Pion-Nucleon Scattering and the Structure of the Pion

Shigeru MACHIDA

Department of Physics, Rikkyo University, Tokyo

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Effects of the “Structure” of the pion to the pion-nucleon scattering are discussed. A formal expression of the interaction kernel for the nucleon-antinucleon propagator is given, from which the “effective Hamiltonian” for the emission and absorption of one pion and that for two pions are derived respectively. Effects of two pion vertex, which is characteristic of the structure of pion, seem to be not necessarily negligible compared with the effects of one pion vertex, according to a very rough estimate of the order of magnitude of the “coupling constant” appearing in the interaction kernel for the nucleon-antinucleon propagator.

At present two standpoints are recognized with respect to the theory of pion, i.e., the compound theory of the pion and the conventional local pion field theory. However, we intended to discuss some important properties hurried under both theories by using the above method, and obtained the qualitative properties concerning the isospin dependence of the pion-nucleon scattering matrix elements.

Quantitative evaluation of the possible effects are not made, since for that purpose it is necessary to attack the details of the structure of the pion. Such an attempt is not made in the present paper.

It is shown, in the Appendix, that the crossing symmetry for the pion-nucleon scattering holds even if the pion is considered to be composed of arbitrary number of fermion pairs and bosons, provided that the primary interaction Lagrangian satisfies the integrability condition to assure the existence of propagators.

§ 1. Introduction

It is well known that the main features of low energy P-wave pion-nucleon scattering are well reproduced by the charge independent meson theory with fixed extended source. Concerning the S-wave scattering, however, we cannot explain the experiments even qualitatively.

It seems doubtless that the effects of virtual nucleon-antinucleon pairs are important for the understanding of the behaviors of the S-wave pion-nucleon scattering as well as the electromagnetic properties of the nucleons, including the anomalous magnetic moments, neutron-electron interaction, and the high energy electron-nucleon scattering.

It is the purpose of this paper to discuss one of the effects of the nucleon-antinucleon pairs, i.e. the effects of the structure of the pion, to the low energy pion-nucleon scattering. We regard the physical pion to be a “compound” state of nucleon pairs. However, the following discussions are not restricted to the compound theory of the pion, but may be applicable to the pseudoscalar meson theory which seems now standard, as an approximation with an appropriate consideration.

In most of the compound theories of “elementary” particles, pions are identified with suitable bound states composed of a nucleon pair. However, more exactly speaking,
they should be considered to be composed of equal numbers of particles and antiparticles belonging to the nucleon family. We will not go into the details of the primary interaction between constituent particles, which should cause the desired bound states, but only assume the existence of such an interaction. It might be worthwhile to point out in this connection that it is very difficult to draw some conclusion on the primary interaction from the binding energy data, since it is needed to treat the interaction very rigorously even at very small distances for the calculation of the binding energy.

The following formalism is also applicable to the amplitudes of nucleon pair components in the case of the conventional local theory of field, where both nucleons and pions are regarded as structureless particles. Such a treatment should be a good approximation to phenomena where main contributions are given by states where only nucleon pairs exist. Interactions between nucleons and antinucleons are due, in this case, to the intermediary of pion field.

In principle, there must be differences between the consequences of the "compound" pion theory, where pions are assumed to be appropriate bound states of more fundamental particles, and the "elementary" pion theory, where pions are assumed to be fundamental particles. It seems, however, very difficult to show it, since the differences between the consequences of the two standpoints appear always quantitatively but not qualitatively. This fact is most easily understood by seeing that the primary interaction in one theory may be derived by the other, although these detailed functional forms may be different. Therefore, in the following, our standpoint need not be restricted to one of the above mentioned two theories.

Experimental values of the low energy S-wave scattering phase shifts* are characterized by their small magnitudes and the large isospin dependence, to both of which any reasonable explanations are not yet given from the pion field theory. The large isospin dependence seems, at first sight, hard to be explained in connection with the crossing symmetry\(^{6}\), which is expressed as

\[ T_{i\mu}(p', p; q' q) = T_{i\ell}(p', p; -q, -q'), \tag{1.1} \]

where \( T_{i\mu}(p', p; q' q) \) is the scattering amplitude for the scattering of an \( i \)-th meson in momentum state \( q \) to a \( j \)-th meson in state \( q' \), the scattering nucleon going from \( p \) to \( p' \). It is easily seen from eq. (1.1) that the isospin dependent term of the scattering amplitude contains a factor \( \mu/M (\mu \) and \( M \) are masses of a pion and a nucleon respectively) since for \( q' = -q = 0 \) (which is only possible for \( \mu = 0 \) \( T_{\mu} = T_{\ell} \) and the scattering is independent of isospin. In fact, none of calculations, performed up to present, starting from the charge-independent pseudoscalar meson theory have succeeded in obtaining the adequate isospin dependence\(^{5}\), although it may be well reproduced by a model proposed by Bosco and Stroffolini\(^{10}\) which manifestly destroys the crossing symmetry as well as by semiphenomenological effective Hamiltonian theory with two coupling constants\(^{7}\).

\* \( \delta_1 \approx -0.11k/\mu, \delta_3 \approx 0.16k/\mu \), \( \delta_1 \)'s are the phase shifts of the isospin \( i/2 \) states, \( k \) is the momentum of the incident pion, and \( \mu \) is the mass of a pion.
Pion-Nucleon Scattering and the Structure of the Pion

connection with the above mentioned situation, it seems important to investigate whether or not the crossing symmetry holds in the case of scattering of compound particles. This problem is studied in the Appendix, and it is proved that the crossing symmetry does also hold for the scattering of compound particles if the primary interaction between constituent particles satisfies the integrability condition.

In the present paper, we will investigate the relation between the effective pion-nucleon interaction Hamiltonian and the nucleon-antinucleon interaction kernel. In § 2, a formal representation is given for the nucleon-antinucleon interaction kernel, in § 3 the effective Hamiltonian for the emission or absorption of a pion is derived, and in § 4 the effective Hamiltonian for the emission or absorption of two pions is derived.

