A Theory of Elementary Particles

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(Received March 9, 1957)

A theory of elementary particles formulated without using space-time co-ordinates is given. The causality principle in our theory is given by the relation of the results of two measurements in the ordered observation \( (O_1, O_2) \). Our causality principle contains an undetermined constant \( \alpha \) (a four-vector) \( C(O_1, O_2) \). Our space-time is derived when the undetermined constants in the ordered observation \( (O_1, O_2, O_3, \ldots) \) are given values by the law of probability. The field theory is valid only for "the local ideal system" which has no fluctuation of space-time. The limits of validity of the theory of relativity and the quantum mechanics based on Schrödinger equation are given in view of our theory. The difficulty of nonintegrability of the non-local field theories which aim at avoiding the divergence difficulty disappears into the fluctuation of space-time in our theory. A view is mentioned that the spatial extension of the proton core found in the high energy electron scattering is possibly an observed effect of the fluctuation of the "space-time of the system" in our theory.

§ 1. Introduction

When Einstein had renounced the mechanical theory of light ether and regarded the Maxwell equation of electro-magnetic field as the equation of motion of vacuum, the space-time was considered not only as the frame to describe the natural phenomena but as the fundamental physical substance. Relativistic quantum field theory has considered the Dirac equation of electron and the field equations of other particles as the equations of motion of vacuum and excellently described elementary particles and their interactions as the quanta and their interactions resulted from the quantization of the equation of motion of vacuum containing interactions. Owing to the success of the field theory, the fundamental theory of physics has been governed by the thought that the space-time is the fundamental substance which contains all the natural laws.

The field theory, however, has the divergence difficulty which is related intimately to the infinite numbers of points in space-time. What curbs various attempts to avoid the divergence difficulty is the assumption of existence of the infinitely exact clock and rule in the field theory. None of us knows at present in what way the divergent quantities will be replaced by the finite quantities in the future theory. We must await the future experiments. But at the present stage, if we can conquer the yoke of space-time by reconsiderations about its concept and extend the limit of applicability of the theory of elementary particles, the difficulty will be replaced by the ignorance.

Following these ideas, the author discussed in the previous paper the possibility to formulate the fundamental law of nature without using the space-time co-ordinates, and
to derive the space-time as a statistical concept such as the temperature in the kinetic theory of gases. Our theory started from the existence and the description of the elementary particles, and attempted to formulate the quantum mechanics of the system of elementary particles without using the space-time co-ordinates. The essential point of our theory was the introduction of an undetermined constant \( C(O_1, O_2) \) into the fundamental law which replaced the Schrödinger equation. At this point the theory differs essentially from the theory of second quantization starting from the particle aspect.

It is the purpose of this paper to formulate the theory more completely following the program given at the end of the previous paper.\(^1\) The author believes that our theory gives a new concept of space-time and has the possibility of inclusion of finite theory and suggests that the structure of elementary particles must be discussed in the co-ordinate spaces independent of the space-time.

§ 2. Existence and degrees of freedom of elementary particles

Our theory assumes the existence of elementary particles and starts from the description of their degrees of freedom in the co-ordinate spaces independent of the space-time. We assume the following \((A_1), (A_2), (A_3)\) and \((A_4)\).

\((A_1)\) * Elementary particles have intrinsic quantities \( m, \sigma \) and are divided into kinds \( k(m, \sigma) \) which are distinguished by intrinsic quantities, \((m, \sigma)\).

Intrinsic quantities are quantities which we believe from observations to be intrinsic of the elementary particles. Intrinsic quantities will increase and may change in the progress of physics. At present, mass, magnitude of spin and isospin, etc., are considered to be intrinsic quantities. For convenience, we denote mass by \( m \) and other intrinsic quantities by \( \sigma = (\sigma^{(1)}, \sigma^{(2)}, \ldots) \). \( m \) is the mechanical mass (We mention the reason at the end of the next section).

