Propagating and incorporating the error in anisotropy-based inclination corrections

Dario Bilardello,¹ Josef Jezek² and Kenneth P. Kodama³

¹Department of Earth and Environmental Sciences, LMU, Theresienstrasse 41, 80333 Munich, Germany; E-mail: dabc@lehigh.edu
²Institute of Applied Mathematics and Information Technologies, Faculty of Science, Charles University, Albertov 6, 128 43 Praha 2, Czech Republic
³Department of Earth and Environmental Sciences, Lehigh University, 31 Williams Dr., 18015 Bethlehem PA, USA

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SUMMARY

Sedimentary rock palaeomagnetic inclinations that are too shallow with respect to the ambient field inclination may be restored using anisotropy-based inclination corrections or techniques that rely on models of the past geomagnetic field. One advantage of the anisotropy technique is that it relies on measured parameters (declinations, inclinations, bulk rock magnetic fabrics and particle magnetic anisotropy) that have measurement errors associated with them, rather than relying on a geomagnetic field model and statistical treatment of the data.

So far, however, the error associated with the measurements has not been propagated through the corrections and the reported uncertainties are simply the \( \alpha_{95} \) 95 per cent confidence circles of the corrected directions.

In this paper we outline different methodologies of propagating the error using bootstrap statistics and analytic approximations using the case example of the Shepody Formation inclination correction. Both techniques are in good agreement and indicate a moderate, \( \sim 15 \) per cent, uncertainty in the determination of the flattening factor (\( f \)) used in the correction. Such uncertainty corresponds to an \( \sim 0.31^\circ \) increase of the confidence cone and a bias that steepens the mean inclination by \( 0.32^\circ \). For other haematite-bearing formations realistic uncertainties for \( f \) ranging from 0 and 30 per cent were used (together with an intermediate value of 15 per cent) yielding a maximum expected increase in the confidence cones and steepening of the inclinations of \( \sim 1^\circ \). Such results indicate that for moderate errors of \( f \) the inclination correction itself does not substantially alter the uncertainty of a typical palaeomagnetic study.

We also compare the uncertainties resulting from anisotropy-based corrections to those resulting from the elongation/inclination (E/I) technique. Uncertainties are comparable for studies with a large sample number (> 100), otherwise the anisotropy-based technique gives smaller uncertainties.

When anisotropy data are not available, it is possible to estimate a correction using flattening factors (\( f \)) obtained from the literature. A range of flattening factors has been observed for both magnetite and haematite-bearing rocks (0.4 \( \leq f \leq 1 \) for haematite and 0.54 \( \leq f \leq 1 \) for magnetite), but the exact value is specific to the anisotropy of the formation. To evaluate the maximum effects of inclination shallowing, the smallest \( f \) (for magnetite or haematite) should be used.

Key words: Magnetic fabrics and anisotropy; Palaeomagnetism applied to tectonics; Palaeomagnetism applied to geologic processes; Rock and mineral magnetism.

1 INTRODUCTION

Inclination shallowing is one of the most important problems in palaeomagnetism of sedimentary rocks because accurate measurement of the inclination of the palaeomagnetic field is crucial to determine palaeolatitudes.

Red sedimentary rocks are particularly prone to shallowing (Garcés et al. 1996; Gilder et al. 2001; Tan & Kodama 2002; Tan et al. 2003; Tauxe & Kent 2004; Kent & Tauxe 2005; Tan et al. 2007; Bilardello & Kodama 2010a,c) but considerable inclination shallowing has also been described in magnetite-bearing rocks (Jackson et al. 1991; Kodama & Davi 1995; Tan & Kodama 1998; Kodama 2009; Bilardello & Kodama 2010b).

Palaeomagnetic inclination is a function of the palaeolatitude of deposition and the magnitude of the inclination error has been found to vary with field inclination according to the relation (King 1955;
Griffiths et al. (1960; King & Rees 1966)
\[ \tan(I_m) = f \tan(I_c), \]

where \( I_m \) is the measured (remanent) inclination, \( I_c \) is the corrected (field) inclination and \( f \) is a flattening factor, found to range between 0.4 and 0.83 in haematite-bearing rocks and between 0.54 and 0.79 in magnetite-bearing rocks from a compilation of inclination correction studies (Bilardello & Kodama 2010b). This relationship predicts the largest inclination errors for intermediate inclinations, which occur at intermediate latitudes.

Shallower than expected palaeomagnetic inclinations may be caused by a variety of processes occurring during or after deposition. Mechanical processes such as compaction may affect the remanence after deposition and lead to shallow inclinations (Anson & Kodama 1987; Sun & Kodama 1992). Depositional processes, however, such as rolling of particles after they have settled or flattening of platy particles due to gravity (see review by Verosub 1977) have also been proposed as a cause of shallower than expected palaeomagnetic inclinations. Many experimental studies performed in still and running water have shown that depositional shallowing does indeed occur (e.g. King 1955; Barton et al. 1980; Tauxe & Kent 1984).