§ 2. Interaction Kernel

Let us suppose the physical pion to be composed of the same number of nucleons and antinucleons. We shall illustrate by making use of the shaded area in Fig. 1 the interaction kernel of the relativistic integral equation satisfied by one nucleon and one antinucleon amplitude, \( \langle 0 | T(\psi(x)\bar{\psi}(y)) | \pi \rangle \), where \( | \pi \rangle \) and \( | 0 \rangle \) means physical one pion and vacuum state vector respectively, \( \psi \) is the nucleon field operator in the Heisenberg representation, and \( T(\ ) \) means \( T \)-product defined by Wick\(^6\). The interaction kernel may be given, in principle, once the Lagrangian is given explicitly\(^9\). It is given by the sum of contributions from the Feynman diagrams shown in Fig. 2 in the compound theory of the pion and by that in Fig. 3 in the ordinary “elementary” theory of the pion. We will not, however, restrict our standpoint to one of the above mentioned two, but start from the formal expression of the “interaction kernel” corresponding to Fig. 1, i.e.,

\[
\int K d^4X = g_s \int d^4x_1 \, d^4x_2 \, d^4x_3 \, d^4x_4 \, \Phi(x_1, x_2, x_3, x_4) \cdot \bar{\psi}(x_1) \, \Omega_1 \psi(x_3) \cdot \bar{\psi}(x_2) \, \Omega_2 \psi(x_4) ;
\]

the interaction kernel is characterized by \( g_s \), \( \Omega_i \)'s, \( I \), \( I \)'s, and \( \Phi(x_1, x_2, x_3, x_4) \), where \( g_s \) is effective coupling constant, \( \Omega_i \)'s are operators involving Dirac and differential operators,
I and P are operators in isospin space, and \( \Phi(x_1, x_2, x_3, x_4) \) is the invariant form factor. Definitions of other notations are as follows:

\[
\begin{align*}
\bar{X} &= (x_1 + x_2 + x_3 + x_4) / 4, \\
\phi &= (\phi^p, \phi^n), \\
\phi' &= C\phi = \begin{pmatrix} -i\phi^p \\ i\phi^n \end{pmatrix}, \\
C &= i\hat{J}^{d_2^*},
\end{align*}
\]

and \( \phi' \)'s represent symbolically the outgoing nucleon lines in Fig. 1, and anticommute with each other. Contributions to the interaction between a nucleon and an antinucleon from the other amplitudes, e.g., that from the states where many nucleon pairs exist, are included in eq. (2.1).

The form factor, \( \Phi(x_1, x_2, x_3, x_4) \), must satisfy following general requirements. In the first place, it must be invariant with respect to arbitrary displacements of the origin of the system of space-time coordinates. This condition imposes a factor \( \delta^{4}(l^1 + l^2 + l^3 + l^4) \) upon the Fourier transform \( F(l^1, l^2, l^3, l^4) \) of the function \( \Phi(x_1, x_2, x_3, x_4) \). Accordingly, putting \( F(l^1, l^2, l^3, l^4) = \delta^{4}(l^1 + l^2 + l^3 + l^4) \cdot G \), we get

\[
\Phi(x_1, x_2, x_3, x_4) = \Phi(\bar{X}, x_{12}, x_{34}) \]

\[
= (2\pi)^{-12} \int G(L, l^{12}, l^{34}) \exp i(LX + l^{12}x_{12} + l^{34}x_{34}) \, d^4L \, d^4l^{12} \, d^4l^{34}, \tag{2.2}
\]

where

\[
\begin{align*}
X &= (x_1 + x_2) / 2 - (x_3 + x_4) / 2, \\
x_{12} &= x_1 - x_2, \\
x_{34} &= x_3 - x_4,
\end{align*}
\]

and

\[
\begin{align*}
L &= (l^1 + l^2) / 2 - (l^3 + l^4) / 2, \\
l^{12} &= (l^1 - l^2) / 2, \\
l^{34} &= (l^3 - l^4) / 2.
\end{align*}
\]

Next, since (2.1) must be Hermitian, a further restriction can be imposed:

\[
\Phi(\bar{X}, x_{12}, x_{34}) = \Phi^*(\bar{X}, -x_{34}, -x_{12}), \tag{2.3}
\]

or, in the Fourier representation,

\[
G(L, l^{12}, l^{34}) = G^*(L, l^{34}, l^{12}). \tag{2.3'}
\]

Finally, \( \Phi \) and \( G \) should be invariant with respect to Lorentz transformations.* We will conveniently normalize \( \Phi \) by

* These conditions have been discussed by Kristensen and Møller for the case of three point nonlocal interaction.
or in the Fourier representation by

\[ G(0, 0, 0) = 1. \]

For simplicity, we will hereafter not consider the case where \( \Omega_i \)'s include differential operators, but assume only five \( \Omega_i \)'s,

\[ \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5 = 1, \gamma_\alpha, \sigma_{\alpha \beta}, i\gamma_\beta \gamma_\lambda, i\gamma_\lambda. \]

Moreover, assuming charge independence, we take general forms for \( I \) and \( I' \) as

\[ I = \psi(x, \gamma, \tau_4) \quad \text{and} \quad I' = -\bar{\psi}(x, \gamma, \tau_4), \]

where \( \gamma_i \)’s are arbitrary real constants corresponding to five \( \Omega_i \), and \( \tau \) and \( \tau_4 \) are ordinary Pauli-type vector operator and unit operator in the nucleon isospin space, respectively.