\((A_2)\) Elementary particles have degrees of freedoms which are characterized by the intrinsic quantities \( m, \sigma \). We describe the freedoms in the co-ordinate spaces \( \sum^{(m)}, \sum^{(1)}, \sum^{(2)}, \ldots \sum^{(f-1)} \). The freedom of momentum of elementary particles are described in four dimensional \( p_m \)-space which we denote by \( \sum^{(m)} \). In this space, the momentum of the particle of mass \( m \) lies on the surface:

\[
\sum^{(m)}(m) = \{ p_0, p_1^2 + p_2^2 + p_3^2 + p_4^2 = -m^2 \},
\]

where \( p_i = ip_0, \ p_0, \ p_1, \ p_2, \ p_3 \) are all real,

which are restricted by mass \( m \).

The freedom of spin is described in the space of discrete points \( s^{(1)} \) which we denote by \( \sum^{(1)} \). In this space, the freedom of spin of the particle of spin \( \sigma^{(1)} \) is described by the assembly of points:

\[
\sum^{(1)}(\sigma^{(1)}) = \{ s^{(1)} ; |s^{(1)}| \leq \sigma^{(1)} \}
\]

* The alphabetical letters designating the main assumptions and postulates keep the correspondence to those in the previous paper. \(^1\)
which are restricted by $\sigma^{(1)}$.

Other freedoms are described similarly by $\sum^{(0)}(\sigma^{(0)})$'s which are characterized and restricted by $\sigma^{(0)}$'s. That the degree of freedom of elementary particles is characterized by intrinsic quantity restricting its freedom is the criterion for the degree of freedom of the elementary particle in our theory. Space-time localization or orbital angular momentum are not considered to be the degree of freedom of elementary particles because they are not characterized by intrinsic quantities. This is a motive of our attempt to formulate the fundamental law without using the space-time co-ordinates.

(A₃) To a state of the system of elementary particles corresponds the assembly of integers (containing zero) $n_k(p, s)$ which specify the state of the system. $n_k(p, s)$'s are the variables of our theory. We call $n_k(p, s)$ as the numbers of the elementary particles of the kind $k$ and of the co-ordinates which are specified by the points $p$ and $s^{(0)}$ in the co-ordinate spaces $\sum^{(\infty)}$ and $\sum^{(0)}$.

(A₄) Elementary particles are divided into two groups of Fermi and Bose particles and the possible values of $n_k(p, s)$'s are restricted as

$$n_k(p, s) = 0, 1 \quad \text{for Fermi particles},$$

$$n_k(p, s) = 0, 1, 2, \ldots \quad \text{for Bose particles}.$$

$n_k(p, s)$'s are not the numbers of observed particles, but the numbers of bare particles. It is a central problem in the Tomonaga-Schwinger theory to discuss the true vacuum and observed particles. In relation to the idea of renormalization two standpoints are possible at present. One anticipates that the future theory is described in terms of only observed quantities, and has tried to formulate the theory in the renormalized quantities. The other expects that we can calculate unrenormalized quantities from observed quantities. This standpoint considers the renormalization technique as the phenomenological theory which calculates the observed quantities which are insensible to the detailed structure of the elementary particles. The attempts which have been tried to calculate the neutron-proton mass difference stand on the latter viewpoint. We take the latter standpoint and this is the reason why we choose the numbers of bare particles as variables. The problems of the true vacuum or physical particles are the problems of the observables discussed in the next section. (In our theory, we can choose in principle the representation in which the numbers of physical particles are diagonal from the transformation theory mentioned in § 5.)

§ 3. Quantum mechanics of the system of elementary particles

We start from the quantum theory, not from the classical theory. Correspondence principle must be merely the heuristic method in our theory. We can postulate the kinematical part of the quantum mechanics, but we cannot use the dynamical part, because the Schrödinger equation assumes the existence of the exact clock and rule. We have no clock or rule at the beginning. Therefore we must have different description of dynamics.
1. **Hilbert space**

The state of the system of elementary particles is described by the vector in a Hilbert space spanned by the functions of the form,

\[ \psi'[\eta'_j(p, s)] \]

\( \psi \) has the physical meaning of the probability amplitude of the system, i.e. the square of its norm gives the probability of finding the system in the state specified by the assembly of \( \eta'_j(p, s) \), and it defines the physical meaning of the superposition principle.