Notwithstanding the process involved, inclination shallowing is a function of magnetic anisotropy and therefore two parameters are necessary for inclination corrections, the magnetic fabric and the individual particle anisotropy of the remanence carrier (Jackson et al. 1991; Tan & Kodama 2003). The rock fabric can be measured using a variety of techniques (e.g. AMS, AAR and AIR), the choice of which will depend on the magnetic carrier of the targeted magnetic inclinations. Many experimental studies performed in still and running water have shown that depositional shallowing does indeed occur (e.g. King 1955; Barton et al. 1980; Tauxe & Kent 1984).

Both parameters have an associated measurement error and also exhibit variability (e.g. the variation of bulk anisotropy within a site) that is typically higher than the measurement error. Regardless of the source of the variability for the parameters used in the inclination corrections, it produces an additional uncertainty in the corrected results and should be determined by error propagation. In this paper we report on the error propagation of recently reported inclination corrections of haematite-bearing rocks. We also assess a means of determining the uncertainty of inclination corrections that are estimated on the basis of compiled shallowing factors for both haematite- and magnetite-bearing sedimentary rocks (Bilardello & Kodama 2010b).

2 ERROR PROPAGATION FOR THE CARBONIFEROUS SHEPODY FORMATION

We will demonstrate the technique using the example of the error propagation for an inclination correction of the Shepody Formation (Shepody Fm.) of Nova Scotia and New Brunswick (Bilardello & Kodama 2010a).

Haematite is the main carrier of the magnetic remanence of the Shepody Fm. and the field inclination may be restored using the equation of Tan & Kodama (2003)

\[ f = q_x (2a + 1) - 1 \]

\[ q_x = q_z (2a + 1) - 1 \]

where \( q_z \) and \( q_x \) are the maximum and minimum normalized anisotropy principal axes of the magnetic fabric and \( a \) is the individual haematite particle anisotropy. In Table 1 we report the site-mean statistics for the measured directions and principal anisotropy of the formation. At each of the 19 sites, magnetic anisotropy was measured on selected samples and the values of \( q_z \) and \( q_x \) in Table 1 are computed from the mean of the normalized anisotropy tensor. Values of the parameter \( a \) measured from magnetic extracts of different haematite-bearing rock formations are reported in Table 2. Site-mean directions are shown in Fig. 1(a).

Table 1. Determination of the shallowing factor \( f \) and relative uncertainty (\( \delta f \)) for the Shepody Fm. of New Brunswick/ Nova Scotia (Bilardello & Kodama 2010a) by both bootstrap and analytic solutions: \( q_z \) and \( q_x \) are the minimum and maximum normalized principal anisotropy axes, respectively, while \( \delta q_z \) and \( \delta q_x \) are their associated uncertainties. \( f_{\text{bootstrap}} \) and \( \delta f_{\text{bootstrap}} \) are the bootstrap-simulated values of \( f \) and associated uncertainty as determined through eq. 3(a) (please refer to text for detail).

<table>
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<tr>
<th>Site</th>
<th>( q_z )</th>
<th>( q_x )</th>
<th>( \delta q_z )</th>
<th>( \delta q_x )</th>
<th>( f_{\text{bootstrap}} )</th>
<th>( \delta f_{\text{bootstrap}} )</th>
<th>( f_{\text{analytic}} )</th>
<th>( \delta f_{\text{analytic}} )</th>
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<td>0.350668688</td>
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<td>0.002425756</td>
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<td>0.6271</td>
<td>0.0918</td>
<td>0.6309</td>
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<td>0.350955886</td>
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<td>0.6309</td>
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</tr>
<tr>
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<td>0.350689206</td>
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<td>0.000579672</td>
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<td>Shep5</td>
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<td>0.351642211</td>
<td>0.001075849</td>
<td>0.002559931</td>
<td>0.533</td>
<td>0.094</td>
<td>0.5555</td>
<td>0.0928</td>
</tr>
</tbody>
</table>
Table 2. Individual particle anisotropy values measured from magnetic extractions of haematite-bearing rocks (please refer to Kodama 2009 and Bilardello & Kodama 2010a,b for details on the extraction procedure).

<table>
<thead>
<tr>
<th>Haematite</th>
<th>a</th>
<th>Reference</th>
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<tbody>
<tr>
<td>Mauch Chunk Fm.</td>
<td>1.45</td>
<td>Kodama (2009)</td>
</tr>
<tr>
<td>Shepody Fm.</td>
<td>1.39</td>
<td>Bilardello &amp; Kodama (2010a)</td>
</tr>
<tr>
<td>Maringouin Fm.</td>
<td>1.34</td>
<td>Bilardello &amp; Kodama (2010a)</td>
</tr>
<tr>
<td>Kapusaliang Fm.</td>
<td>1.45</td>
<td>Kodama (2009)</td>
</tr>
<tr>
<td>Newark Basin</td>
<td>1.37</td>
<td>Tan et al. (2007)</td>
</tr>
<tr>
<td>Haematite crystal</td>
<td>1.28</td>
<td>Tan et al. (2007)</td>
</tr>
<tr>
<td>Mean</td>
<td>1.38</td>
<td>Bilardello &amp; Kodama (2009b)</td>
</tr>
<tr>
<td>δ</td>
<td>0.06</td>
<td>Bilardello &amp; Kodama (2009b)</td>
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</tbody>
</table>