In order to proceed with our calculation it seems convenient to express \( \bar{\psi}' \) and \( \psi \) in (2.1) by \( \psi \) and \( \bar{\psi} \) as follows:

\[
\bar{\psi}'(x_2) \Omega_{\bar{\psi}} \left( \frac{-\tau}{\tau_4} \right) \psi'(x_2) = \epsilon_i \bar{\psi}(x_i) \Omega_{\psi} \left( \frac{\tau}{\tau_4} \right) \psi(x_i),
\]

where

\[ \epsilon_i = \begin{cases} 
1 & \text{for } i = 1, 4, 5, \\
-1 & \text{for } i = 2, 3.
\end{cases} \]

Eq. (2.1) may, then, be written very explicitly as

\[
\int K d^4 \vec{X} = g_i \epsilon_i \int d^4 \vec{X} d^4 x_{12} d^4 x_{34} \Phi(X, x_{12}, x_{34}) \cdot \left[ \tilde{\psi}_a^k(x_1) \Omega_{\tilde{\psi}} \tau^{kl} \phi'_{\lambda}(x_2) \cdot \bar{\psi}_m^l(x_1) \Omega_{\bar{\psi}} \tau^{mn} \bar{\psi}_m^l(x_2) \right. \\
+ \left. \gamma_i \bar{\psi}_a^k(x_1) \Omega_{\tilde{\psi}} \tau^{kl} \phi'_{\lambda}(x_2) \cdot \bar{\psi}_m^l(x_1) \Omega_{\bar{\psi}} \tau^{mn} \bar{\psi}_m^l(x_2) \right], \tag{2.4}
\]

where indices of \( \psi \) and \( \bar{\psi} \), \( k, l, m \) and \( n \) vary as 1 for proton and 2 for neutron.

§ 3. One pion vertex

We consider the contribution of the interaction kernel, (2.4), to the one pion vertex (Fig. 4). One obtains from the first term of the right side of eq. (2.4),

\[
g_i \epsilon_i \int d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 \Phi(X, x_{12}, x_{34}) \cdot \left[ -\tilde{\psi}_a^k(x_1) \Omega_{\tilde{\psi}} \tau^{kl} \phi'_{\lambda}(x_2) \cdot \bar{\psi}_m^l(x_1) \Omega_{\bar{\psi}} \tau^{mn} \bar{\psi}_m^l(x_2) \right. \\
- \bar{\psi}_a^k(x_1) \Omega_{\bar{\psi}} \tau^{kl} \phi'_{\lambda}(x_2) \cdot \bar{\psi}_m^l(x_1) \Omega_{\bar{\psi}} \tau^{mn} \bar{\psi}_m^l(x_2) \right. \\
+ \left. \bar{\psi}_a^k(x_1) \Omega_{\tilde{\psi}} \tau^{kl} \phi'_{\lambda}(x_2) \cdot \bar{\psi}_m^l(x_1) \Omega_{\bar{\psi}} \tau^{mn} \bar{\psi}_m^l(x_2) \right] \tag{3.1}
\]
where
\[ \phi_{\gamma \gamma}^{\text{fin}}(x_3|x_4) = \langle 0 | T(\phi_{\gamma}^{+}(x_3) \phi_{\gamma}^{-}(x_4)) | \pi \rangle \]  
(3.2)
is the Bethe-Salpeter amplitude for the existence of a nucleon at \( x_3 \) and an antinucleon at \( x_4 \) respectively in the physical pion.\(^*\) Interchanging the notations in the second and fourth terms in (3.1) as follows,

\[ x_1 \leftrightarrow x_4, \; x_2 \leftrightarrow x_3; \; \alpha \leftrightarrow \gamma, \; \beta \leftrightarrow \delta; \; k \leftrightarrow m, \; l \leftrightarrow n, \]

(3.1) becomes
\[ 2g_{\epsilon_4} \epsilon_i \int d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 \bar{\Phi}(X, x_{12}, x_{34}). \]

\[ \cdot \left[ -\phi^{+}_{\alpha}(x_1) \Omega_{\epsilon \sigma \gamma \delta} \phi^{-}_{\beta}(x_3) \Omega_{\gamma \tau \delta \epsilon} \bar{\phi}_{\gamma \gamma}^{\text{fin}}(x_3|x_4) \right. \]
\[ + \phi^{+}_{\alpha}(x_1) \Omega_{\epsilon \sigma \gamma \delta} \phi^{-}_{\beta}(x_3) \Omega_{\gamma \tau \delta \epsilon} \bar{\phi}_{\gamma \gamma}^{\text{fin}}(x_3|x_4), \]
(3.3)

where
\[ 2 \bar{\Phi}(X, x_{12}, x_{34}) = \Phi(X, x_{12}, x_{34}) + \Phi(-X, -x_{34}, -x_{12}) \]
\[ = 2 \bar{\Phi}(X, x_{12}, x_{34}); \]

the last equation comes from eq. (2.3).

We may expand \( \phi_{\gamma \gamma}^{\text{fin}}(x_3|x_4) \) in the following way:
\[ \phi_{\gamma \gamma}^{\text{fin}}(x_3|x_4) = \sum_{\delta=1}^{16} \sum_{\mu=1}^{4} \phi^{\mu}_{\delta}(x_3|x_4) \gamma_{\delta}^{(\mu)} \gamma_{\gamma}^{(\mu)}, \]
(3.4)

where \( \gamma^{(\mu)} \)'s are linearly independent sixteen Dirac operators, and \( \gamma_{\gamma}^{(\mu)} \)'s are linearly independent four isospin operators. Substituting (3.4) in (3.3), the bracket in (3.3) becomes
\[ \sum_{\gamma} \sum_{j} \left[ -\bar{\phi}(x_1) \Omega_{\epsilon \gamma \gamma} \phi(x_3) \phi^{+}(x_3|x_4) \cdot Tr(\Omega_{\epsilon \gamma \gamma}^{(\mu)}) Tr(\tau \gamma) \right] \]
\[ + \phi(x_1) \Omega_{\epsilon \gamma \gamma}^{(\mu)} \Omega_{\epsilon \gamma \gamma}^{(\mu)} \tau \gamma \phi(x_3) \phi^{+}(x_3|x_4) \right]. \]
(3.3')

Using
\[ \tau T \tau (\tau \gamma_j) = \begin{cases} 2 \gamma_j & \text{for } j = 1, 2, 3, \\ 0 & \text{for } j = 4, \end{cases} \]
\[ \tau \gamma_j \tau = \begin{cases} -\gamma_j & \text{for } j = 1, 2, 3, \\ 3 & \text{for } j = 4, \end{cases} \]

we can write (3.1) as
\[ 2g_{\epsilon_4} \epsilon_i \int d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 \bar{\Phi}(X, x_{12}, x_{34}) \cdot \left[ -2 \phi(x_1) \Omega_{\epsilon \gamma \gamma} \phi(x_3) \phi^{+}(x_3|x_4) Tr(\Omega_{\epsilon \gamma \gamma}^{(\mu)}) \right]. \]

\(^*\) Although we use here the Bethe-Salpeter amplitude for its familiarity, it may be more appropriate to use the connected Feynman amplitude as has been discussed in detail by Maki.\(^{10}\) However, our discussions would also be meaningful in that case, if appropriately interpreted.
Pion-Nucleon Scattering and the Structure of the Pion

\[ -\bar{\psi}(x_1) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \phi_s(x_5|x_4) \]
\[ + 3\bar{\psi}(x_1) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \phi_s(x_5|x_4) \], (3.5)

where the bold-faced letter \( \phi_s \) means a vector in three dimensional isospin space.