2. **Observable**

We introduce the matrices \( a_k(p, s) \) and \( a_k^*(p, s) \),

\[
\begin{align*}
  a_k(p, s) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, &
  a_k^*(p, s) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
  a_k(p, s) &= \begin{pmatrix} 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, &
  a_k^*(p, s) &= \begin{pmatrix} 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots \end{pmatrix}
\end{align*}
\]
for Fermi particles,

\[
\begin{align*}
  a_k(p, s) &= \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \end{pmatrix}, &
  a_k^*(p, s) &= \begin{pmatrix} 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots \end{pmatrix}
\end{align*}
\]

for Bose particles.

We can get all the operators which move the state vector in the space spanned by the functions \( \psi'[\eta'_j(p, s)] \)'s, making the linear combinations of the product of \( a_k(p, s) \) and \( a_k^*(p, s) \), and we call the hermitians in these operators as observables which we denote by \( \Omega \).

We assume that every physical quantity corresponds to a observable. The most part of theoretical physics concerns with the problem of finding observables which correspond to physical quantities. The field theory utilizes the correspondence principle to find observables. Our theory has a correspondence to the field theory as we mention in § 6, so that our theory has also a correspondence to the classical theory. But, in fact, the elementary particles have many quantities which do not correspond to classical quantities. And even in the field theory, almost all the physical quantities are not given their observables explicitly or exactly, because the vacuum or physical particles and interactions between them are not simple. Generally, at present, we must rely on the result of observations and the physical intuition to get observables which correspond to physical quantities.

We assume the existence of the observables which are important in our theory:

\[
\begin{align*}
  \mathcal{E}_\lambda &= \sum p_\lambda a_k^*(p, s) a_k(p, s) + \mathcal{E}'_\lambda, \\
  \lambda &= 1, 2, 3, 4,
\end{align*}
\]

which correspond to the total energy and momentum of the system we observe and we assume that \( \mathcal{E}_\lambda \) is a four vector in \( \Sigma_\eta^\infty \) \( \mathcal{E}'_\lambda \) will have the form
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(D) \[ \hat{S}_1 = \sum \mathcal{A}_1 G \]

where \( \mathcal{A}_1 \) is a product of \( a_k(p, s) \) and \( a_k^*(p, s) \) and \( G \) is a function of co-ordinates which appear in \( \mathcal{A} \). The form of \( G \) is considered to be determined by the future observations and future theoretical considerations on \( \sum^{(m)} \) and \( \sum^{(o)} \), but at present we are free to presume the form of \( G \) to get a finite theory. This is the possibility of our theory for the inclusion of finite theories.

3. Observation

Our theory gives the fundamental law in the relation between the results of two measurements. Our fundamental law is based on the following assumptions.

(E1) We have the measurement which is characterized by the observable \( Q \) and the system we measure which has a four vector \( \psi \) in \( \sum^{(m)} \).

(E2) We have measurements. Measurements of the same system are able to be ordered. We denote the order of measurements on the same system as \( 0_1, 0_2, 0_3, \ldots \), and call the ordered measurements on the same system as the ordered observation \( (0_1, 0_2, 0_3, \ldots) \). The concept of the ordering of measurements is more primitive* than the concept of space-time in physics.

(E3) When we measure a system, and the observable \( Q \) is expressed for the system as the direct sum

\[ Q = Q_1 + Q_2 + \cdots, \]

we call the measurement of \( Q \) as the simultaneous measurements of \( Q_1, Q_2, \ldots \). Simultaneous measurements are assumed to belong to the same \( O \), and to be measured independently.