2.1 The error of the factor $f$

To assess the site-variability (uncertainty) of the factor $f$, the site-variability of the parameters $q_x$, $q_z$ and $a$ is needed. The parameter $a$ represents the anisotropy of a haematite single crystal. Although curve-fitting techniques can be used to accurately estimate individual particle anisotropy (Tan & Kodama 2002; Tan et al. 2003; Kodama 2009), it can be measured directly using remanence anisotropy of synthetic samples prepared from magnetic extracts (Kodama 2009; Bilardello & Kodama 2010a,b). Multiple measurements of the anisotropy of the magnetic extracts will allow calculation of an average particle anisotropy and associated standard deviation. Haematite anisotropy results from a competition between magnetocrystalline and magnetoelastic anisotropy (Dunlop & Özdemir 1997; Özdemir & Dunlop 2005). Assuming that the anisotropy is mainly controlled by magnetocrystalline forces, particle anisotropy should be very similar for any haematite crystal. A compilation of $a$ factors measured from extracts obtained from seven different red bed formations (Bilardello & Kodama 2009b) confirms that haematite $a$ factors are indeed very similar with a mean value of 1.38 and a standard deviation ($\delta$) of 0.06 (Table 2). We will use this average and its associated standard deviation for the propagation of the error in our inclination corrections.

For $q_x$ and $q_z$, the standard deviations computed from each site show high variability and are probably not an accurate estimate of site-variability. However, the between-site mean values of standard deviations ($\delta_{q_x} = 0.0042$, $\delta_{q_z} = 0.0043$) are in good agreement with standard deviations computed from all samples from the whole formation ($\delta_{q_x} = 0.0043$, $\delta_{q_z} = 0.0051$). We will use the latter estimates of $q_x$ and $q_z$’s standard deviations for all sites not to risk underestimating the uncertainty.

2.2 Bootstrap of the $f$-factor

The uncertainty of $f$ may be assessed by means of parametric bootstraps (Tauxe et al. 2010). We use a normal distribution for all parameters ($q_x$, $q_z$ and $a$) in all sites. Although the real distributions of the parameters may not be normal (Tauxe et al. 2010) we do not have sufficient data to characterize them. However, our estimate of the error of the formation mean will be based on averaging, therefore for the central limit theorem the individual distributions of parameters would not play a significant role. The parameters $q_x$ and $q_z$ should be generated as correlated. For the whole formation we compute the covariance to be $c = -1.41 \times 10^{-5}$ and the correlation

Figure 1. (a) Site-mean directions of the Shepody Fm. and 95 per cent Fisherian confidence cone for the formation mean. (b) Corrected site-mean directions and the formation confidence cone. (c) Simulated site-mean directions using the bootstrapped variability of the $f$ factor. (d) Formation means computed from the simulated site-mean directions (see Fig. 2 for more detail).

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to be \(\rho = -0.642\) and use these values to generate new correlated parameters \(q_x\) and \(q_z\). To avoid the possibility that by random choice we obtain higher values for \(q_x\) than for \(q_z\), or they are not the minimum and maximum principal values, respectively, we always also compute \(q_i = 1 - q_x - q_z\) and check that \(q_i\) and \(q_z\) are, respectively, the minimum and maximum of the triplet. If not we exchange them accordingly. Doing so slightly distorts the normality of the chosen \(q_x\) and \(q_z\) and also influences the final distribution of the factor \(f\). Nevertheless, differences from a normal distribution were not visible on histograms of \(q_x\), \(q_z\), and \(f\). Based on this procedure we have bootstrapped the factor \(f\) and its standard deviation \(\delta f\) for all sites (Table 1).

### 2.3 Analytic expression

Results of the bootstrap technique can be compared to an analytic expression. The \(f\)-factor determined by eq. (2) is linear in \(q_x\) and only slightly non-linear in \(q_z\) and \(\alpha\) (considering the range of their values), which allows using an estimate based on classic schemes of error propagation (e.g. Roddick 1987). Assuming that the uncertainties at a site of individual particle and bulk anisotropies are not too high and taking into account the fact that \(q_x\) and \(q_z\) are correlated, the final uncertainty of the factor \(f\) can be approximated by

\[
\delta f = \sqrt{\left(\frac{\delta f}{\partial a}\right)^2 \delta a^2 + \left(\frac{\delta f}{\partial q_x}\right)^2 \delta q_x^2 + \left(\frac{\delta f}{\partial q_z}\right)^2 \delta q_z^2 + 2c \left(\frac{\delta f}{\partial q_x}\right) \left(\frac{\delta f}{\partial q_z}\right)}.
\]

(3a)

where \(c\) is the covariance, or, using relative errors

\[
\frac{\delta f}{f} = \sqrt{P \left(\frac{\delta a}{a}\right)^2 + Q \left(\frac{\delta q_x}{q_x}\right)^2 + R \left(\frac{\delta q_z}{q_z}\right)^2 + S \left(\frac{\delta q_x}{q_x}\right) \left(\frac{\delta q_z}{q_z}\right)}.
\]

(3b)

where

\[
P = \left(\frac{a}{f}\right)^2 \left(\frac{\delta f}{\partial a}\right)^2, \quad Q = \left(\frac{q_x}{f}\right)^2 \left(\frac{\delta f}{\partial q_x}\right)^2, \quad R = \left(\frac{q_z}{f}\right)^2 \left(\frac{\delta f}{\partial q_z}\right)^2, \quad S = 2p \left(\frac{q_x}{f}\right) \left(\frac{q_z}{f}\right) \left(\frac{\delta f}{\partial q_x}\right) \left(\frac{\delta f}{\partial q_z}\right).
\]

The second approach (eq. 3b) allows assessing the role of each individual term and its relative contribution to the error. By numerical analysis it was found that all terms are important (non-negligible) and should be considered.