Contributions from the second term of the right side of eq. (2.4) are given by (3.4) if one replaces \( \tau \) by \( \tau_s = 1 \) and multiplies by \( \eta_i \), resulting in

\[
2g_\varepsilon \cdot \eta_i \int d^4\bar{X} d^4X d^4x_{12} d^4x_{54} \bar{\Phi}(X, x_{12}, x_{54}) \cdot \sum_s \left[ -2\bar{\psi}(x_1) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \psi(x_5|x_4) \right] \psi(x_3) \psi(x_6) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \phi_s(x_5|x_4)
\]
\[ + \bar{\psi}(x_1) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \psi_s(x_5|x_4) \]
\[ + \bar{\psi}(x_1) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \phi_s(x_5|x_4) \]. (3.6)

Adding (3.5) and (3.6), one obtains

\[
\int H^{(1)}_{\text{eff}} d^4\bar{X} = 2g_\varepsilon \cdot \eta_i \int d^4\bar{X} d^4X d^4x_{12} d^4x_{54} \bar{\Phi}(X, x_{12}, x_{54}) \cdot \sum_s \left[ -2\bar{\psi}(x_1) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \psi(x_5|x_4) \right] \psi(x_3) \psi(x_6) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \phi_s(x_5|x_4)
\]
\[ + \bar{\psi}(x_1) \Omega \gamma^{(o)} \Omega \tau \psi(x_2) \psi_s(x_5|x_4) \}
\[ + (1 - \gamma_i) \bar{\psi}(x_2) \phi_s(x_5|x_4) \}
\[ + (3 + \gamma_i) \bar{\psi}(x_2) \phi_s(x_5|x_4) \}
\[ \psi(x_2) \]
\[ (3.7)
\]

where one may call \( H^{(1)}_{\text{eff}} \), the "effective Hamiltonian density" for emission or absorption of a pion.

We have, as yet, put no restriction on the transformation property of \( \phi_s \). We will retain, in the following, only the charge triplet function \( \phi_s \) and discard \( \phi_s \), considering it not to correspond to the pion. Due to the requirement that the Lagrangian be a scalar, the neutral pion field should transform as a "first kind pseudoscalar" in terms of Watanabe\(^{12}\), that is, the transformation rule is given by

\[
\phi \rightarrow \phi' = \sigma \phi \quad (x'_u \rightarrow x'_u),
\]
\[ (3.8)
\]

where

\[
\sigma = \partial (x', y', z', t') / \partial (x, y, z, t).
\]

Contrary to the case of the neutral pion field, we cannot infer any definite specific "kind" of the charged pion fields, because of the unknown difference of the phases which appear in the transformation of the neutron and the proton fields. We assume, however, that the charged pion fields are also first kind pseudoscalars.

Then, if \( \varphi \) is a pion field wave function itself, it should have the following transformation properties:

\[
\varphi \rightarrow \varphi' = \gamma_4 \varphi \gamma_4 = -\varphi \quad \text{for space reflection},
\]
\[ (3.9a)\]

and

\[
\varphi \rightarrow \varphi' = \gamma_5 \gamma_4 \varphi \gamma_5 = -\varphi \quad \text{for time reflection}.
\]
\[ (3.9b)\]

Therefore, we need to retain only \( \gamma_5 \) among sixteen \( \gamma^{(o)} \)'s. The \( \phi_s \) corresponding to \( \gamma_5 \) and \( \tau \) in eq. (3.4) will hereafter be denoted as \( \phi \).
\( \varphi \) could, in general, be derivatives of the wave function of the pion. We will consider the case of \( \varphi \) being the first derivative of the wave function of the pion, discarding the case of higher derivatives for simplicity. \( \varphi \) should, then, be a "first kind pseudovector" for the neutral pion field. We assume, as before, that \( \varphi \) has the same transformation property both for charged pion and for neutral pion. Then, only \( \gamma_5 \gamma_\mu \) of \( \gamma^{(\nu)} \) survives in eq. (3.4), and we denote the \( \phi_\mu \) in this case as \( \phi_\mu \). \( \phi_\mu \) is a first derivative with respect to certain space-time coordinate of \( \phi \) times a quantity with dimension of length, since \( \phi_\mu \) has the same dimension with \( \phi \).

Consequently, eq. (3.7) becomes

\[
\int H_{\text{eff}}^\gamma d^4 \overline{X} = 2 \int d^4 \overline{X} d^4 X d^4 x_{12} d^4 x_{34} \overline{\Phi} (X, x_{12}, x_{34}) \cdot \\
\cdot \left[ -2^5 g_i \overline{\psi}(x_i) \gamma_5 \tau \phi (x_2 | x_3) \psi (x_3) + 2^4 g_i \overline{\psi}(x_i) \gamma_5 \gamma_\mu \tau \phi_\mu (x_2 | x_3) \psi (x_3) \\
- g_i \epsilon \varepsilon (1 - \gamma_5) \overline{\psi}(x_i) \gamma_5 \tau \phi (x_2 | x_3) \psi (x_3) - g_i \epsilon \varepsilon (1 - \gamma_5) \overline{\psi}(x_i) \gamma_5 \gamma_\mu \tau \phi_\mu (x_2 | x_3) \psi (x_3) \right],
\]

(3.10)

where \( \xi, \xi' \) are defined by

\[
\Omega_\xi \gamma_5 = \xi \gamma_5, \quad \Omega_\xi \gamma_\mu \gamma_5 = \xi' \gamma_\mu \gamma_5,
\]

their numerical values being

\[
\xi = 1, -4, 6, -4, 1 \\
\xi' = 1, 2, 1, -2, -1
\]

for \( i = 1, 2, 3, 4, 5 \).