We have often the system for which an observable is approximately expressed as direct sum of other observables. This is the reason why we find that physicists discuss the structure of a nucleus in the same way wherever we go in the world. However, we have no observable which can be expressed as the direct sum of more than two observables exactly. This is the reason, as we mention in § 4, why we cannot have in principle an exact clock and rule.

4. Results of measurements

We postulate:

(E4) When the system is the state represented by \( \psi \), the expectation value of the result of the measurement of the quantity which corresponds to the observable \( Q \) is given by

\[ \langle \psi \mid Q \mid \psi \rangle. \]

\( \langle \mid \) and \( \mid \rangle \) are bra and ket of Dirac.

(E5) The result of measurement of \( Q \) is one of eigenvalues of \( Q \).

5. The principle of causality in our theory is given by the relation between the results of an ordered observation \( (0_1, 0_2) \). We postulate:

(F) When the result of \( Q_1 \) in \( 0_1 \) is \( \omega \), then the expectation value of the result of measurement of \( Q_2 \) in \( 0_2 \) is

\[ \langle \psi' \mid Q_2 \mid \psi' \rangle, \]

where

* Exact clock and rule can order measurements, but the ordering of measurements cannot imply the exact clock and rule.
\[ (G) \quad \psi' = \exp(i \sum_j \xi_j C_j(O_1, O_2)) \psi_\omega \quad \text{and} \quad \Omega_1 \psi_\omega = \omega \psi_\omega. \]

\( C(O_1, O_2) \) is a constant four vector in \( \sum^{(w)} \). \( C(O_1, O_2) \) is undetermined except that it is independent of \( \Omega_1, \Omega_2 \) and \( \xi_j \), and depends only on \( O_1 \) and \( O_2 \). Hence, if the measurement of \( \Omega_2 \) is the simultaneous measurements \( \{\Omega\} \) of \( \Omega_{21} + \Omega_{22} + \cdots \), the expectation values of \( \Omega_{21}, \Omega_{22}, \cdots \) in \( O_1 \),

\[ \langle \psi' | \Omega_{21} | \psi' \rangle, \quad \langle \psi' | \Omega_{22} | \psi' \rangle, \cdots \]

have the same \( C(O_1, O_2) \), but the value of \( C(O_1, O_2) \) is undetermined.

Our fundamental law \( (G) \) replaces the Schrödinger equation containing time. We have got the fundamental law which contains no space-time coordinate. We have conquered the space-time by the introduction of undetermined constant \( C(O_1, O_2) \) into the fundamental law.

Our theory insists that we cannot generally predict the expectation values of the result of measurement of \( O_2 \) even when we know the result of measurement of \( O_1 \) no more than the values containing undetermined quantity \( C(O_1, O_2) \). \( C(O_1, O_2) \) reveals a fundamental unknowable. The meaning of \( C(O_1, O_2) \) and our intention in what way we want to describe the nature will be explained by the figurative illustration of the “rope-dancing”. Now we imagine that the many equal ropes are extended through the air. On each rope a rope-dancer is performing his rope-dance. They are tireless but fall down due to uncontrollable causes. We have no watches. Rope-dancers are classified into A, B, C, \cdots according to their abilities. If the abilities of classes is given only in the ratio \( a_A : a_B : a_C : \cdots \), our knowledge contains an undetermined constant C. If rope-dancers correspond to elementary particles, C will correspond to \( C(O_1, O_2) \) in our theory. In what way of description can we predict the fate of a rope-dancer B if we have no concept of time? The way we can find is as follows. When there are many rope-dancers of the same class A, the probability of surviving is \( \alpha \) when the ratio of the number of survivors and the original numbers of A-rope-dancers is \( \beta \). The value of \( \beta \) is the statistical quantity having fluctuations. When the number of A is very large, \( \{\beta\} \) will play a rôle of a graduation of time.