In Table 1 we compare the errors of the factor \(f\) computed using eq. (3a) with those obtained by bootstrapping. Both estimates of the error are of the order of \(\sim 15\) per cent and mutually consistent. By bootstrapping the site-relative errors \(\delta f/f\) have a mean of 15.59 per cent, (minimum 12.19, maximum 19.14), while the estimate obtained through eq. (3a) is very similar, with a mean of 15.62 per cent (minimum 13.29, maximum 18.96). We have used both bootstrap and analytic estimates for the following computations, however, since the results are similar we always report the results based on the errors of \(f\) computed by analytic approximation.

### 2.4 Assessment of the additional uncertainty of the formation mean by bootstrap

The uncertainty in the \(f\)-factor introduces an additional variability to the mean formation direction. We will assess this additional variability by bootstrap, based on established errors \(\delta f\). The Shepody Fm. inclinations were corrected at the site level by eqs (1) and (2) and the result is shown in Fig. 1(b). As is typically done, we characterize the mean direction by the resultant with components

\[
X = \sum \cos D_i \cos I_i, \quad Y = \sum \sin D_i \cos I_i \quad \text{and} \quad Z = \sum \sin I_i
\]

(4)

and length

\[
R = \sqrt{X^2 + Y^2 + Z^2}
\]

and by a circular 95 per cent confidence cone based on the assumption of Fisher’s (1953) distribution, whose radius is

\[
\alpha_{95} = \arccos \left(1 - \frac{n - R}{R} \left(2g^2 n - 1\right)\right).
\]

An estimate of the precision parameter \(\kappa\) (Fisher et al. 1987) is

\[
\kappa = \frac{n - 1}{n - R}.
\]

A comparison of Figs 1(a) and (b) shows that the inclination correction causes an elongation of the site-mean directions. We have therefore also fitted an elliptical cone corresponding to the Kent (1982) distribution, with semi-axes

\[
\zeta_{95} = \arcsin \left(\tau_2 \sqrt{g}\right), \quad \eta_{95} = \arcsin \left(\tau_1 \sqrt{g}\right),
\]

where \(g = -2 \log(0.05) n/R^2\), and \(\tau_2\) and \(\tau_3\) are the second and third eigenvalues of the orientation matrix (see Tauxe et al. 1991 for details). Both circular and elliptical confidence cones are shown in Fig. 2, where the respective confidence angles are

\[
\alpha_{95} = 7.75^\circ, \quad \zeta_{95} = 9.30^\circ \quad \text{and} \quad \eta = 4.72^\circ.
\]

Note that there is a small negligible difference in the centres of the circular and elliptical cone, since the first is given by the resultant and the second by the first eigenvector of the orientation matrix. Furthermore, the long axis of the elliptical cone is almost parallel to a longitudinal line that passes through the centre of the cone (they include a small angle \(\sim 3.5^\circ\)).

Figure 2. Detail of the simulated formation means (marked by the symbol x). Their elliptical scatter represents the additional uncertainty in the estimation of the formation mean direction due to the error of the \(f\) factor. The black dots are Shepody Fm. corrected site-means, the circle is the Fisherian confidence cone, the ellipse is Kent’s confidence cone, the line is a longitudinal line passing through the mean formation directions (all projected on the upper hemisphere of a unit sphere).
We use the previously estimated site-variability of the $f$-factor (Table 1) to estimate the uncertainty of the inclination correction. From the previous bootstrap of the $f$-factor we already know that we can assume that the distribution of the errors is approximately normal. Using site-mean values of $f$ and their standard deviations $\delta f$ (Table 1) we randomly and independently generate a new value of $f$ and a new site mean for each site and then compute a corresponding formation mean from the new site means. We repeat this procedure many times (>10 000 times). Eqs 1 and 2 indicate that only the inclination is affected. This causes the simulated site means to spread along lines of longitude towards the centre of the stereogram in Fig. 1(c). The corresponding formation means are shown in Fig. 1(d) and in more detail in Fig. 2. They exhibit a small elliptical variability also subparallel to a line of longitude. The orientation matrix gives the orientation of the axes of the elliptical confidence cloud. The difference between the long axis and longitudinal line is only 0.6°. Note that the centre of the cloud is slightly shifted to higher inclinations with respect to the centres of the Fisher (1953) and Watson (see Tauxe et al. 1991) cones of confidence. Repeated simulations show that this shift is systematic and on the order of some tenths of a degree. This detail will be studied later.

