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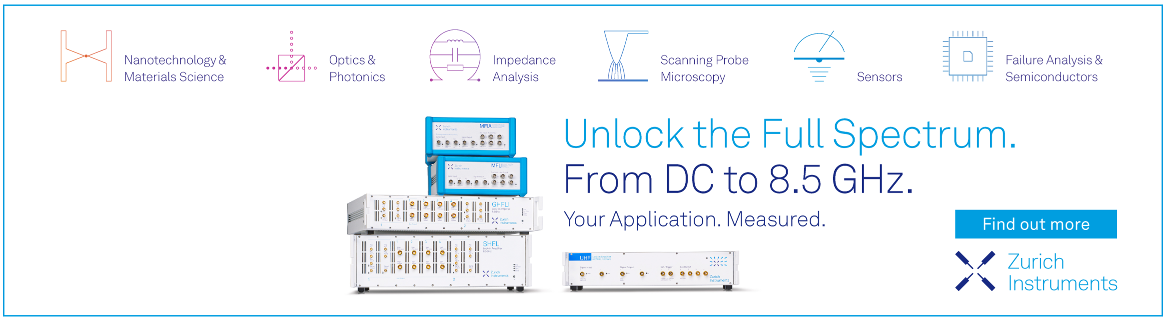

Cosmological singularity

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Abstract. The talk represents a review of the old results and contemporary development on the problem of cosmological singularity.

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PREFACE

This problem was born in 1922 when A.Friedmann wrote his famous cosmological solution for the homogeneous isotropic Universe. However, during the next 35 years researches devoted their attention mainly to the physical processes after the Big Bang and there was no serious attempts to put under a rigorous analysis the phenomenon of the cosmological singularity as such. The person who inspired the beginning of such analysis was L.D.Landau. In the late 1950s he formulated the crucial question whether the cosmological singularity is a general phenomenon of General Relativity or it appears only in particular solutions under the special symmetry conditions. The large amount of work have been done in Landau school before an answer emerges in 1969 in the form of the so-called “BKL conjecture” (the present-day terminology). The basic reviews[1] covering also the contemporary development are [1]-[14]. The BKL conjecture has its foundation in a collection of results and, first of all, it asserts that the general solution containing the cosmological singularity exists. This fundamental question of existence of such solution was the principal goal of our work, however, we succeeded also in describing the analytical structure of gravitational and matter fields in asymptotic vicinity to the singularity and we showed that in most general physical settings such solution has complicated oscillatory behaviour of chaotic character.

In order to avoid misunderstandings let’s stress that under cosmological singularity we mean the singularity in time, when singular manifold is space-like, and when the curvature invariants together with invariant characteristics of matter fields (like energy density) diverge on this manifold. An intuitive feeling that there are no reasons to doubt in existence of the general solution with cosmological singularity we have already in 1964 but another five years passed before the concrete structure have been discovered. In 1965 appeared the important theorem of Roger Penrose [15], saying that under some conditions the appearance of incomplete geodesics in space-time is unavoidable. This is also singularity but of different type since, in general, incompleteness does not means that invariants diverge. In addition the theorem can say nothing about the analytical structure of the fields near the points where geodesics terminate. Then Penrose’s result was not of a direct help for us, nevertheless it stimulated our search. Today it is reasonable to consider that the BKL conjecture and Penrose theorem represent two sides of the phenomenon but the links are still far to be understandable. This is because BKL approach deal with asymptotic in the vicinity to the singularity and Penrose theorem has to do with global space-time.

It is worth to stress that some misleading statements are to be found in the literature in relation to the aforementioned results. First of all, from the BKL theory as well as from Penrose theorem not yet follows that cosmological singularity is inevitable in General Relativity. BKL showed that the general solution containing such singularity exists but general in the sense that initial data under which the cosmological singularity is bound to appear represent a set of nonzero measure in the space of all possible data. However, we don’t know “how big” this measure is and we have no proof that this set can cover

[1] Some of these articles should not be considered merely as reviews since they contain also the original results. This is relevant especially for the papers [3], [8] and [13]. For example, the first construction of the general non oscillatory solution with power asymptotic near singularity for the case of perfect liquid with stiff matter equation of state and investigation of the general oscillatory regime in presence of the SU(2) Yang-Mills field can be found only in [3]. The Iwasawa decomposition for the metric tensor was first introduced namely in [8]. Many original details of the dynamical system approach to cosmological evolution can be found in [13]. The papers [13] and [14] we mentioned for completeness in order to remind to a reader on the existence of an alternative approach to the analysis of the character of cosmological singularity based on the representation of the Einstein equations in the form of dynamical system and the serch for the description of its attractor in vicinity to the singularity. This is powerful method which can be considered as dual to the "cosmological billird approch" to which the present talk is mainly dedicated. The restricted time for the talk prevented to include a discussion also on this important aspect of the theory.

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the totality of initial data. In the non-linear system can be many general solutions (that is, each containing maximal number of arbitrary functional parameters) of different types including also a general solution without singularity. Moreover there is the proof [16] of the global stability of Minkowski spacetime which means that at least in some small (but finite) neighbourhood around it exists a general solution without any singularity at any time. The same is true in relation to the all versions of Penrose theorem: for these theorems to be applicable the nontrivial initial/boundary conditions are strictly essential to be satisfied, but an infinity of solutions can exists which do not meet such conditions. The thorough investigation of applicability of Penrose theorem as well as all its subsequent variations the reader can find in [17, 18].

