On the Theory of the Unstable Particle, II

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A model which is more convenient than the Lee model for studying the theory of the unstable particle is proposed. In this model, the decayed particles are distinguished from the parental ones. The production and decay of an unstable particle is investigated by the stationary treatment. The conclusion we reached is that it is possible to interpret the renormalization constant as the probability in the same way as in the case of the stable particles.

Recently, many authors investigated the theory of the unstable particle by using the Lee model. It seems, however, that we have not yet found any satisfactory way of judging whether we can interpret the renormalization constant as representing the probability for finding the bare state in a physical unstable particle. The cause which makes this question complicated is due to the finite lifetime of the unstable particle and the uncertainties originated from this lifetime. It is worth while to make a research of the properties of physical unstable particles in order to answer the above mentioned question. In the previous paper of the present author, some arguments were made concerning this problem by using the Lee model. This model was, however, inconvenient for studying this problem because of the fact that the parental particles which produce the unstable V-particles are identical with the particles to which the V's decay. In the stationary treatment of the decay process, if the plural particles are produced by some collision, we can discuss the stability of the produced particles by introducing the relative co-ordinate of these particles, and by investigating the behaviour of the wave amplitude of the unstable particle without taking into account the decayed products. In this case, it is easily seen that the final particles produced from the unstable one should necessarily be different from the first particles from which the plural particles (at least one of them is unstable) were produced.

In order to distinguish the decayed particles from the parental ones, we shall consider, following Lee, the four kinds of fictitious fermions, $N_1$, $N_2$, $N_3$, $N_4$, together with three kinds of bosons, $\theta_1$, $\theta_2$, and $\theta_3$, with the mutual interaction defined by the total Hamiltonian:

$$H = H_1 + H_2,$$

where

$$H_1 = \sum_{\ell=1}^{4} m_{\ell} \int \! dp \, \psi_\ell^*(p) \psi_\ell(p) + \sum_{\ell=1}^{3} \int \! dk \, a_\ell^*(k) a_\ell(k),$$

and

$$H_2 = (2\pi)^{-3/2} \int \! dp dk \, \frac{1}{\sqrt{2\omega_2}} \left( g_1 f_1(\omega_2) \psi_2^*(p) \psi_1(p-k) a_1(k) + g_2 f_2(\omega_2) \psi_2^*(p) \psi_2(p-k) a_2(k) + g_3 f_3(\omega_2) \psi_2^*(p) \psi_3(p-k) a_3(k) \right).$$
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$+g_3 f_3(\omega_k) \phi_3^*(p) \phi_3(p-k) a_3(k) + g_3 f_3(\omega_k) \phi_4^*(p) \phi_4(p-k) a_4(k)$

Here, $m_i$ is the unrenormalized mass of the $i$-th fermion and the masses of bosons are, for simplicity, taken to be equal to each other. The magnitude of the mass of each physical particle is assumed in such a way that $N_3$ is unstable and can decay into $N_4 + \theta_3$ while others are stable ones.

In the present paper, we shall investigate the behaviour of $N_3$ associated with $\theta_3$ produced by an $N_1 - \theta_1$ collision and discuss the relationship between the renormalization constant and the probability for finding the bare $N_3$ in its physical state in a similar way to that used in the case of stable particles.

The model stated above has the following six constants concerning the numbers of particles:

$$N_1 + N_2 + N_3 + N_4 = \text{constant},$$
$$\theta_1 + N_2 + N_3 + N_4 = \text{constant},$$
$$N_1 + N_2 + N_3 + \theta_3 = \text{constant},$$
$$\theta_1 + N_2 + N_3 + \theta_3 = \text{constant},$$
$$N_1 + N_2 + \theta_2 = \text{constant},$$
$$\theta_1 + N_2 + \theta_2 = \text{constant},$$

where $N_i$ or $\theta_i$ is the total number of $N_i$- or $\theta_i$-particle. Accordingly, the stationary state describing the $N_i - \theta_i$ collision can be expressed as

$$|Z\rangle = \left( \int dk \alpha(k) \phi_1^*(p-k) a_1^*(k) + \beta \phi_2^*(p) \int dk \gamma(k) \phi_3^*(p-k) a_3^*(k) a_5^*(k') \right) |0\rangle. \quad (1)$$