The simulated formation means shown in Fig. 2 represent an additional uncertainty of the previously estimated formation mean, which is caused by the error of the factor $f$, and should be added to its uncertainty (i.e. increasing the confidence cones in Fig. 2). Assuming elliptical uncertainty described by the orientation matrix, the angles defining the elliptical 95 per cent confidence cone of the bootstrapped formation means in Fig. 2 are estimated as quantiles. We obtain

$$\alpha_{eis} = 2.20° \quad \text{and} \quad \beta_{eis} = 0.35°.$$

Since we are interested in how different errors of $\delta f$ affect the overall uncertainty ellipse for the corrected formation mean, we have also performed a series of simulations with artificially chosen relative errors $\delta f$ between 5 and 30 per cent (in each simulation the relative error is equal for all sites). Results are shown in Fig. 3 (marked by asterisks). Note that the formation mean is computed by averaging which causes the distribution of the error of $f$ not to have a significant effect on the error propagation. For example, if we change the assumption of a normal distribution of the error $\delta f$ and use a uniform distribution (width $\pm 2\delta f$), the elliptical character of the resulting distribution of the bootstrapped formation mean does not change and the values of angles $\alpha_{eis}$ and $\beta_{eis}$ change minimally (marked by open circles in Fig. 3).

### 2.5 Analytical assessment of the additional variability

We also want to determine an analytical expression for the added uncertainty of the formation mean caused by the error of $f$. We first assess how the corrected inclination in eq. (1), that is, $I_c = \arctan(\tan(I_m)/f)$ changes with deviations of $f \pm \delta f$. In Fig. 4 we show these changes together with their approximations by first and second derivatives

$$\delta I_c = \frac{dI_c}{df} \delta f = -\frac{\tan I_m}{\tan^2 I_m + f^2} \delta f,$$

$$\delta^2 I_c = -\frac{\tan I_m}{\tan^2 I_m + f^2} \delta f + \frac{f \tan I_m}{(\tan^2 I_m + f^2)^2} \delta f^2.$$

For relative errors of $f$ up to 15–20 per cent approximation (5) is acceptable. For higher errors, up to ~35 per cent, the second derivative approximation works better. The deviations $\delta I_c$ caused by equal positive and negative deviations of $f$ are not equal (in absolute values), the latter being larger. This will cause a bias of the formation mean, which will be analysed in more detail below.

Due to the almost longitudinal character of the correction (Fig. 1c), it is possible to make a rough estimate of the angle $\alpha_{eis}$ directly from the eq. (1). Using the mean inclination of the Shepody $I_m = 20.4°$, the mean $f = 0.618$, the relative standard deviation 0.15 per cent ($\delta f = 0.093$) we compute $\delta I_c$ by eq. (5). Taking this...
estimate as valid for all sites, the angle $\alpha_{\text{sys}}$ is given by

$$\alpha_{\text{sys}} \approx \sqrt{\frac{\chi^2}{n}} |\delta I_i| = \sqrt{\frac{\chi^2}{n}} \tan^2 I_i + f^2 \delta f,$$  

(7)

where $\chi^2$ is a 95 per cent quantile of Chi-square distribution. The result is $\alpha_{\text{sys}} \approx 1.19^\circ$.

To refine the simple estimate from eq. (7) we use eq. (5) with the specific site inclinations. We obtain

$$\alpha_{\text{sys}} \approx \sqrt{\sum \frac{\delta I_i^2}{n^2}},$$  

(8)

where $\delta I_i$ is the result of eq. (5) for the $i$th site. The resulting angle $\alpha_{\text{sys}} = 2.47^\circ$ is slightly higher than that previously estimated by bootstrapping. In Fig. 3 the upper curve shows the estimate given by eq. (8) as a function of the error $\delta f$.

We examine another estimate of the additional uncertainty of the formation mean, that is more rigorous and also provides the angle for the short axis of the confidence ellipse. From eq. (4) for the resultant we calculate the inclination and declination of the formation mean direction, and then we find by differentiation how every site’s individual inclination error influences the position of the formation mean

$$\delta I_F = \frac{\cos I_i}{\sqrt{X^2 + Y^2}} \delta I_i,$$

$$\delta D_F = -X \sin D_i \sin I_i + Y \cos D_i \sin I_i,$$

where $I_i$ and $\delta I_i$ are the corrected inclination site-means and their error. Assuming the errors $\delta I_i$ are small, the $I_F$ and $D_F$ variations are assumed to lie in a plane tangent to the unit sphere at the formation mean direction (point $I_F, D_F$). Summing the variances we obtain, for the whole formation

$$\delta I_F = \frac{1}{\sqrt{X^2 + Y^2}} \sqrt{\sum (\cos I_i \delta I_i)^2},$$

$$\delta D_F = \frac{1}{X^2 + Y^2} \sqrt{\sum (-X \sin D_i \sin I_i + Y \cos D_i \sin I_i)^2}.$$

The angles defining the 95 per cent elliptical cone then become

$$\alpha_{\text{sys}} \approx \sqrt{\chi^2 \delta I_F}, \quad \beta_{\text{sys}} \approx \sqrt{\chi^2 \delta D_F}.$$  

(9)

Resulting angles are $\alpha_{\text{sys}} = 2.01^\circ$ and $\beta_{\text{sys}} = 0.40^\circ$.