The second delusion is that the general solution with singularity can be equally applied both to the singularity in future (Big Crunch) and to the singularity in past (Big Bang) ignoring the fact that these two situations are quite different physically. To describe what are going near cosmological Big Crunch (as well as near the final stage of gravitational collapse of an isolated object in its co-moving system) one really need the general solution since in the course of evolution inescapably will arise the arbitrary perturbations and these will reorganize any regime into the general one. The Big Bang is not the same phenomenon. We don't know initial conditions at the instant of singularity in principle and there are no reasons to expect that they should be taken to be arbitrary. For example, we can not ruled out the possibility that the Universe started exactly with the aid of the Friedmann solution and it may be true that this does not means any fine tuning from the point of view of the still unknown physics near such exotic state. Of course, the arbitrary perturbations familiar from the present day physics will appear after Big Bang but this is another story. The conclusion is that if somebody found the general cosmological solution this not yet means that he knows how Universe really started, however he has grounds to think that he knows at least something about its end.

Sometimes one can find in literature the statement that in the BKL approach only the time derivatives are important near singularity and because of this the asymptotic form of Einstein equations became the ordinary differential equations with respect to time. Such statement is a little bit misleading since space-like gradients play the crucial role in appearing the oscillatory regime. One of the main technical advantage of the BKL approach consists in identification among the huge number of the space gradients those terms which are of the same importance as time derivatives. In the vicinity to the singularity these terms in no way are negligible, they act during the whole course of evolution (although from time to time and during comparatively short periods) and namely due to them oscillations arise. The subtle point here is that asymptotically these terms can be represented as products of effective scale coefficients, governing the time evolution of the metric, and some factors containing space-like derivatives. This nontrivial separation springing up in the vicinity to the singular point produce gravitational equation of motion which effectively are the ordinary differential equations in time because all factors containing space-like derivatives enter these equations solely as external parameters, though dynamically influent parameters. Owing to these ordinary differential equations the asymptotic evolution can be described as motion of a particle in some external potential. The aforementioned dominating space gradients create the reflecting potential walls responsible for the oscillatory regime. For the case of homogeneous cosmological model of the Bianchi IX type such potential have been described by Misner [19]. The literal assertion that “only the time derivatives are important near singularity” is correct just for those cases when the general solution is of non oscillatory character and has simple power asymptotic near singularity as, for instance, for the cases of perfect liquid with stiff-matter equation of state [3, 20, 21], pure gravity in space-time of dimension more than ten [22], or some other classes of "subcritical" field models [23].

BASIC STRUCTURE OF COSMOLOGICAL SINGULARITY

The character of the general cosmological solution in the vicinity to the singularity can most conveniently be described in the synchronous reference system, where the interval is of the form

$$-ds^2 = -dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta \qquad (1)$$

We use a system of units where the Einstein gravitational constant and the velocity of light are equal to unity. The Greek indices refer to three-dimensional space and assume the values 1,2,3. Latin indices i,k will refer to four-dimensional space-time and will take the values 0,1,2,3. The coordinates are designated as $(x^0, x^1, x^2, x^3) = (t, x, y, z)$.

The Einstein equations in this reference system take the form

$$R_0^0 = \frac{1}{2}\dot{\kappa} + \frac{1}{4}\kappa_\beta^\alpha \kappa_\alpha^\beta = T_{\ 0}^{0} - \frac{1}{2}T\ , \qquad (2)$$

$$R_\alpha^0 = \frac{1}{2}(\kappa_{,\alpha} - \kappa_{\alpha;\beta}^\beta) = T_{\ \alpha}^{0}\ , \qquad (3)$$

$$R_\alpha^\beta = \frac{1}{2\sqrt{g}}(\sqrt{g}\kappa_\alpha^\beta)^{\dot{}} + P_\alpha^\beta = T_{\ \alpha}^{\beta} - \frac{1}{2}\delta_\alpha^\beta T\ , \qquad (4)$$

where the dot signifies differentiation with respect to time t and

$$\kappa_{\alpha\beta} = \dot{g}_{\alpha\beta} \ , \quad g = \det g_{\alpha\beta} \ . \tag{5}$$

The tensorial operations on the Greek indices, as well as covariant differentiation in this system are performed with respect to the three-dimensional metric $g_{\alpha\beta}$. The quantity κ is a three-dimensional contraction:

$$\kappa = \kappa_\alpha^\alpha = (\ln g)\dot{} \ . \tag{6}$$

P_α^β is a three-dimensional Ricci tensor, expressed in terms of $g_{\alpha\beta}$ in the same way as R_i^k is expressed in terms of g_{ik}. The quantities $T_0^{\ 0}, T_\alpha^{\ 0}$ and $T_\alpha^{\ \beta}$ are components of the energy-momentum tensor $T_i^{\ k}$ four-dimensional contraction of which is designated by T .

$$T = T_k^k = T_0^{\ 0} + T_\alpha^{\ \alpha} \ . \tag{7}$$

It turn out that the general cosmological solution of Eqs. (2)-(4) in the asymptotic vicinity of a singularity with respect to time is of an oscillatory nature and may be described by an infinite alternation of the so-called Kasner epochs. The notions of a Kasner epoch and of the succession of two of these epochs are the key elements in the dynamics of the oscillatory regime. It is most convenient to study their properties in the example of empty space, when $T_i^{\ k} = 0$ and then take into account all the changes that may be observed in the presence of matter. This procedure is reasonable since, in general, the influence of matter upon the solution in the vicinity of the singularity appears to be either negligible or can be put under the control.