The eigenvalue $E$ of this state is given by

$$F_1 + \omega_k - E \right) \alpha(k) + (2\pi)^{-3/2} \int \frac{dk f_1(\omega_k)}{\sqrt{2} \omega_k} \beta = 0,$$

$$(m_2 - E) \beta + (2\pi)^{-3/2} \int \frac{dk g_2 f_2(\omega_k) \alpha(k)}{\sqrt{2} \omega_k} (2\pi)^{-3/2} \int \frac{dk g_4 f_4(\omega_k) \gamma(k)}{\sqrt{2} \omega_k} = 0,$$

$$F_3 + \omega_k - E \right) \gamma(k) + (2\pi)^{-3/2} \int \frac{dk g_3 f_3(\omega_k)}{\sqrt{2} \omega_k} \beta$$

$$+ (2\pi)^{-3/2} \int \frac{dk' g_5 f_5(\omega_k)}{\sqrt{2} \omega_k'} \lambda(k, k') = 0$$

and

$$(m_4 + \omega_{kr} - E) \lambda(k, k') + (2\pi)^{-3/2} \frac{g_3 f_3(\omega_k')}{\sqrt{2} \omega_{kr}} \gamma(k) = 0.$$
We solve eqs. (2) with the boundary condition that \( \alpha(k) \) represents the incident plane wave plus the outgoing spherical wave and \( \lambda(k,k') \) the outgoing one. After the straightforward calculations, we get

\[
\lambda(k,k') = - (2\pi)^{-\frac{3}{2}} g_2 g_1 (\omega_{k}) \frac{f_2(\omega_{k})}{\sqrt{2} \omega_{k}} \gamma(k) (m_4 + \omega_k + \omega_{k'} - E - i\epsilon)^{-1},
\]

(3)

\[
\gamma(k) = - (2\pi)^{-3} g_1 f_2 (\omega_{k}) \beta \left( m_3 - \gamma (E - \omega_k + i\epsilon) \right)^{-1},
\]

(4)

\[
\alpha(k) = \delta(k - k_0) - (2\pi)^{-3} g_1 f_2 (\omega_{k}) \beta \left( m_1 + \omega_k - E - i\epsilon \right)^{-1}
\]

(5)

and

\[
\beta = - (2\pi)^{-3} g_1 f_2 (\omega_{k}) \left[ m_3 - E - (2\pi)^{-\frac{3}{2}} \int dk \frac{g_2^2 f_2^2 (\omega_k)}{2 \omega_k} \left( m_3 - \gamma (E - \omega_k + i\epsilon) \right)^{-1} \right]^{-1},
\]

(6)

where

\[
\gamma(x) = x + (2\pi)^{-3} \int dk \frac{g_2^2 f_2^2 (\omega_k)}{2 \omega_k} \frac{1}{m_4 + \omega_k - x}
\]

(7)

and

\[
m_1 + \omega_{k_0} - E = 0.
\]

When \( N_3 \) is unstable, \( m_3 - \gamma (E - \omega_k + i\epsilon) \) has no zero point in the real value of \( E - \omega_k \) (cf. Fig. 1). Therefore in solving the last two equations in (2) with respect to \( \gamma(k) \), there occurs no ambiguity in dividing the both sides of the above equations by the factor \( m_3 - \gamma (E - \omega_k + i\epsilon) \), namely, we need no boundary condition for determining the \( \gamma(k) \) provided that \( \lambda(k,k') \) is determined by its boundary condition. This situation is due to the fact that \( |\gamma(k)|^2 \) represents the probability of finding the state of \( N_3 \) and \( \theta_3 \) traveling away from the source and the wave function in the co-ordinate space corresponding to the above \( \gamma(k) \) rapidly vanishes in the remote point from the source as it should be in the case of the unstable state. On the contrary, in the case of stable particle, since the factor \( m_3 - \gamma (E - \omega_k + i\epsilon) \) has a real zero point, we need a boundary condition for removing the ambiguity occuring in the formal solution of \( \gamma(k) \).