The lower curve in Fig. 3 shows the estimate by eq. 9 for the angle $\alpha_{\text{sys}}$ that is closer to the bootstrapped estimates for values of $\delta f$ up to 20 per cent than the previous estimate by eq. (8). However, for higher values of $\delta f$ the estimate by eq. (8) shows better agreement. In the next procedure we will use the estimate by eq. (8), which is simpler to compute and provides a satisfying approximation in the whole range of $\delta f$ up to 30 per cent.

2.6 Increasing the confidence cones

Given the relatively small angles defining both confidence cones (the original cone for the formation mean and the cone for the additional uncertainty of the formation mean) and the fact that their long axes are almost parallel we can again estimate by a fair approximation the final formation confidence cone as if working in a plane. We obtain

$$\alpha_{95} = \sqrt{\alpha^2_{\text{sys}} + \alpha^2_{\text{sys}}} \quad \text{and} \quad \beta_{95} = \sqrt{\beta^2_{\text{sys}} + \beta^2_{\text{sys}}}$$  

(10)

for the original Fisherian cone (which becomes elliptical), and

$$\zeta_{95} = \sqrt{\zeta^2_{\text{sys}} + \zeta^2_{\text{sys}}} \quad \text{and} \quad \eta_{95} = \sqrt{\eta^2_{\text{sys}} + \eta^2_{\text{sys}}}$$  

(11)

for the Kent’s cone. The results for the Shepody Fm. are

$$\alpha_{95} = 8.06^\circ, \quad \beta_{95} = 7.76^\circ,$$

$$\zeta_{95} = 9.56^\circ, \quad \eta_{95} = 4.76^\circ$$

(computed with $\alpha_{95} = 7.75^\circ, \zeta_{95} = 9.30^\circ, \eta_{95} = 4.72^\circ, \alpha_{\text{sys}} = 2.20^\circ$ and $\beta_{\text{sys}} = 0.35^\circ$).

We can also express the increase of the confidence angle, for example the Fisherian cone, as

$$\delta \alpha_{95} = \sqrt{\alpha^2_{95} + \alpha^2_{\text{sys}} - \alpha_{95} \approx \frac{\alpha^2_{\text{sys}}}{2\alpha_{95}},}$$  

(12)

For the Shepody Fm. we find that the Fisherian confidence cone would be enlarged by $\delta \alpha_{95} = 0.31^\circ$. In Fig. 3 we also show the angle $\delta \alpha_{95}$ computed for a range of values of $\delta f$.

2.7 Bias of the formation mean

We have already mentioned that negative deviations of $f$ have a greater effect on the change of the corrected inclination than positive ones. This means that the propagated uncertainty in $f$ will cause a bias towards higher values of the corrected inclinations (making inclinations steeper than the corrected inclination without proper error propagation) and therefore introducing a bias of the formation mean. We can quantify this bias as half the difference between the results of eq. (6) for positive and negative $\delta f$. We obtain an approximation of the bias from the second derivative of $\delta f$

$$\delta I_F = \frac{f \tan I_m}{(\tan^2 I_m + f^2)^2} \delta f^2,$$  

(13)

where $I_m$ is the formation-mean measured inclination and $f, \delta f$ are the formation-mean values of $f$ and its error. Nevertheless, a better approximation results from an average of the site-mean values for $I_m, f$ and $\delta f$.

$$\delta I_F = \frac{1}{n} \sum \frac{f \tan I_m}{(\tan^2 I_m + f^2)^2} \delta f^2.$$  

(14)

This angle reflects the steepening of the formation mean due to the errors of $f$. In Fig. 5 we compare the estimates by eqs (13) and (14) to the bootstrapped values from simulations already shown in Fig. 3. The estimate by eq. (14) is very close to the value computed by bootstrap, $\delta I_F = 0.35^\circ$, while by eq. (13) we obtain a higher value $\delta I_F = 0.42^\circ$.

2.8 Summary of the Shepody Fm.

We found that the additional uncertainty caused by the error of the factor $f$ caused an increase of the confidence cone by $0.31^\circ$ and a bias that steepens the formation mean by $0.35^\circ$. This illustrates that an error of the $f$-factor on the order of 15 per cent does not substantially affect the estimate of the mean direction of the formation in the context of a typical palaeomagnetic study. Furthermore, we cannot exclude the possibility of overestimating the final error of the formation mean by overestimating the errors of $q_x$ and $q_z$. 

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We have assessed the additional errors in the formation mean due to errors in the $f$-factor by both bootstrapping and using analytic expressions, and they are found to be similar in magnitude. Nevertheless, the approaches described (especially the analytic expressions) rely on the assumptions of small angles and moderate within-site errors of the $f$-factor, which must be checked when applied to other studies. For formations with confidence cones smaller than $\sim 10^\circ$ the analytical equations can provide reliable results.

The final error of the formation mean can be simulated directly in one bootstrap. We have split the task to separately assess the error of the $f$-factor, how it is propagated into the inclination correction and subsequently into the confidence cone for the formation mean.