Eq. (3.10) is the desired "effective Hamiltonian" for emission or absorption of a pion, where the terms including \( \phi \) correspond to the pseudoscalar coupling in the local pion theory, and those including \( \phi_\mu \) to the pseudovector coupling. It should, however, be noted that \( \phi_\mu \) may be a derivative of \( \phi \) with respect to the relative coordinate of nucleon pair composing the pion as well as a derivative with respect to the center of mass coordinate.

To compare eq. (3.10) with the local pion theory, we consider the case of \( \gamma_5 = 1 \), \( g_4 = 0 \) and \( g_5 \neq 0 \) in a little more detail. In such a case eq. (3.10) takes a simple form

\[
\int H_{\text{eff}}^\gamma d^4 \overline{X} = -2^4 g_5 \int d^4 \overline{X} d^4 X d^4 x_{12} d^4 x_{34} \overline{\Phi} (X, x_{12}, x_{34}) \cdot \\
\cdot \overline{\psi}(x_i) \gamma_5 \tau \phi (x_2 | x_3) \psi (x_3).
\]

(3.11)

Substituting the following expressions in eq. (3.11):

\[
\overline{\Phi} (X, x_{12}, x_{34}) = (2\pi)^{-12} \int d^4 L d^4 q_{12} d^4 q_{34} \\
\cdot \exp i (L X + q_{12} x_{12} + q_{34} x_{34}) \cdot \hat{G} (L, q_{12}, q_{34}),
\]

\[
\cdot \exp i (L X + q_{12} x_{12} + q_{34} x_{34}) \cdot \hat{G} (L, q_{12}, q_{34}),
\]

\[
\cdot \exp i (L X + q_{12} x_{12} + q_{34} x_{34}) \cdot \hat{G} (L, q_{12}, q_{34}),
\]

where

\[
\hat{G} = G (L) \cdot \hat{G} (q_{12}) \cdot \hat{G} (q_{34}),
\]
Pion-Nucleon Scattering and the Structure of the Pion

\[ \bar{\psi}(x) = (2\pi)^{-4} \int d^4 p \cdot \exp (-ip\cdot x) \tilde{\psi}(p), \]  
\[ \psi(x) = (2\pi)^{-4} \int d^4 p \cdot \exp (ip\cdot x) \psi(p), \]  
\[ \phi(x_1|x_2) = (2\pi)^{-4} \int d^4 k d^4 q \cdot \exp (i k X_{24} - i q x_{24}) \phi(k, q), \]

where \( X_{24} = (x_2 + x_4)/2, x_{24} = x_2 - x_4 \), the "effective Hamiltonian" is given in the momentum representation by

\[ \int H_{el} d^4 \bar{X} = -2^4 (2\pi)^{-12} g_5 \int d^4 p' d^4 p d^4 k \delta^4(p+k-p'), \]
\[ \cdot \int d^4 q \tilde{G}(p + \frac{k}{2} + q, \frac{p'}{2} + \frac{q}{2}, -\frac{p}{2} + \frac{k}{2} + \frac{q}{2}) \cdot \tilde{\psi}(p') \gamma_5 \phi(k, q) \psi(p). \]  

In the local limit, \( \Phi(X, x_{12}, x_{56}) \rightarrow \delta^4(X) \delta^4(x_{12}) \delta^4(x_{56}) \) and \( \tilde{G}(L, l^{12}, l^{56}) \rightarrow 1 \), one obtains

\[ \int H_{el} d^4 \bar{X} = -2^4 (2\pi)^{-12} g_5 \int d^4 p' d^4 p d^4 k \delta^4(p+k-p') \cdot \tilde{\psi}(p') \gamma_5 \phi(k, q) \psi(p), \]

where

\[ \tilde{\phi}(k) = (2\pi)^{-4} \int d^4 q \phi(k, q) \]
\[ = (2\pi)^{-4} \int d^4 q \int d^4 r \exp (iqr) \phi(k, r) = \phi(k, r=0) \]  

is the amplitude of the nucleon pair existing at the same space-time point. \( \tilde{\phi}(k) \) is related to the ordinary local pion field operator, denoted as \( u(k) \), by the following equation

\[ \tilde{\phi}(k) = iN^{-1} u(k), \]  

where \( N \) is a quantity with dimension of the square of length and may depend on the detailed structure of the pion. We will not try to determine the value of \( N \) in this paper and only mention that Maki(5) has determined a similar quantity to the above \( N \) by solving the covariant integral equation for the composite pion in a very rough approximation. Ordinary dimensionless coupling constant, \( f' \), is given by

\[ f' = 2^4 N^{-1} g_5. \]

§ 4. Two pion vertex

The interaction kernel for the pion-nucleon propagator may be illustrated by a diagram in Fig. 5, if the pion is represented by the nucleon pair amplitude. In this section, we shall consider the processes given by Fig. 5 except the "three body" interaction. They are illustrated by the diagrams given by Fig. 6, and give rise to pion-nucleon
scattering matrix elements which include terms of first order in \( g_s \), contrary to the contribution from the one pion vertex, eq. (3·10), which are the second and higher orders in \( g_s \). The processes shown in Fig. 6 are similar to the deuteron stripping or pick up process, and one may treat the former processes in the same way as the latter\(^1\).

Although the two body interaction kernel, (2·1), is rather complex, the isospin dependence of the “effective Hamiltonian” for the processes in Fig. 6 is determined rather definitely by the charge independence hypothesis.