To investigate the production of the unstable \( N_3 \), let us examine the behaviour of \( \gamma(x) = \int dk \exp(ik \cdot x) \gamma(k) \) for the sufficiently large \( r = |x| \). A short calculation leads to

\[
\gamma(x) = (2\pi)^{-1} g_2 g_1 \int_{-\infty}^{\infty} kdke^{ik \cdot r} \frac{f_2(\omega_k)}{\sqrt{2} \omega_k} \left( m_3 - \gamma (E - \omega_k + i\epsilon) \right)^{-1}.
\]

(8)
The path of this integral is shown in Fig. 2.

In order to discuss the behaviour of \( \gamma(x) \), it is of use to mention about the functional properties of \((m_3 - \gamma(z))^{-1}\). The function \((m_3 - \gamma(z))^{-1}\) has one branch point on the real axis of \(z\)-plane corresponding to the creation of the \(\theta\)-meson. This situation forces us to put a cut on the \(z\)-plane from the branch point \(z_0\) to \(\infty\) or alternatively a cut on the complex \(k\)-plane from \(-k_0\) to \(+k_0\) along the real axis, where \(z = E - \sqrt{k^2 + \mu^2}\).

Besides these branch points, according to the paper of Araki et al., if \(E\) is sufficiently large, the function \((m_3 - \gamma(E - \omega_k + i\epsilon))^{-1}\) (the imaginary number \(i\epsilon\) is necessary for determining the Riemannian plane) has a pole \(k_1 + i k_2 (k_1 > 0, k_2 > 0)\) and another \(-k_1 - i k_2\) on the \(k\)-plane where the former can be connected with the latter by a curve crossing the cut (cf. Fig. 2).

The pole on the \(E - \omega_k\) plane corresponding to the above mentioned poles is \(M - i\Gamma/2\) \((M > 0, \Gamma > 0)\) which is given by

\[
M = \frac{i}{2\pi} \int d\omega_k \frac{g_3^2 f_3^2(\omega_k)}{\omega_k} \frac{1}{m_3 + \omega_k - M + (i\Gamma/2)} + \frac{i}{2\pi} \sqrt{(M - (i\Gamma/2) - m_3)^2 - \mu^2} g_3^2 f_3^2 (M - (i\Gamma/2) - m_3) = m_3. \tag{9}
\]

We assume that the magnitude of \(\Gamma\) is much smaller than \(M\), since \(M\) and \(\Gamma\) are the mass and the reciprocal of the lifetime of the \(N_3\)-particle respectively.

For a sufficiently large \(r\), owing to the rapid oscillation of \(\exp(ikr)\), the integrand for a large \(k\) has no contribution to \(\gamma(x)\). In the vicinity of the pole \(k_1 + i k_2\), the integrand approximately has a form as

\[
\frac{\chi}{k - (k_1 + i k_2)} e^{ikr} = \frac{\chi}{\sqrt{(k - k_1)^2 + k_2^2}} e^{i(kr + \phi)},
\]

where

\[
\chi = \sqrt{\frac{E - M + i\Gamma/2}{2}} f_2 \left(E - M + \frac{i}{2} \Gamma\right) \left[1 + (2\pi)^{-3} \int dk \frac{g_3^2 f_3^2(\omega_k)}{2\omega_k} \frac{1}{(m_3 + \omega_k - M + i\Gamma/2)^2}\right]^{1/2}
\]

\[
+ \frac{i}{2\pi} \frac{\partial}{\partial M} \sqrt{(M - i\Gamma/2 - m_3)^2 - \mu^2} g_3^2 f_3^2 (M - \frac{i}{2} \Gamma - m_3) \left[1 - i(k - k_1)\right]^{-1}
\]

and

\[
\tan \phi = k_2/(k - k_1).
\]