A confirmation that our estimates are correct can be given by a comparison to the bootstrap results using larger errors of $f$. In Fig. 6 we show a complete bootstrapping example of the Shepody formation mean and its 95 per cent confidence cone, assuming a 30 per cent error of $f$. We have first bootstrapped the mean using the original site-means without considering the error of $f$. The formation mean direction was estimated using the orientation matrix and angles of the 95 per cent confidence cone were found as 95 per cent quantiles (open circles and smaller ellipse in Fig. 6). The resulting formation mean inclination is $I_F = 30.21^\circ$, and confidence angles $a_F = 8.92^\circ$, $b_F = 4.81^\circ$. New site means were then randomly simulated using a 30 per cent error of $f$ (symbol x and larger ellipse in Fig. 6). The formation mean changed to $I_F = 31.53^\circ$ ($1.32^\circ$ steeper) and the new confidence cone had angles $a_F = 10.18^\circ$ ($1.26^\circ$ larger), $b_F = 4.82^\circ$.

The resulting confidence angles and the steepening of the mean direction are reported in Table 3 together with estimates given by eqs (8), (10) and (14). The comparison shows that these analytical estimates work sufficiently well, so that they can be used as a first-order approximation. Their advantage is that they can be easily computed without the tedious and time-consuming bootstrap.

### 3 Other Haematite-Based Inclination Corrections

We have semi-quantitatively evaluated what the effect of the anisotropy-based inclination correction is on the final confidence ellipse of inclination corrections made to other haematite-bearing sedimentary rocks. Bearing in mind that the study of Shepody Fm. revealed an error for $f$ of $\sim 15$ per cent, we have performed simulations on the other formations assuming zero, 15 and also 30 per cent errors of $f$. Calculations using the analytical estimates given by eqs 8 and 10 have also been performed (Table 3). Results indicate that even when a 30 per cent error is assumed the final confidence ellipses of all formations increase by $\sim 1^\circ$ (Fig. 7). The bias of the formation mean, $\delta I_F$, given by bootstrapping is also reported in Table 3 together with the bias of the formation means as estimated by eq. 14.

Given that the 15 per cent error of $f$ of the Shepody Fm. is already a slight overestimate arising from the overestimation of the errors of $q_x$ and $q_y$, we feel that testing the effect of the inclination correction on other formations using a 30 per cent error provides a reasonable highest end-member for the uncertainty. When applying inclination corrections to haematite-bearing rocks it is reasonable to assume an uncertainty in the $f$-factor intermediate between 0 and 30 per cent which will enlarge the confidence ellipses by less than $1^\circ$. Therefore, for moderate errors of $f$ the effect on the formation mean can be neglected.

Inclination corrections have also been performed on magnetite-bearing rocks. Although the equation for $f$ is slightly different due to different orientation distributions expected for oblate haematite and prolate magnetite particles [please refer to Tan & Kodama (2003) for a discussion on the two equations], inclination correction studies have similar uncertainties in the determination of $q_x$, $q_y$ and $a$, therefore it is very reasonable to assume that the final propagated uncertainty is comparable to that of haematite-bearing rocks.

### 4 Comparison with Other Inclination Correction Techniques

The elongation inclination technique (E/I) of Tauxe & Kent (2004) is a method for correcting palaeomagnetic inclinations that relies on models of the geomagnetic field. The main advantage of the
Results of the propagated errors through corrected haematite-bearing formations: $\delta f$ is the relative error of the $f$ factor (equal at all sites). $\alpha_{95}$ and $\beta_{95}$ are the long and short semi-axes of the confidence cone (in degrees), respectively, derived by both bootstrap and analytical techniques (eqs 8 and 10). $\delta f$ is the bias (steepening) of the mean direction (in degrees) derived using eq. 14 (please refer to text for details on the simulations and equations used).

<table>
<thead>
<tr>
<th>Formation</th>
<th>$\alpha_{95}$ (per cent)</th>
<th>$\beta_{95}$ (per cent)</th>
<th>$\delta f$ 15 (per cent)</th>
<th>$\delta f$ 30 (per cent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shepody</td>
<td>8.90</td>
<td>4.80</td>
<td>9.15</td>
<td>10.16</td>
</tr>
<tr>
<td>Maringouin</td>
<td>15.11</td>
<td>10.15</td>
<td>15.41</td>
<td>16.43</td>
</tr>
<tr>
<td>Mauch Chunk</td>
<td>9.57</td>
<td>5.56</td>
<td>9.8</td>
<td>10.38</td>
</tr>
<tr>
<td>Kapusaliang</td>
<td>6.36</td>
<td>5.95</td>
<td>6.63</td>
<td>7.63</td>
</tr>
</tbody>
</table>

$\delta f$, relative error of the $f$ factor (equal at all sites). $\alpha_{95}$ and $\beta_{95}$, long and short semi-axes of confidence cone (in degrees). $\delta f$, bias (steepening) of the mean direction (in degrees).

Figure 7. Bootstrap simulations of different haematite-bearing formations: (A) Shepody Fm., (B) Maringouin Fm., (C) Mauch Chunk Fm., (D) Kapusaliang Fm. All plots show results of $10^6$ simulations (only first 300 plotted) using zero (circles) and 30 per cent (symbol x) error of $f$ with corresponding (smaller and larger) confidence ellipses. The difference between the large and small confidence ellipses is $\sim 1^\circ$, the inclination bias given by the increased error is of the same order (the ellipse is shifted towards higher inclinations with increasing error of $f$; for exact values please refer to Table 3). All plots have the same scale.