We will calculate the pion-nucleon scattering matrix elements of

\[
\int K d^4 x \cdot \overline{\psi}(x) \sigma(x) d^4 x,
\]

which is the interaction kernel of the three particle integral equation shown in Fig. 6, where \( K \) is the two-body kernel given by eq. (2·1), and \( \overline{\psi}(x) \sigma(x) d^4 x \) denotes symbolically the change of the state of a nucleon as shown by a straight line from bottom to top in Fig. 6. For example, in Fig. 6 (b) it represents a transition from a free nucleon state to a bound state.

By straightforward calculations similar to that in § 3, one obtains the “effective Hamiltonian density” for emission or absorption of two pions, \( H_{\text{eff}}^{(2)} \), as follows:

\[
\int H_{\text{eff}}^{(2)} d^4 x = g_s \varepsilon_1 \int d^4 \overline{\psi}(x) \sigma(x) \cdot d^4 x
\]

\[
\cdot \left[ (A_1) + (A_2) + (B_1) + (B_2) + (C_1) + (C_2) + (C_3) + (C_4) \right],
\]

where

\[
(A_1) = - (1 - \eta) \overline{\psi}(x) \mathcal{O} \gamma^{(0)} \mathcal{O} \gamma^{(4)} \left\{ \tau \phi^{(0)}_0 (x) \tau \phi^{(4)}_1 (x) \psi(x) \right\} \psi(x)
\]

\[
+ \tau \phi^{(4)}_0 (x) \tau \phi^{(0)}_1 (x) \psi(x) \cdot \frac{Tr(\mathcal{O} \gamma^{(4)} \mathcal{O} \gamma^{(0)})}{1 - \eta}.
\]

\[
(A_2) = - (1 - \eta) \overline{\psi}(x) \mathcal{O} \gamma^{(0)} \mathcal{O} \gamma^{(4)} \left\{ \tau \phi^{(0)}_0 (x) \tau \phi^{(4)}_1 (x) \psi(x) \right\} \psi(x)
\]

\[
+ \tau \phi^{(4)}_0 (x) \tau \phi^{(0)}_1 (x) \psi(x) \cdot \frac{Tr(\mathcal{O} \gamma^{(4)} \mathcal{O} \gamma^{(0)})}{1 - \eta}.
\]

\[
\left( B_1 \right) = \left( B_2 \right) = \left( C_1 \right) = \left( C_2 \right) = \left( C_3 \right) = \left( C_4 \right).
\]

\[
\]

* The fact that the contributions from Fig. 6 might not be so small even in low energy pion-nucleon scattering was first noted by Hayakawa\(^{13}\).
Pion-Nucleon Scattering and the Structure of the Pion

\[ -2 \bar{\psi}(x_2) \Omega \gamma^{(\nu)} \{ \tau \phi^*_\mu(x_1|x_2) \psi(x_2|x) \cdot \]

+ \bar{\psi}_\mu(x_3|x_1) \tau \phi^*_\nu(x_2) \} \psi(x) \cdot Tr(\Omega \gamma^{(\mu)}) ,

\begin{align*}
(B_1) &= -(1 - \gamma_\mu \gamma_\nu) \bar{\psi}(x) \gamma^{(\nu)} \Omega \gamma^{(\nu)} \{ \tau \phi^*_\mu(x_1|x_2) \\
&+ \bar{\psi}_\mu(x_3|x_1) \tau \phi^*_\nu(x_2) \} \psi(x) \cdot Tr(\Omega \gamma^{(\mu)}) ,
\end{align*}

\begin{align*}
(B_2) &= -(1 - \gamma_\mu \gamma_\nu) \bar{\psi}(x) \gamma^{(\nu)} \Omega \gamma^{(\nu)} \{ \tau \phi^*_\mu(x_1|x_2) \\
&+ \bar{\psi}_\mu(x_3|x_1) \tau \phi^*_\nu(x_2) \} \psi(x) .
\end{align*}

\begin{align*}
(C_1) &= -(1 - \gamma_\mu \gamma_\nu) \bar{\psi}(x) \gamma^{(\nu)} \Omega \gamma^{(\nu)} \{ \tau \phi^*_\mu(x_1|x_2) \\
&+ \bar{\psi}_\mu(x_3|x_1) \tau \phi^*_\nu(x_2) \} \psi(x) \cdot \psi(x) ,
\end{align*}

\begin{align*}
(C_2) &= -2 \bar{\psi}_\mu(x_1) \Omega \tau \phi^*_\mu(x_2|x) \times \phi^*_\mu(x_2|x) \\
&+ \phi^*_\mu(x_2|x) \times \psi(x_3) \tau \phi^*_\mu(x_2|x) \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu)})
\end{align*}

\begin{align*}
&-2 \bar{\psi}(x_1) \Omega \gamma^{(\mu')} \gamma^{(\nu')} \gamma^{(\nu')} \{ \phi^*_\mu(x_1|x_2) \psi_\nu(x_2|x) \\
&+ \phi^*_\nu(x_1|x_2) \phi_\mu(x_2|x) \} \cdot \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu')}) \\
&+ \phi^*_\mu(x_1|x_2) \phi_\mu(x_2|x) \} \cdot \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu')})
\end{align*}

\begin{align*}
&-2 \bar{\psi}(x_1) \Omega \gamma^{(\mu')} \gamma^{(\nu')} \gamma^{(\nu')} \{ \phi^*_\mu(x_1|x_2) \psi_\nu(x_2|x) \\
&+ \phi^*_\nu(x_1|x_2) \phi_\mu(x_2|x) \} \cdot \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu')}) \\
&+ \phi^*_\mu(x_1|x_2) \phi_\mu(x_2|x) \} \cdot \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu')})
\end{align*}

\begin{align*}
&+ 4 \bar{\psi}(x_1) \Omega \gamma^{(\mu')} \gamma^{(\nu')} \gamma^{(\nu')} \{ \phi^*_\mu(x_1|x_2) \psi_\nu(x_2|x) \\
&+ \phi^*_\nu(x_1|x_2) \phi_\mu(x_2|x) \} \cdot \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu')}) \\
&+ \phi^*_\mu(x_1|x_2) \phi_\mu(x_2|x) \} \cdot \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu')})
\end{align*}

\begin{align*}
&+ 4 \bar{\psi}(x_1) \Omega \gamma^{(\mu')} \gamma^{(\nu')} \gamma^{(\nu')} \{ \phi^*_\mu(x_1|x_2) \psi_\nu(x_2|x) \\
&+ \phi^*_\nu(x_1|x_2) \phi_\mu(x_2|x) \} \cdot \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu')}) \\
&+ \phi^*_\mu(x_1|x_2) \phi_\mu(x_2|x) \} \cdot \psi(x_3) \cdot Tr(\Omega \gamma^{(\mu')})
\end{align*}

and \( \gamma^\nu \)'s are \( \gamma_\mu \) or \( \gamma_\mu \gamma_\nu \). Since \( \phi^i \) does not correspond to the pion, it has been omitted in the above equations, and the same assumption on the properties of \( \varphi \) as in \( \S \) 3 is made.