\( \S \ 4. \ Space-time \)

1. \textit{Space-time of the system determined by} \( \{\Omega\} \)

When the system allows or approximately allows a number of simultaneous measurements of the observables \( \{\Omega\} \) which do not commute with the total energy momentum operators of the system, we can, by the law of probability, give the values to the \( C(O_1, O_2) \) in the ordered observation \( O_1, O_2, \cdots \). The aggregate of the values of \( C(O_1, O_2) \) determined thus statistically in all the possible ordered observations is the “space-time of the system in the measurements of observables \( \{\Omega\} \)” in our theory.

The system which has the space-time can be used as a clock or a rule when we can observe the system with the other systems simultaneously. Suppose that the system is given which is composed of independent systems 1, 2, \cdots N, so that
We denote the total energy momentum of the system as \( \mathbf{S}_\lambda \). And we assume that we have the observables,

\[
\mathbf{Q} = \mathbf{Q}^{(1)} + \mathbf{Q}^{(2)} + \mathbf{Q}^{(3)} + \mathbf{Q}^{(4)}
\]

where

\[
\mathbf{Q}^{(\lambda)} = \mathbf{Q}_1^{(\lambda)} + \mathbf{Q}_2^{(\lambda)} + \cdots + \mathbf{Q}_N^{(\lambda)}, \quad \lambda = 1, 2, 3, 4
\]

and \( \mathbf{Q}^{(\lambda)} \) is defined to satisfy the following commutation relations.

\[
[\mathbf{Q}^{(k)}, \mathbf{S}_l] \neq 0, \quad [\mathbf{Q}^{(k)}, \mathbf{S}_l] = 0 \quad \text{for} \quad l \neq k, \quad l, k = 1, 2, 3.
\]

If we measure \( \mathbf{Q}^{(4)} \) in the ordered observation \( (O_1, O_2) \), we can compare the results of the measurements and the expectation values of the measurements in \( O_2 \).

\[
\langle \psi_4^{(1)} \vert \exp\left(-i\mathbf{S}_1 C_1(O_1, O_2) \mathbf{Q}^{(4)} \exp(i\mathbf{S}_1 C_1(O_1, O_2)) \right) \psi_4^{(1)} \rangle
\]

\[
i = 1, 2, \cdots N,
\]

where \( \psi_4^{(1)} \) is the state vector prepared by the measurement in \( O_1 \). When \( N_1 \) is large, we can give the most probable value to \( C_1(O_1, O_2) \) by the law of probability.* We denote this statistical value as \( \overline{C}_4(O_1, O_2) \). If we measure \( \mathbf{Q}^{(4)} \) in the ordered measurements of \( O_1^4, O_2^4, O_3^4, \cdots \), we can get in a similar way

\[
\overline{C}_i(O_1^4, O_2^4), \quad \overline{C}_i(O_3^4, O_4^4), \quad \overline{C}_i(O_5^4, O_6^4), \cdots
\]

If the above mentioned \( \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{Q}^{(3)} \) and \( \mathbf{Q}^{(4)} \) exist which allow simultaneous measurements in the ordered observation \( (O_1, O_2, O_3, \cdots) \) we can give to every pair of \( (O_t, O_{t+1}) \) the four values \( \overline{C}_\mu(O_t, O_{t+1}) \), \( \mu = 1, 2, 3, 4 \). We call the assembly of \( \overline{C} \) in all the possible ordered observations, as the space-time of the system in the measurements of \( \mathbf{Q}^{(\alpha)} \), \( \mu = 1, 2, 3, 4 \).

When we consider the system \( \psi_M \) as a composition of two systems, i.e. \( \psi_M = \psi_{M_1} \psi_{M_2} \), we can get two assemblies of \( \overline{C}_\mu(O_t, O_{t+1}) \) and \( \overline{C}_\mu(O_t, O_{t+1}) \) in an ordered observation \( O_1, O_2, O_3, \cdots \).