So let's assume that the tensor $T_i^{\ k}$ in Eqs.(2)-(4) equals zero. A Kasner epoch is a time interval during which in Eq. (4) the three-dimensional Ricci tensor P_α^β may be neglected in comparison with the terms involving time differentiation. Then from (2) and (4) we obtain the following equations in this approximation:

$$(\sqrt{g}\kappa_\alpha^\beta)\dot{} = 0 \ , \qquad \dot{\kappa} + \frac{1}{2}\kappa_\beta^\alpha \kappa_\alpha^\beta = 0 \ . \tag{8}$$

Here and elsewhere we shall assert that the singularity corresponds to the instant $t = 0$ and we shall follow the evolution of the solution towards the singularity, i.e., the variation of time as it decreases from certain values $t > 0$ down to $t = 0$. Eq. (3) in the general case is of no interest for the dynamics of the solution, since its role is reduced to the establishment of certain additional relations on arbitrary three-dimensional functions resulting from the integration of the Eqs.(2),(4) (that is of supplementary conditions for initial data).

The general solution of Eqs. (8) may be written down in the form

$$g_{\alpha\beta} = \eta_{AB} l_\alpha^A l_\beta^B, \quad \eta_{AB} = diag(t^{2p_1}, t^{2p_2}, t^{2p_3}) \tag{9}$$

where by the big Latin letters A, B, C we designate the three-dimensional frame indices (they take the values 1,2,3). The exponents p_1, p_2, p_3 and vectors l_α^A are arbitrary functions of the three-dimensional coordinates x^α. We call the directions along l_α^A as Kasner axis, the triad l_α^A represents the common eigenvectors both for metric $g_{\alpha\beta}$ and second form $\kappa_{\alpha\beta}$. The exponents p_1, p_2, p_3 satisfy two relations:

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1. \tag{10}$$

It ensues from these relations that one of the exponents p_A is always negative while the two others are positive. Consequently, the space expands in one direction and contracts in two others. Then the value of any three-dimensional volume element decreases since, according to (9)-(10), the determinant of $g_{\alpha\beta}$ decreases proportionally to t^2.

Of course, the solution (9)-(10) sooner or later will cease to be valid because the three-dimensional Ricci tensor P_α^β contain some terms which are growing with decreasing of time faster than the terms with time derivatives and our assumption that P_α^β can be neglected will become wrong. It is possible to identify these "dangerous" terms in P_α^β and include them into the new first approximation to the Einstein equations, instead of (8). The remarkable fact is that the asymptotic solution of this new approximate system can be described in full details and this description is valid and stable up to the singularity. The result is that the evolution to the singularity can be represented by a never-ending sequence of Kasner epochs and the singularity $t = 0$ is the point of its condensation. The durations of epochs tend to zero and transitions between them are very short comparatively to its durations. The determinant of the metric tensor $g_{\alpha\beta}$ tends to zero. On each Kasner epoch the solution take the form (9)-(10) but each time with new functional parameters $\acute{p}_A$ and $\acute{l}_\alpha^A$. On each epoch the exponents $\acute{p}_A$ satisfy the same relations (10), that is the space expands in one direction and contracts in two others, however, from epoch to epoch these directions are different, i.e. on each new epoch the Kasner axis rotate relatively to their arrangement at the preceding one.

The effect of rotation of Kasner axis make its use inconvenient for an analytical description of the asymptotic oscillatory regime because this rotation never stops. However, it turn out that another axis exist (they are not eigenvectors for the second form $\kappa_{\alpha\beta}$) , rotation of which are coming to stop in the limit $t \rightarrow 0$ and projection of the metric tensor into such "asymptotically frozen" (terminology of the authors of Ref. [8]) triad still is a diagonal matrix. The components of this matrix have no limit since their behaviour again can be described by the never-ending oscillations of a particle against some potential walls. This is an efficient way to reduce the description of asymptotic evolution of six components of the metric tensor to the three oscillating degrees of freedom. For the homogeneous model of Bianchi type IX this approach was developed in [24, 25] where the three-dimensional interval has been represented in the form $g_{\alpha\beta}dx^{\alpha}dx^{\beta} = (\tilde{R}\Gamma R)_{AB}(l^A_{\alpha}dx^{\alpha})(l^B_{\beta}dx^{\beta})$ with the standard Bianchi IX differential forms $l^A_{\alpha}dx^{\alpha}$ (where l^A_{α} depends only on x^{α} in that special way that $\partial_{\nu}l^C_{\mu} - \partial_{\mu}l^C_{\nu} = C^C_{AB}l^A_{\mu}l^B_{\nu}$ with only non-vanishing structural constants $C^1_{23} = C^2_{31} = C^3_{12} = 1$). The diagonal matrix Γ and three-dimensional *orthogonal* matrix R depend only on time (tilde means transposition). Remarkably, the gravitational equations for this model shows that near singularity all three Euler angles of matrix R tends to some arbitrary limiting constants and three components of Γ oscillate between the walls of potential of some special structure.