In the case of treating the low energy pion phenomena, we shall be allowed to approximate the form factor \( \Phi \) by a \( \delta \)-function, and replace the wave function \( \phi_\mu(x|x') \),
of the pion by $\phi_s(x|\lambda)$, since the spatial extent of the internal wave function of the pion would be very small compared with the de Broglie wave length of the pion. All terms with $s=s'$, then, give isospin independent contributions, and terms with $s \neq s'$ and containing $\tau$ explicitly give isospin dependent contributions. Putting $\gamma_i=1$ for simplicity, and retaining terms with $s \neq s'$, one has

$$g_i e_i \times (A_i) \approx -2g_0 \phi^*_i(x) \left\{ \phi(x|\lambda) \right. \tau \phi_s^*(x|\lambda) \left. \phi(x) \right\} \psi(x)$$

$$+ 2g_0 \phi^*_i(x) \left\{ \phi(x|\lambda) \right. \tau \phi_s^*(x|\lambda) \left. \phi(x) \right\} \psi(x)$$

where space-time extent of the internal wave function of the pion and also that of the interaction kernel were neglected. Discarding the odd Dirac operators, the above equation turns to

$$g_i e_i \times (A_i) \approx -2g_0 \phi^*_i(x) \left( \phi \cdot \tau + i \tau \cdot [\phi \times \tau] \right) \phi(x)$$

$$+ 2g_0 \phi^*_i(x) \left( \phi \cdot \tau + i \tau \cdot [\phi \times \phi] \right) \phi(x)$$

(4.3)

where $\tau$ is the conjugate momentum to $\phi$. The $\tau \cdot [\tau \times \phi]$ terms give rise to the desired isospin dependence of low energy S-wave pion-nucleon scattering phase shifts. Similar results are obtained from the remaining terms in eq. (4.2). These terms arise from the terms with $i=2$, 4, and 5 in the interaction kernel (2.1).

To calculate the low energy S-wave scattering phase shifts, one also needs to know the magnitude of isospin independent parts. These terms have about the same magnitudes as that in eq. (4.2). Moreover, contributions from the one pion vertex, eq. (3.10), must be calculated, which are at least of second order in $g_i$, contrary to the two pion vertex obtained in this section. Relative magnitudes of the contributions from these two processes are not so easy to be evaluated, since it needs the more detailed information about the structure of the pion as was briefly mentioned at the end of § 3. According to a very rough estimation such as given by Finkelstein, i.e., the order of magnitudes of $g_i$'s being $10^{-42} \sim 10^{-44}$ erg cm$^3$, contributions from the two pion vertex derived in this section may not necessarily be neglected compared with that from the repetition of the one pion vertex.

Next, we give Fourier transform of eq. (4.1). It is sufficient to consider only one term in eq. (4.1) explicitly. Putting $\gamma_i=1$, $\gamma^{(o)} = \gamma^{(e)} = \gamma_5$, the contribution to $\int H^{(p)}_0 d^4X$ from $(A_i)$ is

$$-2g_0 \int d^4X d^4X' d^4x_{12} d^4x_{34} d^4x \Psi(X, x_{12}, x_{34}) \cdot$$

$$\left\{ \tau \phi(x_1|\lambda) \tau \phi^*(x_2|\lambda) + \tau \phi^*(x_3|\lambda) \tau \phi(x_4|\lambda) \right\} \psi(x).$$

(4.5)

Substituting eq. (3.12) in the above expression, one obtains
Pion-Nucleon Scattering and the Structure of the Pion

\[
-2^5 g_\pi (2\pi)^{-1} \int d^4p \, d^4p' \, d^4k \, d^4k' \, \delta^4 (p+k-p'-k') \cdot \\
\cdot \bar{\psi} (p') \left\{ \left[ \int d^4q \cdot G_1 \tau \phi (k, q) \right] \cdot \tau \phi^* (k', p - \frac{k'}{2}) \right. \\
+ \left. \left[ \int d^4q' G_2 \tau \phi^* (k', q') \right] \cdot \tau \phi (k, -p - \frac{k}{2}) \right\} \psi (p),
\]

where

\[
G_1 = G \left( \frac{p' - k}{2} + q, \frac{p' + k}{2} - \frac{q}{2}, \frac{-p}{2} + \frac{k}{2} + \frac{q}{2} \right),
\]

\[
G_2 = G \left( \frac{p' + k'}{2} - q', \frac{p' - k'}{2} + \frac{q'}{2}, \frac{-p}{2} - \frac{k}{2} - \frac{q}{2} \right).
\]

In the local limit, eq. (4·6) turns to

\[
-2^5 g_\pi (2\pi)^{-1} \int d^4p \, d^4p' \, d^4k \, d^4k' \delta^4 (p+k-p'-k') \cdot \\
\cdot \bar{\psi} (p') \left\{ \bar{\phi} (k) \phi^* (k', p - \frac{k'}{2}) + \bar{\phi}^* (k') \phi \left( k, -p - \frac{k}{2} \right) \right\} \psi (p),
\]

where \( \bar{\phi} (k) \), being given by eq. (3·15), represents the amplitude for a nucleon and an antinucleon to be at the same space-time point.

