \quad (67)$$

Equation (67) is a problem of zero degree of difficulty, yielding the following optimal weights:

$$w_1^* = \frac{2(m+5)}{7m+5} \quad (68)$$

$$w_3^* = \frac{5(m-1)}{7m+5}. \quad (69)$$

In the present problem for  $m = 1.62$ , the optimal weight  $w_3^* = 0.1897$ . That is, the share of the establishment cost (in the optimal zone cost per cumec) is about 19%. The corresponding optimal cost and zone size, respectively, are

$$F_1^* = \frac{(7m+5)nk_m}{10\pi\sigma} \left(\frac{64\pi k_T \rho f \sigma^3}{3mn^3 k_m}\right)^{2m/(7m+5)} \times \left[\frac{2E}{(m-1)nk_m}\right]^{[5(m-1)]/(7m+5)} \quad (70)$$

$$L^* = \left(\frac{3mn^3 k_m}{64\pi k_T \rho f \sigma^3}\right)^{m/(7m+5)} \left[\frac{2E}{(m-1)nk_m}\right]^{(m+5)/(7m+5)}. \quad (71)$$

By substituting  $m = 1$  in Eq. (70) a thumb rule for the optimal cost per cumec is obtained as

$$F_1^* = 1.2k_m \left(\frac{64k_T \rho f n^3}{3\pi^5 k_m \sigma^3}\right)^{1/6}. \quad (72)$$

In the foregoing developments the friction factor  $f$  has been considered as constant. The variation of the friction factor can be considered iteratively by first designing the system with constant  $f$  and revising it by using the following equation valid in the laminar flow, turbulent flow and the transition in between them (Swamee 1993):

$$f = \left\{ \left(\frac{64}{R}\right)^8 + 9.5 \left[ \ln\left(\frac{\varepsilon}{3.7D} + \frac{5.74}{R^{0.9}}\right) - \left(\frac{2500}{R}\right)^6 \right]^{-16} \right\}^{0.125} \quad (73)$$

**Table 1** | Design details

<i>n</i>	$w_1$	$w_2$	$w_3$	$F_1^*$ (S\$/m <sup>3</sup> )	<i>L</i> * (m)	<i>B</i> * (m)	<i>A</i> * (km <sup>2</sup> )
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	0.3657	0.3657	0.2685	49 416 $k_m$	11 727	241	5.652
2	0.3657	0.3657	0.2685	41 024 $k_m$	8 508	200	6.809
3	0.0369	0.6363	0.3268	78 821 $k_m$	938		2.883
4	0.0489	0.6265	0.3247	65 815 $k_m$	1 050		3.464
5	0.0606	0.6263	0.3226	57 370 $k_m$	1 126		3.986
6	0.0721	0.6073	0.3205	51 387 $k_m$	1 192		4.464
7	0.0835	0.5980	0.3185	46 898 $k_m$	1 250		4.907
8	0.0946	0.5888	0.3166	43 331 $k_m$	1 301		5.319

where  $\varepsilon$  = average height of roughness projections and  $R$  = Reynolds' number given by

$$R = \frac{4Q}{\pi\nu D} \quad (74)$$

where  $Q$  = discharge flowing in the pipe,  $D$  = pipe diameter and  $\nu$  = kinematic viscosity of water.

### SALIENT POINTS

In the case of a circular zone, **Table 1** shows that the zone area  $A$  gradually increases with the number of branches. However, the area remains less than that of a strip zone. Thus, a judicious value of  $A$  can be selected and the input points in the water distribution network area can be placed at its center. The locations of the input points are similar to optimal well-field configurations (Swamee *et al.* 1999). Keeping the input points as the center and consistent with the pipe network geometry the zones can be demarcated approximately as circles of diameter  $2L$ . These zones can be designed as independent entities and nominal connections provided for inter-zonal water transfer.

### CONCLUSIONS

The optimal area of a water supply zone has been obtained that minimizes the cost of the water supply system per unit discharge. By dividing the entire urban area into several

zones the water supply system can be designed as several independent entities having nominal linkage for inter-zonal water transfer. Such a system is reliable as failures are always partial failures. It has been possible to reduce the optimal zone size problem to geometric programming problems having zero and a single degree of difficulty. The solution of the problem yielded closed form equations for the design variables. By passing the service connection cost to consumers it has been found that the optimal establishment cost is 19% of the total cost per unit discharge.

### NOMENCLATURE

$A$	zone area
$B$	width
$D$	pipe diameter
$D_s$	ferrule diameter
$F$	cost function
$F_1$	objective function (primal)
$F_2$	objective function (dual)
$F_A$	annual averaging factor
$F_D$	daily averaging factor
$f$	friction factor
$g$	gravitational acceleration
$H$	terminal head
$h_0$	pumping head
$k_m$	pipe cost parameter
$k_s$	ferrule cost parameter
$k_p$	pump cost parameter
$k_T$	pumping cost parameter
$L$	length, zone radius
$M$	pipe cost parameter
$m_s$	ferrule cost parameter
$n$	number of radials
$Q$	discharge
$Q_T$	discharge pumped
$R_E$	cost of energy per kWh
$r$	interest rate expressed as proportion per year
$s_b$	standby fraction
$x$	radial distance from input point
$z_0$	elevation of entrance point
$z_L$	elevation of peripheral point
$\varepsilon$	average height of roughness projections
$\eta$	combined efficiency



$\nu$	kinematic viscosity of water
$\rho$	mass density of fluid
$\sigma$	peak water demand per unit area
$\xi$	nondimensional distance

### Superscript

- optimal

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### APPENDIX. MATHEMATICAL TERMS

- **Degree of difficulty** For a posynomial having the number of terms equal to the number of design variables

plus one, a closed form optimal solution in terms of the exponents of the design variables is obtained. If the number of terms is increased, such a closed form solution is not possible. This increased number of terms is called the degree of difficulty.

- **Design variables** These are the variables that are determined in a design process.
- **Dual objective function** It is a function having the same maximum (minimum) as that of the minimum (maximum) of the primal objective function.
- **Normality condition** The summation of the weights is equal to unity.
- **Orthogonality conditions** These conditions at optimality of the dual objective function state that the powers of the design variables should equate to zero.
- **Posynomial** It is a polynomial in design variables having real exponents and positive coefficients.
- **Primal objective function** It is a function expressing the requirements of an engineering system like cost, efficiency, power consumption, etc. When this function is optimized (minimized or maximized), it yields the best engineering design.
- **Single degree of difficulty** The single degree of difficulty arises when the number of terms in a posynomial is equal to the number of design variables plus two.
- **Zero degree of difficulty** The zero degree of difficulty arises when the number of terms in a posynomial is equal to the number of design variables plus one.

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