We never tried to generalize this approach (namely with orthogonal matrix R) to the inhomogeneous models but the recent development of the theory showed that even in most general inhomogeneous cases (including multidimensional spacetime filled by different kind of matter) there is analogous representation of the metric tensor leading to the same asymptotic freezing phenomenon of "non-diagonal" degrees of freedom and reducing the full dynamics to the few "diagonal" oscillating scale factors. This is so-called Iwasawa decomposition first used in [8] and thoroughly investigated in [26]. The difference is that in general inhomogeneous case instead of orthogonal matrix R it is more convenient to use an upper triangular matrix N (with components N^A_{α} where upper index A numerates the rows and lower index α corresponds to columns)

$$N = \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix} \qquad (11)$$

and to write three-dimensional interval in the form $g_{\alpha\beta}dx^{\alpha}dx^{\beta} = (\tilde{N}\Gamma N)_{\alpha\beta}dx^{\alpha}dx^{\beta})$. The diagonal matrix Γ as well as matrix N are functions of all four coordinates but near the singularity matrix N tends to some time-independent limit and components of Γ oscillate between the walls of some potential. This asymptotic oscillatory regime has the well defined Lagrangian. If one writes matrix Γ as $\Gamma = diag(e^{-2\beta^1}, e^{-2\beta^2}, e^{-2\beta^3})$ then the asymptotic equations of motion for the scale coefficients β^A became the ordinary differential equations in time (separately for each point x^{α} of three-dimensional space) which follow from the Lagrangian:

$$\begin{aligned} &L = G_{AB}\tfrac{d\beta^A}{d\tau}\tfrac{d\beta^B}{d\tau} - V(\beta^A), \\ &V(\beta^A) = C_1e^{-4\beta^1} + C_2e^{-2(\beta^2-\beta^1)} + C_3e^{-2(\beta^3-\beta^2)}\ . \end{aligned} \qquad (12)$$

Here we use the new time variable τ instead of original synchronous time t. In asymptotic vicinity to the singularity the link is $dt = -\sqrt{\det g_{\alpha\beta}}d\tau$ where differentials should be understood only with respect to time, considering the coordinates x^{α} in $\det g_{\alpha\beta}$ formally as fixed quantities. Since $\det g_{\alpha\beta}$ tends to zero approximately like t^2 it follows that singular limit $t \to 0$ corresponds to $\tau \to \infty$. The metric G_{AB} of three-dimensional space of scale coefficients β^A are defined by the relation $G_{AB}d\beta^A d\beta^B = \sum(d\beta^A)^2 - (\sum d\beta^A)^2$. This is flat Lorenzian metric with signature (-,+,+) which can be seen from transformation $\beta^1 = \acute{\beta}^1 + \acute{\beta}^2 + \acute{\beta}^3$, $\beta^2 = \acute{\beta}^1 - \acute{\beta}^2 + \acute{\beta}^3$, $\beta^3 = -\acute{\beta}^3$ after which one get $G_{AB}d\beta^A d\beta^B = -2(d\acute{\beta}^1)^2 + 2(d\acute{\beta}^2)^2 + 2(d\acute{\beta}^3)^2$. All coefficients $C_A(x^{\alpha})$ are time-independent and positive; with respect to the dynamics they play a role of external fixed parameters. Apart from the three differential equations of second order for β^A which follow from the Lagrangian (12) there is well known additional constraint

$$G_{AB}\frac{d\beta^A}{d\tau}\frac{d\beta^B}{d\tau} + V(\beta^A) = 0, \qquad (13)$$

which represents the $\binom{0}{0}$ component of the Einstein equations. In particular case of homogeneous model of Bianchi type IX equations (12) and (13) gives exactly the same system which was described in [24, 25], in spite of the fact that in these last papers the asymptotical freezing of "non-diagonal" metric components has been obtained using an orthogonal matrix R instead of Iwasawa's one N. Analysis of the eqs. (12)-(13) shows that in the limit $\tau \to \infty$ the exponents $\beta^A(\tau)$ are positive and all tend to infinity in such a way that the differences $\beta^2 - \beta^1$ and $\beta^3 - \beta^2$ also are positive and tend to infinity, that is each term in the potential $V(\beta^A)$ tends to zero. Then from (13) follows that each trajectory $\beta^A(\tau)$ becomes "time-like" with respect to the metric G_{AB}, i.e. near singularity we have $G_{AB}\frac{d\beta^A}{d\tau}\frac{d\beta^B}{d\tau} < 0$, though this is so only in the extreme vicinity to the potential walls $\beta^1 = 0$, $\beta^2 - \beta^1 = 0$ and $\beta^3 - \beta^2 = 0$. Between the walls where $\beta^1 > 0$, $\beta^2 - \beta^1 > 0$, $\beta^3 - \beta^2 > 0$ the potential is exponentially small and trajectories become "light-like", i.e. $G_{AB}\frac{d\beta^A}{d\tau}\frac{d\beta^B}{d\tau} = 0$. These periods of "light-like" motion between the walls corresponds exactly to the Kasner epochs (9)-(10) (with an appropriate identification of Kasner axis during each period). It is easy to see that the walls itself are "time-like" what means that collisions of a "particle moving in a light-like directions" against the walls are inescapable and interminable.