In general the equalities

\[
\overline{C}_\mu(O_t, O_{t+1}) = \overline{C}_\mu(O_t, O_{t+1}), \quad i = 1, 2, 3 \cdots
\]

and

\[
\overline{C}_\mu(O_t, O_{t+1}) + \overline{C}_\mu(O_{t+1}, O_{t+2}) = \overline{C}_\mu(O_t, O_{t+2}), \quad i = 1, 2, 3, 4 \cdots
\]

are not valid. We call this fact as the fluctuation of space-time. When (I), (I2) are

* The law which gives \( \overline{C}_\mu(O_1, O_2) \) is contained in the definition of the expectation value of \( (E_4) \).
valid we call that the system has the exact space-time. The magnitudes of the fluctuations of the $C_\mu (O_i, O_j)$ $\mu = 1, 2, 3, 4$ give the measure of exactness of the space-time of the system determined by $\{O\}$.

3. **Ideal System**

We call, for convenience, the system of which the state vector and the observables have the form of $(H_1)$ and $(H_2)$ and $N_\mu$ is considered to be infinitely large, as “ideal” (in $\mu$-direction). The system which contains an independent ideal system is an ideal system. The law of large numbers in the theory of probability infers that when we observe the two ideal systems simultaneously in an ordered measurements $O_1, O_2, O_3, \ldots$, the two ideal systems give the same assembly of values of $\bar{C}_\mu (O_i^\mu, O_j^\nu)$. Namely the equality (I) is valid for ideal system. We call that the ideal system is “local” when the equality (I) is valid and “non-local” when the equality (I) is not valid.

4. **Local ideal system**

For the local ideal system, $\bar{C}(O_i, O_j)$ is expressed as a difference of the form,

\[(\text{I}_2) \quad \bar{C}_\mu (O_i, O_j) = \gamma (X_{\mu i} - X_{\mu j}), \quad \mu = 1, 2, 3, 4.\]

where $\gamma$ is a universal constant. Hence, if we have a local ideal system (in four directions), we can give to every $O_i$ in the ordered measurement $O_1, O_2, O_3, \ldots$, four values $X_{\mu i}$ which is specified by a point in a four dimensional space spanned by four $X_\mu$-axis. We call this space as $X$-space for the local ideal system.

Our theory is invariant for the rotation of $\sum^{(m)}$ as we mentioned in a preceding section. If we rotate the $\rho\mu$-axis in $\sum^{(m)}$, the energy-momentum four vector becomes

\[(\text{J}) \quad \mathcal{E}_\lambda' = \sum_{\nu} a_{\lambda \nu} \mathcal{E}_\nu.\]

Since $C(O_i, O_j)$ is a four vector in $\sum^{(m)}$, $\bar{C}(O_i, O_j)$ is in the rotated representation

\[(\text{J}_2) \quad \bar{C}_\lambda' (O_i, O_j) = \sum_{\nu} a_{\lambda \nu} \bar{C}_\nu (O_i, O_j),\]

hence

\[X_\lambda' = \sum_{\nu} a_{\lambda \nu} X_\nu.\]

The invariance property of the theory in $\sum^{(m)}$ implies the invariance in the Lorentz transformation in $X$-space.

The local ideal system can be used as an exact clock or an exact rule when we measure the system with the other system simultaneously. The local ideal system corresponds to the thermometer of ideal gas in thermodynamics. The pi-meson clock mentioned in the previous paper is an illustrative example of the local ideal system (in 4-direction). However as we mention in $\S$ 6, the assumption of the existence of the local ideal system implies the local field theory, and hence the local ideal system has the divergence difficulty. In reality, therefore, we cannot have the local ideal system. And we cannot have the ideal system, because a particle in the system receives “the reaction
of real particle 

The space-time of any system fluctuates. The fluctuation of space-time is the original motive of our theory.

We can say that the theory of relativity is only valid for the case in which the fluctuation of space-time of the system is neglected. This is the limit of applicability of the theory of relativity in view of our theory.

§ 5. Transformation theory

Our theory is based on the quantum mechanics and is described in the relation between operators and vectors in a Hilbert space so that the theory is also invariant in the unitary transformation of the Hilbert space. We mention here the transformations that are important for our discussions.