If one is considering the low energy pion phenomena, the momentum of the center of mass and the relative momentum will be approximately separable, since effects of the Lorentz contraction will be small. We, then, have

\[
\phi (k, q) \approx \bar{\phi} (k) v (q),
\]

where \( v (q) \) is the internal wave function of the pion at rest in the momentum representation, and is normalized according to

\[
(2\pi)^{-1} \int d^4q v(q) = 1.
\]

Apparently there is no isospin dependent terms in eq. (4·7) in this approximation, and this fact corresponds to that in \( x \)-representation shown below eq. (4·2).

§ 5. Conclusions

We have discussed the effects of the nucleon-antinucleon amplitude in the physical pion on the pion-nucleon scattering, having started from the interaction kernel. Usually they have started their discussions from the Lagrangian, which determines the "dynamics" of the system, when they want to attack such a problem as a pair effect. However, it
seems very hard to get a lucid insight if we start from the Lagrangian and does not separate the "kinematical" and "dynamical" aspects of the problem as in the conventional way. Therefore we assumed a formal expression of the interaction kernel for the relativistic integral equation of the nucleon-antinucleon system as given in § 2, which is general enough to cover both the compound pion theory and the conventional local pion theory. We have derived the "effective Hamiltonian" for the emission or absorption of one or two pions in § 3 and in § 4 respectively, neglecting for the latter the "three body" interaction. One can evaluate immediately their effects to the low energy pion-nucleon scatterings quantitatively, if the interaction kernel is given explicitly.

A very rough estimate shows that the "effective Hamiltonian" for the two pion vertex, which is characteristic of the structure of the pion, may not necessarily be neglected even in the low energy pion-nucleon scattering and may give a rather large isospin dependent phase shifts. It is qualitatively in agreement with experiment. We are, however, not able to discuss these points quantitatively, unless we go into the dynamics. Such a problem has not been treated but to be postponed to a future paper.

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Appendix. Crossing symmetry for the scattering of a particle with internal structure

We consider the pion to be composed of the same number of nucleons and antinucleons. In a relativistically invariant local field theory, the propagator for the two nucleon and one antinucleon system may be written as $^9$

$$G(x_1, x_2, x_3; x'_1, x'_2, x'_3) = i\langle T(\phi(x_1)\phi(x_2)\phi'(x_3); \phi(x'_1)\phi'(x'_2)\phi'(x'_3))\rangle$$

$$-\langle T(\phi(x_1)\phi(x_2)\phi'(x_3))\phi'(x'_1)\phi'(x'_2)\phi'(x'_3)\rangle + \langle T(\phi(x_1)\phi(x_2)\phi'(x_3))\phi'\phi'(x'_1)\phi'(x'_2)\phi'(x'_3)\rangle$$

where $\phi(x)$ is the Heisenberg operator for the nucleon field, $\phi'(x)$ is its charge conjugate, and $\langle T(\ )\rangle$ means the true vacuum expectation value of the $T$-product; the second and third terms in the right side of eq. (A.1) merely correct for the fact that the uncoupled nucleon-antinucleon system cannot undergo a virtual annihilation process.

Let us suppose that the nucleon pair $(\phi(x_2), \phi'(x_3), \phi(x'_2), \phi'(x'_3))$ compose a pion, then the pion-nucleon propagator may be written as

$$T_{ij}(p, p'; q, q') = i\int d^4x_i d^4x'_i d^4y d^4y' \exp i(px_i - p'x'_i + qy - q'y') \cdot$$

$$\cdot \int d^4z d^4z' \chi_i(y, z) G(x_1, x_2, x_3; x'_1, x'_2, x'_3) \chi_j(y', z'),$$

(A.2)
Pion-Nucleon Scattering and the Structure of the Pion

where \( p', q' \) are the nucleon and pion incoming four momenta, respectively, and \( p, q \) are the outgoing momenta, \( \gamma = (x_2 + x_3)/2 \), \( z = x_2 - x_3 \), \( \gamma' = (x_2' + x_3')/2 \), \( z' = x_2' - x_3' \), and \( \chi_i(\gamma, z) \), \( \chi_j(\gamma', z') \) are the wave functions of the incoming and outgoing pions, respectively.

Substituting (A·1) in eq. (A·2) we obtain

\[
T_{ij}(p, p'; q, q') = \int d^4x_1 d^4x'_1 d^4y d^4y'.
\]

\[
\exp \left( i(p_1 - p'_1 + q_1 - q'_1) \right) \cdot \int d^4x d^4x' \langle T(\psi(x_1) \bar{\psi}(x'_1) \phi_i(y, z) \phi_j(y', z')) \rangle,
\]  

(A·3)

where

\[
\phi_i(y, z) = \chi_i(\gamma, z) \psi(x_2) \psi'(x_3),
\]

\[
\phi_j(y', z') = \chi_j(\gamma', z') \bar{\psi}(x_2') \bar{\psi}'(x_3').
\]

In eq. (A·3) we write only the first term of eq. (A·1) for simplicity. However, it will be seen easily that the following discussions hold also in the case that the omitted terms are included.

Since \( \phi_i(y, z) \) and \( \phi_j(y', z') \) commute inside a T-product, and since \( \gamma \) and \( \gamma' \) are dummy variables, the right side of eq. (A·3) is equal to

\[
T_{ij}(p, p'; q, q') = \int d^4x_1 d^4x'_1 d^4y d^4y'.
\]

\[
\cdot \exp \left( i(p_1 - p'_1 - q_1 - q'_1 + qy - q'y') \right) \cdot \int d^4x d^4x' \langle T(\psi(x_1) \bar{\psi}(x'_1) \phi_i(y, z) \phi_j(y', z')) \rangle
\]

\[
= T_{ij}(p, p'; -q', -q).
\]  

(A·4)

Consequently, the crossing symmetry holds also for the case of the scattering of a compound particle, provided that the propagator, (A·1), exists. Clearly, the proof given above, which is essentially the same as that given by Feldman and Matthews, can be valid, even if the incoming and outgoing particles are composed of arbitrary number of fermion pairs and bosons. If the propagators such as eq. (A·1) do not exist, we cannot say whether the crossing symmetry holds or not.
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