One of the crucial points discovered in [8] is that in the limit $\tau \to \infty$ the walls become infinitely sharp and of infinite height which simplify further the asymptotic picture and make transparent the reasons of chaoticity of such oscillatory dynamics. Because $G_{AB}\beta^A\beta^B = -2(\beta^1\beta^2 + \beta^1\beta^3 + \beta^2\beta^3)$ and near singularity all β^A are positive we have $G_{AB}\beta^A\beta^B < 0$. Then by the transformation $\beta^A = \rho\gamma^A$ one can introduce instead of β^A the "radial" coordinate $\rho > 0$ ($\rho \to \infty$ when $\tau \to \infty$) and "angular" coordinates γ^A subjected to the restriction $G_{AB}\gamma^A\gamma^B = -1$. The last condition pick out in γ-space the two-dimensional Lobachevsky surface of constant negative curvature and each trajectory $\beta^A(\tau)$ has the radially projected trace on this surface. The free Kasner flights in three-dimensional β-space between the walls are projected into geodesics of this two-dimensional surface. The walls are projected into three curves forming a triangle on the Lobachevsky surface and reflections against these curves are geometric (specular). If we introduce the new evolution parameter T by the relation $d\tau = \rho^2 dT$ then the new Lagrangian (with respect to the “time” T) will be:

$$
\begin{aligned}
&L_T = -\left(\frac{d\ln\rho}{dT}\right)^2 + G_{AB}\,\frac{d\gamma^A}{dT}\frac{d\gamma^B}{dT} - \rho^2 V(\rho\gamma^A),\\
&G_{AB}\gamma^A\gamma^B = -1.
\end{aligned}
\tag{14}
$$

In the limit $\rho = \infty$ the new potential $\rho^2 V(\rho\gamma^A)$ is exactly zero in the region between the walls where $\gamma^1 > 0,\ \gamma^2 - \gamma^1 > 0,\ \gamma^3 - \gamma^2 > 0$ and becomes infinitely large at the points $\gamma^1 = 0,\ \gamma^2 - \gamma^1 = 0,\ \gamma^3 - \gamma^2 = 0$ where the walls are located and behind them where quantities $\gamma^1,\ \gamma^2 - \gamma^1, \gamma^3 - \gamma^2$ are negative. This means that near singularity potential $\rho^2 V$ depends only on γ-variables and ρ can be considered as cyclic degree of freedom. In this way the asymptotic oscillatory regime can be viewed as the eternal motion of a particle inside a triangular bounded by the three stationary walls of infinite height in two-dimensional space of constant negative curvature. The important fact is that the area occupied by this triangle is finite. It is well known (see references in [8], section 5.2.2) that the geodesic motion under the conditions described is chaotic.

It is worth to mention that in case of homogeneous Bianchi IX model the fact that its dynamics is equivalent to a billiard on the Lobachevsky plane was established in [27].

The numerical calculations confirming the admissibility of the BKL conjecture can be found in [6] and [28, 29].

THE INFLUENCE OF MATTER

In papers [3, 20, 30] we studied the problem of the influence of various kinds of matter upon the behaviour of the general solution of the gravitational equations in the neighbourhood of a singular point. It is clear that, depending on the form of the energy-momentum tensor, we may meet three different possibilities: (i) the oscillatory regime remains as it is in vacuum, i.e. the influence of matter may be ignored in the first approximation; (ii) the presence of matter makes the existence of Kasner epochs near a singular point impossible; (iii) Kasner epochs exist as before, but matter strongly affects the process of their formation and alternation. Actually, all these possibilities may be realized.

There is a case in which the oscillatory regime observed as a singular point is approached remains the same, in the first approximation, as in vacuum. This case is realized in a space filled with a perfect liquid with the equation of state $p = k\varepsilon$ for $0 < k < 1$. No additional reflecting walls arise from the energy-momentum tensor in this case.

If $k = 1$ we have the "stiff matter" equation of state $p = \varepsilon$. This is the second of the above-mentioned possibilities when neither Kasner epoch nor oscillatory regime can exist in the vicinity of a singular point. This case has been investigated in [3, 20] where it has been shown that the influence of the "stiff matter" (equivalent to the massless scalar field) results in the violation of the Kasner relations (10) for the asymptotic exponents. Instead we have

$$
p_1 + p_2 + p_3 = 1, \qquad p_1^2 + p_2^2 + p_3^2 = 1 - p_\varphi^2 \tag{15}
$$

where p_φ^2 is an arbitrary three-dimensional function (with the restriction $p_\varphi^2 < 1$) to which the energy density ε of the matter is proportional (in that particular case when the stiff-matter source is realized as a massless scalar field φ its asymptotic is $\varphi = p_\varphi \ln t$ and this is the formal reason why we use the index φ for the additional exponent p_φ).