1. Heisenberg representation

The fundamental law (F) (G) is described in an equivalent way as follows.

(K) In an ordered observation \((O_1, O_2)\), when the result of measurement of \(\Omega_1\) in \(O_1\) is \(\omega\) then the expectation value of measurement of \(\Omega_2\) in \(O_2\) is \(\langle \psi_\omega | \Omega_2' | \psi_\omega \rangle\), where \(\Omega_1 \psi_\omega = \omega \psi_\omega\) and

\[
\Omega_2' = \exp \left( -i \sum_\alpha C_\alpha (O_1, O_2) \right) \Omega_2 \exp \left( i \sum_\alpha C_\alpha (O_1, O_2) \right).
\]

We may call (F) (G) the Schrödinger representation and (K) (L) the Heisenberg representation in our theory.

2. Fock-representation

It is often convenient to describe the theory in the representation as

\[
\psi = \begin{pmatrix}
\psi_0 \\
\psi_1 (s_1, p_1) \\
\psi_2 (s_1, p_1; s_2, p_2) \\
\vdots \\
\psi_n (s_1, p_1; \ldots; s_n, p_n) \\
\end{pmatrix}
\]

where \(| \psi_n (s_1, p_1; \ldots; s_n, p_n) |^2\) is the probability finding the system of \(n\) particles having the co-ordinates \(s_1, p_1; s_2, p_2; \ldots; s_n, p_n\). (For simplicity, we considered one kind of particles.)

3. Fourier transformation

We can describe the theory in the \(x_n\)-space transformed from the \(p\)-space by the Fourier transformation. The state vector in \(x\)-space is expressed in Fock-representation as

---


\[
\phi = \left\{ \begin{array}{c}
\varphi_0 \\
\varphi_1 \left( s_1, \; x_1 \right)
\varphi_2 \left( s_1, \; x_1; \; s_2, \; x_2 \right)
\varphi_3 \left( s_1, \; x_1; \; s_2, \; x_2; \; s_3, \; x_3 \right)
\end{array} \right.
\]

where

\[
\varphi_n \left( s_1, \; x_1; \; s_2, \; x_2; \; \cdots ; \; s_n, \; x_n \right) = \int e^{i \left( p_1 s_1 + p_2 s_2 + \cdots + p_n s_n \right)}
\]

\[
\varphi \left( s_1, \; p_1; \; \cdots ; \; s_n, \; p_n \right) d^4 p_1 \cdots d^4 p_n
\]

\[n = \sum_{\mu=1}^{4} p_\mu x_\mu\].

\(x\)-space can be identified with \(X\)-space for the local ideal system as we mention in the following section. The real system is the non-local and non-ideal system and the space-time in our theory is not identical with \(x\)-space. Our theory insists that the \(x\)-space is not space-time but only Fourier transform of \(p\)-space and the fundamental theory cannot be based on the Schrödinger equation containing \(x\).

The \(x\)-space will be useful also in our theory when we want to give the boundary condition of which the position of boundary is insensitive to the result of the calculation. In this case, on the boundary, we can identify \(x\) with the space-time.

§6. Correspondence between field theory and our theory

The space-time in the field theory is the \(x\)-space in (N). We show here that the \(X\)-space for the local ideal system is identified with \(x\)-space. This is the correspondence between the field theory and our theory.

For the local ideal system our fundamental equation (G) can be expressed as

\[
\Omega \left( X \right) = \exp \left( i \sum_{\lambda} \xi_\lambda X_\lambda \right) \Omega \left( 0 \right)
\]

or in the form of differential equation

\[
\frac{i}{\gamma} \frac{\partial \Omega \left( X \right)}{\partial X_\lambda} = -\xi_\lambda \Omega \left( X \right), \quad \lambda = 1, \; 2, \; 3, \; 4.
\]