The phase change of the above expression at the point \(k \approx k_1\) is approximately
\[ \Delta k \cdot r + \Delta \phi = \Delta k \left( r - \frac{1}{k_3} \right) \]

where \( \Delta k \) is the variation of \( k \) at the point \( k \approx k_1 \). In order to get a non-vanishing contribution to \( \gamma(x) \), the phase change should be much smaller than unity. Accordingly, from the above expression of \( \Delta \phi \), we see that \( \gamma(x) \neq 0 \) if \( r \lesssim k_3^{-1} \) and \( \Delta k \lesssim k_2 \), while if \( \Delta k \gg k_2 \) or \( r \gg k_2^{-1} \) the strong cancellation of the integration occurs by virtue of the rapid oscillation of the phase factor. Therefore the path of the integral can be deformed in such a way as

\[ c_1 + c_2 \rightarrow c_1 + c_2' + c_3 \]

if \( k_0 \approx \Delta k \approx k_0 \) (cf. Fig. 2). It is easily seen that the contribution from the path \( c_1 + c_2' \) vanishes. Thus we have

\[ \gamma(x) \approx -\sqrt{2\pi} \beta g_2 \chi \frac{e^{ikr} e^{-kx}}{r} \]  

(10)

for a large \( r \). In the same way, the asymptotic expression of \( \lambda(x, k) \) is

\[ \lambda(x, k) \approx -(\pi \beta g_3 f_3(\omega_k) \omega_k) \frac{\gamma(k)}{r} e^{ikr} \]

\[ -\gamma(x) \left( 2\pi \right)^{-3/2} \frac{g_3 f_3(\omega_k)}{\sqrt{2\omega_k}} \left( m_4 + \omega_k - M + \frac{i}{2} \Gamma \right)^{-1} \]  

(11)

where

\[ \lambda(x, k) = \int dk' \exp(i k' \cdot x) \lambda(k', k) \]

and \( k_3 \) is the positive root of \( m_4 + \omega_k + \omega_k - E = 0 \).

In the case of the stable \( N_3 \), where the damping factor \( \exp(-kx) \) does not occur, it is obvious how to interpret the each term of the asymptotic expressions of \( \gamma \) and \( \lambda \). Namely, the \( \gamma(x) \) describes the state of the bare \( N_3 \) associated with \( \theta_3 \), while the second term in (11) that of the dressed \( N_3 \), the sum of these corresponds to the state of the physical \( N_3 \) plus \( \theta_3 \). The first term of (11) stands for the outgoing wave of \( \theta_3, \theta_3 \) and \( N_3 \). It may be quite reasonable to assume that the completely similar interpretation is also possible in the case of the unstable \( N_3 \). From this interpretation the probability of finding the \( \theta_3 \)-meson at the point \( x(r \gtrsim k_2^{-1}) \) and at the same time \( N_3 \) at the origin is given by

\[ P(x) = \left( 1 + (2\pi)^{-3} \int dk \frac{g_3 f_3^2(\omega_k)}{2\omega_k} \frac{1}{(m_4 + \omega_k - M)^2 + (\Gamma^2/4)} \right) |\gamma(x)|^2. \]  

(12)

Owing to the damping factor in \( \gamma(x) \), the probability (12) decrease for increasing \( r \) as we expect in the case of the unstable particle.

The probability lost from \( P(x) \) as the point \( x \) goes away may be gained by the state of \( \theta_3, \theta_3 \) and \( N_4 \). That this is the case is seen as follows. Let us put
\[ \lambda(x, k) = \lambda_1(x, k) - \gamma(x) \left( 2\pi \right)^{-\frac{3}{2}} \frac{g_5 f_5(\omega_k)}{\sqrt{2\omega_k}} \frac{1}{m_4 + \omega_k - M - (i\Gamma/2)} \]

and consider the expression
\[ |\gamma(x)|^2 + \int dk |\lambda(x, k)|^2 = \int dk |\lambda_1(x, k)|^2 \]
\[ -2\text{Re} \int dk \left( \lambda_1(x, k) \left( 2\pi \right)^{-\frac{3}{2}} \frac{g_5 f_5(\omega_k)}{\sqrt{2\omega_k}} \frac{1}{m_4 + \omega_k - M - (i\Gamma/2)} \gamma^*(x) \right) + P(x). \] 

(13)