Thanks to (15), in contrast to the Kasner relations (10), it is possible for all three exponents p_A to be positive. In [20] it has been shown that, even if the contraction of space starts with the quasi-Kasner epoch (15) during which one of the exponents p_A is negative, the asymptotic behaviour (9) with all positive exponents is inevitably established after a finite number of oscillations and remains unchanged up to the singular point. Thus, for the equation of state $p = \varepsilon$ the collapse in the general solution is described by monotonic (but anisotropic) contraction of space along all directions. The asymptotic of the general solution near cosmological singularity for this case we constructed explicitly in [3], see also [20, 21, 23]. The disappearance of oscillations for the case of a massless scalar field should be consider as an isolated phenomenon which is unstable with respect to inclusion into the right hand side of the Einstein equations another kind of fields. For instance, in the same paper [20] we showed that if to the scalar field we add a vector one then the endless oscillations reappear.

The cosmological evolution in the presence of an electromagnetic field may serve as an example of the third possibility. In this case the oscillatory regime in the presence of matter is, as usual, described by the alternation of Kasner epochs, but in this process the energy-momentum tensor plays a role as important as the three-dimensional curvature tensor. This problem has been treated by us in [30], where it has been shown that in addition to the vacuum reflecting walls also the new walls arise caused by the energy-momentum tensor of the electromagnetic field. The electromagnetic type of alternation of epochs, however, qualitatively takes place according to the same laws as in vacuum.

In paper [3] we have also studied the problem of the influence of the Yang-Mills fields on the character of the cosmological singularity. For definiteness, we have restricted ourselves to fields corresponding to the gauge group SU(2). The study was performed in the synchronous reference system in the gauge when the time components of all three vector fields are equal to zero. It was shown that, in the neighbourhood of a cosmological singularity, the behaviour of the Yang-Mills fields is largely similar to the behaviour of the electromagnetic field: as before, there appears an oscillatory regime described by the alternation of Kasner epochs, which is caused either by the three-dimensional curvature or by the energy-momentum tensor. If, in the process of alternation of epochs, the energy-momentum tensor of the gauge fields is dominating, the qualitative behaviour of the solution during the epochs and in the transition region between them is like the behavior in the case of free Yang-Mills fields (with the Abelian group). This does not mean that non-linear terms of the interaction may be neglected completely, but the latter introduce only minor, unimportant quantitative changes into the picture we would observe in the case of non-interacting fields. The reason for this lies in the absence of time derivatives of the gauge field strengths in those terms of the equations of motion which describe the interaction.

MULTIDIMENSIONAL SPACETIME AND SUPERGRAVITY

The story resembling the aforementioned effect of dissappearance (for scalar field) and reconstruction (after adding a vector field) of oscillations occurred later in more general and quite different circumstances. In 1985 appeared very interesting and unexpected result [22] that oscillatory regime near cosmological singularity in multidimensional spacetime (for pure gravity) holds for spacetime dimension D up to $D=10$ but for dimension $D \geq 11$ the asymptotic of the general solution follow the smooth multidimensional Kasner power law. Up to now we have no idea why this separating border coincides with dimension so significant for superstring theories, most likely it is just an accident. However, the important point is that if we will add to the vacuum multidimensional gravity the fields of p-forms the presence of which is dictated by the low energy limit of superstring models, the oscillatory regime will reappear. This fact was established in [31, 32] and subsequently has been developed by T.Damour, M.Henneaux, H.Nicolai, B.Julia and their collaborates into the new interesting and promising branch of superstring theories. In articles [31, 32] it was demonstrated that bosonic sectors of supergravities emerging in the low energy limit from all types of superstring models have oscillatory cosmological singularity of the BKL character. Let consider the action of the following general form:

$$S = \int d^D x \sqrt{g} \left[R - \partial^i \varphi \partial_i \varphi - \right. \\ \left. - \frac{1}{2} \sum_p \frac{1}{(p+1)!} e^{\lambda_p \varphi} F^{(p+1)}_{i_1 \dots i_{p+1}} F^{(p+1) i_1 \dots i_{p+1}} \right] \tag{16}$$

where $F^{(p+1)}$ designates the the field strengths generated by the p-forms A_p, i.e. $F^{(p+1)}_{i_1 \dots i_{p+1}} = antisym(\partial_{i_1} A_{i_2 \dots i_{p+1}})$. The real parameters λ_p are coupling constants corresponding to the interaction between the dilaton and p-forms. The tensorial operations in (16) are carring out with respect to D-dimensional metric g_{ik} and $g = |\det g_{ik}|$. Now the small Latin indices refer to D-dimensional space-time and Greek indices (as well as big Latin frame indices A, B and C) correspond to d-dimensional space where $d = D-1$. Also in this theory the Kasner-like epochs exist which are of the form:

$$g_{ik} dx^i dx^k = -dt^2 + \eta_{AB}(t, x^\alpha) l^A_\mu (x^\alpha) l^B_\nu (x^\alpha) dx^\mu dx^\nu, \\ \eta_{AB} = diag[t^{2p_1(x^\alpha)}, t^{2p_2(x^\alpha)}, \dots, t^{2p_d(x^\alpha)}], \tag{17}$$

$$\varphi = p_\varphi (x^\alpha) \ln t + \varphi_0 (x^\alpha). \tag{18}$$

However, in the presence of the dilaton the exponents p_A instead of the Kasner law satisfy the relations analogous to (15):

$$\sum_{A=1}^{d} p_A = 1, \quad \sum_{A=1}^{d} p_A^2 = 1 - p_\varphi^2 \tag{19}$$