And (L) is expressed as

\[
\Omega \left( x \right) = \exp \left( -i \sum_{\lambda} \xi_\lambda X_\lambda \right) \Omega \left( 0 \right) \exp \left( i \sum_{\lambda} \xi_\lambda X_\lambda \right)
\]

or in the differential equation

\[
\frac{i}{\gamma} \frac{\partial \Omega \left( X \right)}{\partial X_\lambda} = [\xi_\lambda, \Omega \left( X \right)].
\]
\( (O_\gamma) \) and \((P_\gamma)\) have the same form with the Schrödinger equation and the Heisenberg equation in the field theory when we put

\[
(I_\delta) \quad \gamma = (hc)^{-1}.
\]

We can say that the quantum mechanics based on the Schrödinger equation is only valid for the case in which the fluctuation of space-time is neglected. This is the limit of applicability of the quantum mechanics.

If we have the Schrödinger equation \((O_\gamma)\), our theory is shown to be identical with the field theory by the theory of second quantization starting from the particle aspect and we can get the quantized field equation in \(X\)-space. Hence we can say that the field theory is valid for the local ideal system. In other words, the field theory is valid for the local system which contains an independent local ideal system, i.e. an exact clock and a rule.

For the real system we cannot put the equation \((O_\gamma)\) or \((P_\gamma)\) in the basis of the theory. Our clock and rule fluctuate in principle. We can say that the field theory is valid only in the case where the fluctuation of space-time can be neglected. This is the limit of the field theory in view of our theory.

\section*{§ 7 Fluctuation of space-time and inconsistency in field theory}

Various theories to avoid the divergence difficulty in the field theory based on the Schrödinger differential equation in \(x\)-space have the serious inconsistency in the following point. The fundamental differential equation (Tomonaga-Schwinger equation) in these theories do not satisfy the integrability condition. In other words, in these theories the state vector determined on a surface in \(x\)-space is to be given but can not be given in fact. This inconsistency disappears into the fluctuation of space-time in our theory, because our theory is based on the fundamental equation formulated without using the space-time co-ordinates.

In our theory, when a four-vector \(\xi\) which leads to a finite theory is given, we have a consistent theory involving no divergence difficulty. The detailed form of \(\xi\) can not be given at present and its determination depends on the results of future experiments. But we can presume at present the form of \(\xi\) and make various theories each of which is characterized by the assumption of the form of \(\xi\).

The system whose \(\xi\) leads to a finite theory is the "non local system". The space-time of the non local system has the fluctuation even when the system is an "ideal system" (§ 4). Different theories, characterized by their assumptions of the forms of \(\xi\), give different space-time respectively, which are characterized by nature of their fluctuations in ideal systems.

The reason why the field theory has the above mentioned inconsistency and our theory has not is that the space-time is assumed in the field theory while it is derived in our theory, or in other words, our theory allows the existence of the fluctuation of space-time while the field theory does not.
Stanford experiments\textsuperscript{2) of the electron-proton scattering have shown that the proton core has the spatial extension of the order of rms radius $0.7 \times 10^{-13}$ cm. The interpretation\textsuperscript{2) that the coulomb law breaks down at small dimension seems to be very attractive, because the interpretation suggests that the proton core extension is an observed effect of the non-localized interaction. The break-down of the coulomb law, however, will imply the break-down of the field theory because we have no consistent non local field theories.

In our theory, however, it is possible that the electromagnetic interaction part of the total energy-momentum operator $\mathcal{H}$ contains or effectively contains the form factor (invariant in $\sum_{\nu}$) which cuts the high momentum part. In this case we may say that the effective radius of the proton core found in the electron scattering experiment is neither the radius of the proton nor the radius of the electron, but the radius of the fluctuation of space-time in the electron-proton scattering system.*

The author is grateful to the members of our institute, Dr. S. Oneda, Messrs. S. Hori and A. Wakasa and Miss K. Saikawa for the pleasant discussions and help on this paper.

The author dedicates this paper to Prof. S. Tomonaga.

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* We want to discuss about this problem elsewhere in more detail.