Now \( \lambda_1(x, k) \) can be rewritten as
\[ -\pi^{\frac{1}{2}} \frac{g_5 f_5(\omega_k)}{\sqrt{\omega_k}} \int_0^\infty dk' \frac{g_5 f_5(\omega_{k'})}{\omega_{k'}} \frac{k' \gamma(k') \delta(\omega_k + \omega_{k'} + m_4 - E)}{E - m_4 - (i\Gamma/2)} \frac{e^{ik' \cdot r}}{r}. \]

Inserting this expression into the second term of (13), we have

the second term of (13)
\[ = \left( 2\pi \right)^{-\frac{1}{2}} \text{Re} \left( \int dk \int_0^\infty dk' \frac{g_5^2 f_5^2(\omega_k)}{\omega_k} \frac{k' \gamma(k') \delta(\omega_k + \omega_{k'} + m_4 - E)}{m_4 + \omega_k - M - (i\Gamma/2)} \frac{e^{ik' \cdot r}}{r} \gamma^*(x) \right) \]
\[ = 2\text{Re} \left( \int_0^\infty dk' k' \gamma(k') \frac{g_5^2 f_5^2(E - m_4 - (i\Gamma/2))}{E - \omega_{k'} - M - (i\Gamma/2)} \frac{\sqrt{E - m_4 - (i\Gamma/2)} - \mu^2}{\mu^2} \frac{e^{ik' \cdot r}}{r} \gamma^*(x) \right), \] 

(14)

where \( a = \sqrt{(E - m_4 - \mu)^2 - \mu^2} \). Now recalling the strong cancellation occurred in the evaluation of (10), we see that the domain of integration in (14) can be extended from \(-\infty\) to \(+\infty\) without causing any change of the value of (14), if a pole of the integrand does not exist in the vicinity of the path of integral added. In the case of the stable \( N_3 \), since the integrand of (14) has no pole in the domain \((0, a)\) we see that (14) vanishes owing to the above mentioned cancellation without extending the domain of integral. On the other hand, if \( N_3 \) is unstable, we have a pole in the domain \((0, a)\) which give the result

the second term of (13)
\[ = -\frac{2}{\Gamma} \text{Re} \left( \frac{1}{2\pi} g_5^2 f_5^2(M - (i\Gamma/2) - m_4) \sqrt{M - m_4 - (i\Gamma/2)^2 - \mu^2} \right) |\gamma(x)|^2 \] 

(15)

by following the line of reasoning stated in (10).

Now the imaginary part of (9) leads to

\[ \frac{\Gamma}{2} \left( 1 + (2\pi)^{-3} \int dk \frac{g_5^2 f_5^2(\omega_k)}{2 \omega_k} \frac{1}{(m_4 + \omega_k - M)^2 + (\Gamma^2/4)} \right) \]
\[ = \text{Re} \left( \frac{1}{2\pi} g_5^2 f_5^2(M - (i\Gamma/2) - m_4) \sqrt{M - (i\Gamma/2) - m_4} - \mu^2 \right). \]

(16)

Inserting (12), (15) and (16) into (13), we see that the second and third terms in
cancel each other. This cancellation is nothing but the conservation of the total probability.

From (12), the probability of finding the bare \( N_3 \)-state in the physical \( N_3 \)-particle is written as

\[
N^2 = \left( 1 + (2\pi)^{-3} \int d\mathbf{k} \frac{g_3^2 f_3^2(\omega_k)}{2\omega_k} \cdot \frac{1}{(m_4 + \omega_k - M)^2 + (I^2/4)} \right)^{-1}.
\]

(17)

We can easily confirm that \( N \) is taken as the renormalization constant for \( g_3 \). The expression (17) essentially depends on the method of approximation in evaluating \( \gamma(x) \). Accordingly a slight difference in the expression \( N \) or \( I' \) in the behaviour of \( \gamma(x) \) can be expected when another method of approximation would be used, but we can say that the difference may be of the order of \( I'/M \) and impossible to detect (cf. Araki et al.). Anyhow, it must be noted that the final expressions for \( N \) and \( P(x) \) have the type (17) and (12) respectively without regard to the kinds of approximations owing to the equation (3). Therefore, we arrive at the conclusion that, even in the case of the unstable particle, we can interpret the renormalization constant as the probability.

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