The approximate solution (17)-(19) follows from the D-dimensional Einstein equations by neglecting the energy-momentum tensor of p-forms, d-dimensional curvature tensor P^β_α and spatial derivatives of φ. Now one has to do the work analogous to that one for 4-dimensional gravity: it is necessary to identify in all neglected parts of the equations those "dangerous" terms which will destroy the solution (17)-(19) in the limit

$t \rightarrow 0$. Then one should construct the new first approximation to the equations taking into account also these “dangerous” terms and try to find asymptotic solution for this new system. This is the same method which have been used in case of the 4-dimensional gravity with electromagnetic field and it works well also here. Using the Iwasawa decomposition for d-dimensional frame metric $\eta_{AB} = (\tilde{N}\Gamma N)_{AB}$ where $\Gamma = diag(e^{-2\beta^1}, e^{-2\beta^2}, ..., e^{-2\beta^d})$ it can be shown [8] that near singularity again the phenomenon of freezing of "non-diagonal" degrees of freedom of the metric tensor arise and the foregoing new approximate system reduces to the ordinary differential equations (for each spatial point) for the variables $\beta^1, ..., \beta^d$ and β^{d+1} where $\beta^{d+1} = -\varphi$. It is convenient to use the $d+1$-dimensional flat superspace with coordinates $\beta^1, ..., \beta^{d+1}$ and correspondingly new indices $\bar{A}, \bar{B}$ running over values $1, ..., d+1$. The metric $G_{\bar{A}\bar{B}}$ in this superspace is

$$G_{\bar{A}\bar{B}} d\beta^{\bar{A}} d\beta^{\bar{B}} = \sum_{A=1}^{d} (d\beta^A)^2 - (\sum_{A=1}^{d} d\beta^A)^2 + \left(d\beta^{d+1}\right)^2. \tag{20}$$

The asymptotic dynamics for β-variables follows from the Lagrangian of the form similar to (14):

$$\begin{aligned} &L_T = -\left(\frac{d\ln\rho}{dT}\right)^2 + G_{\bar{A}\bar{B}} \frac{d\gamma^{\bar{A}}}{dT}\frac{d\gamma^{\bar{B}}}{dT} - \rho^2 \textstyle\sum_b C_b e^{-2\rho w_b(\gamma)}, \\ &G_{\bar{A}\bar{B}} \gamma^{\bar{A}} \gamma^{\bar{B}} = -1. \end{aligned} \tag{21}$$

Again $\binom{0}{0}$ component of the Einstein equations gives additional condition to the equations of motion following from this Lagrangian:

$$-\left(\frac{d\ln\rho}{dT}\right)^2 + G_{\bar{A}\bar{B}} \frac{d\gamma^{\bar{A}}}{dT}\frac{d\gamma^{\bar{B}}}{dT} + \rho^2 \sum_b C_b e^{-2\rho w_b(\gamma)} = 0. \tag{22}$$

Here $\beta^{\bar{A}} = \rho\gamma^{\bar{A}}$ and time parameters T and τ are defined by the evident generalization to the multidimensional spacetime of their definitions we used in case of 4-dimensional gravity: $dt = -\sqrt{\det g_{\alpha\beta}}\, d\tau$, $d\tau = \rho^2 dT$. All functional parameters $C_b(x^\alpha)$ in general are positive. The cosmological singularity corresponds to the the limit $\rho \rightarrow \infty$ and in this limit potential term in Lagrangian can be considered as ρ-independent, asymptotically it vanish in the region of this space where $w_b(\gamma) > 0$ and is infinite where $w_b(\gamma) < 0$.The sum in the potential means summation over all relevant (dominating) impenetrable barriers located at hypersurfaces where $w_b(\gamma) = 0$ in the hyperbolic d-dimensional γ-space. All $w_b(\gamma)$ are linear functions on γ therefore $w_b(\gamma) = \rho^{-1} w_b(\beta)$. The free motion of $\beta^{\bar{A}}(\tau)$ between the walls in the original $d+1$-dimensional β-superspace is projected onto a geodesic motion of $\gamma^{\bar{A}}(T)$ on hyperbolic d-dimensional γ-space, i.e. to the motion between the corresponding projections of the original walls onto γ-space. These geodesic motions from time to time are interrupted by specular reflections against the infinitely sharp hyperplanes $w_b(\gamma) = 0$. These hyperplanes bound a region in γ-space inside which a symbolic particle oscillates and the volume of this region, in spite of its non-compactness, is finite. The last property is of principle significance since it leads to the chaotic character of the oscillatory regime.

Of course, one of the central point here is to find all the aforementioned dominant walls and corresponding "wall forms" $w_b(\beta)$. This depends on the spacetime dimension and menu of p-forms. In papers [33, 7, 8] the detailed description of all possibilities for the all types of supergravities (i.e., eleven-dimensional supergravity and those following from the known five types of the superstring models in ten-dimensional spacetime) can be found. It was shown that in all cases there is only 10 relevant walls governing the oscillatory dynamics. The large number of other walls need no consideration because they are located behind these principal ten and have no influence on the dynamics in the first approximation. The mentioned above region in γ-space where a particle oscillate is called "billiard table" and collection of its bounding walls forms the so-called Coxeter crystallographic simplex, that is, in the cases under consideration, polyhedron with 10 faces in 9-dimensional γ-space with all dihedral angles between the faces equal to the numbers π/n where n belongs to some distinguished set of natural numbers (or equal to infinity). This is very special geometric construction which (when combined with the specular laws of reflections against the faces) lead to the nontrivial huge symmetry hidden in the asymptotic structure of spacetime near cosmological singularity which symmetry coexists, nevertheless, with chaoticity.

DAMOUR-HENNAUX-NICOLAI HIDDEN SYMMETRY CONJECTURE

The mathematical description of the symmetry we are talking about can be achieved in the following way. Consider the trajectories $\beta^{\bar{A}}(\tau)$ of a particle moving between the walls $w_b(\beta) = 0$ in the original 10-dimensional β-superspace with coordinates $\beta^{\bar{A}}$ and metric $G_{\bar{A}\bar{B}}$ (20). These trajectories are null stright lines with respect to the Lorenzian metric $G_{\bar{A}\bar{B}}$. Wall forms $w_b(\beta)$ are linear function on β, that is $w_b = w_{b\bar{A}}\beta^{\bar{A}}$ where the set of constants $w_{b\bar{A}}$ depends on the choice of a supergravity model and on the type of the wall (index b) in the chosen model. We see that for each wall $w_b = 0$ the constants $w_{b\bar{A}}$ represent components of the vector orthogonal to to this wall. We can imagine all these vectors (for different b) as arrows starting at the origin of the β-space. All these vectors have fixed finite norm $G^{\bar{A}\bar{B}} w_{b\bar{A}} w_{b\bar{B}}$ ($G^{\bar{A}\bar{B}}$ is inverse to $G_{\bar{A}\bar{B}}$) and one can arrange the scalar products $(w_a \bullet w_b) = G^{\bar{A}\bar{B}} w_{a\bar{A}} w_{b\bar{B}}$ for each supergravity model in the form of the matrix:

$$A_{ab} = 2\frac{(w_a \bullet w_b)}{(w_a \bullet w_a)} \quad \text{(no summation in } a\text{)}. \tag{23}$$

The crucial point is that, independently of a supergravity model, A_{ab} is the Cartan matrix of indefinite type, i.e. with one negative principal value [8, 33, 34, 35]. Any Cartan matrix can be associated with some Lie algebra and particular matrix (23) corresponds to the so-called Lorenzian hyperbolic Kac-Moody algebra of the rank 10. As was shown in [31] the particle's velocity $v^{\bar{A}} = d\beta^{\bar{A}}/d\tau$ after the reflection from the wall $w_{a\bar{A}}\beta^{\bar{A}} = 0$ changes according to the universal (i.e. again independent of the model) law:

$$\begin{aligned} &(v^{\bar{A}})_{after} = (v^{\bar{A}})_{before} - 2\frac{(v^{\bar{B}})_{before} w_{a\bar{B}}}{(w_a \bullet w_a)} w_a^{\bar{A}}, \\ &w_a^{\bar{A}} = G^{\bar{A}\bar{B}} w_{a\bar{B}}, \quad \text{(no summation in } a\text{)}. \end{aligned} \tag{24}$$

This transformation is nothing else but the already mentioned specular reflection of a particle by the wall orthogonal to the vector $w_{a\bar{A}}$. Now it is clear that one can formally identify the ten vectors $w_{a\bar{A}}$ with the simple roots of the root system of Kac-Moody algebra, the walls $w_{a\bar{A}}\beta^{\bar{A}} = 0$ with the Weyl hyperplanes orthogonal to the simple roots, the reflections (24) with the elements of the Weyl group of the root system and the region of β-superspace bounded by the walls (where a particle oscillates) with the fundamental Weyl chamber. For the readers less familiar with all these notions of the theory of generalized Lie algebras (especially in application to the question under consideration) we can recommend the exhaustive review [10] which is well written also from pedagogical point of view.

The manifestation of Lie algebra means that the corresponding Lie symmetry group must somehow be hidden in the system . The hidden symmetry conjecture [35, 36, 37] proposes that this symmetry might be inherent for the exact superstring theories (assuming that they exist) and not only for their classical low energy limits of their bosonic sectors in the vicinity to the cosmological singularity. The limiting structure near singularity should be considered just as an auxiliary instrument by means of which this symmetry is coming to light. As of now we have no comprehension where and how exactly the symmetry would act (could be as a continuous infinite dimensional symmetry group of the exact Lagrangian permitting to transform the given solutions of the equations of motion to the new solutions). If true the hidden symmetry conjecture could create an impetus for the third revolution in the development of the superstring theories.

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