The technique is that it does not require measurements of the magnetic fabric and particle anisotropy, bypassing time-consuming laboratory procedures. However it does require a very large number of site means for the E/I technique to yield statistically significant results (Tauxe & Kent 2004). Therefore, the method works well for studies that have been designed specifically for the E/I technique or that already have larger numbers of site means, like magnetostratigraphic studies or drill core analysis. Of course no typical palaeomagnetic study exists, since the sampling scheme is dictated by the goals of the study, rock type, accessibility of exposure, together with the statistical properties of the magnetization. Site numbers are often less than the >100 directions that are ideally required by the E/I technique (Tauxe & Kent 2004), so when the E/I technique is applied to a palaeomagnetic study taken from the literature the confidence interval is usually high.

For example, if we compare the results of the anisotropy-based correction of the Shepody Fm. (Bilardello & Kodama 2010a) to those of an E/I correction on the 19 site means of the same formation, we find that while the $\alpha_{95}$ of the former, inclusive of the bootstrap-propagated correction error, is 8.06° ($I_c = 30.1^\circ$), the E/I technique yields a shallower corrected inclination with a much broader confidence interval (26.2°, low and high inclinations of 18.1° and 41.8°, respectively), the low and high uncertainties, are thus 8.1° and 15.6°, much higher than for the anisotropy-based correction.

In sedimentary rocks it can be assumed that each sample represents a geological instant in time, therefore it is reasonable to use the E/I technique on individual sedimentary rock directions, granted that the sites are evenly distributed in the section. When we apply the E/I technique to the 104 directions measured for the Shepody
Fm, we obtain an uncorrected inclination of 20.7°, which corrects to 30°. Low and high inclinations are 21.7° and 37.6°, respectively, indicating uncertainties (low and high) of 8.3° and 7.6°. The correction now agrees nicely with the anisotropy-based correction and the uncertainties are also of similar magnitudes.

These results confirm those of Tauxe et al. (2008) who state that for sample numbers smaller than ~100–150 the confidence bounds are large, limiting the reliability of the E/I method, but that for larger sample numbers the E/I technique may be easier to apply than anisotropy-based corrections. The advantage of the E/I technique is that it does not require measurements of the fabric and of the individual particle anisotropy. However, while exact measurement of the fabric may be laborious (e.g. McCabe et al. 1985; Jackson 1991; Bilardello & Kodama 2009a), at least for haematite a mean individual particle anisotropy value of 1.38 can be used with confidence (reported in an earlier section). For magnetite, the individual particle anisotropy must be determined by direct measurement or estimated using curve fitting techniques (Tan & Kodama 2002; Tauxe et al. 2003) for each study.

An added benefit of the anisotropy technique is that the study of magnetic fabrics provides insight into the origin of the remanence (e.g. Tauxe et al. 1990; Jackson 1991; Tarling & Hrouda 1993).

5 ESTIMATED CORRECTIONS

It is difficult to determine the magnitude of an inclination correction without making the necessary fabric and individual particle anisotropy measurements. However, from the ranges of observed flattening f-factors from haematite- and magnetite-bearing rocks it is possible to estimate what the maximum effect of an inclination correction may be. Flattening f factors have been compiled for haematite- and magnetite-bearing rocks and range between 0.4–0.83 and 0.54–0.79, respectively (Bilardello & Kodama 2010b). This compilation comes from studies that confirm inclination shallowing. While depositional inclination shallowing is consistently observed in redeposition experiments and modelled numerically (e.g. King 1955; Scherbakov & Scherbakova 1983; Lavelle & Torsvik 1984; Tauxe & Kent 1984; Mitra & Tauxe 2009), one must exercise caution: for instance some low-latitude sedimentary rocks appear to have undergone little to no shallowing (Schimdt et al. 2009). It is possible, therefore, that inclination shallowing may range from a maximum value (f = 0.4 for haematite, 0.54 for magnetite) to zero (f = 1). These observations lead to a broader range of possible inclination corrections for haematite than for magnetite.

Recently, to evaluate the possible effects of inclination shallowing, researchers have used a ‘mean’ or ‘representative’ value of f = 0.55 (Domeier et al. 2009; Kent & Irving 2010). Such an f value derives primarily from the older studies of King (1955) and Tauxe & Kent (1984).

We want to stress here that no ‘mean’ value of f exists, because f is dependent on the specific anisotropy of the rock formation, following eqs (2) and (3). If one needs to evaluate the maximum possible effect of inclination shallowing in rocks for which anisotropy data do not exist, then the smallest f factor for that rock type (haematite- or magnetite-bearing) must be used. The true inclination will lie anywhere between the corrected and the measured inclination. As more magnetic fabric studies are performed for inclination shallowing corrections, the observed range of f values will be refined.

6 DISCUSSION AND CONCLUSION

In this paper we have outlined a procedure for propagating the uncertainties in anisotropy-based inclination corrections of haematite-bearing rocks by using bootstrap simulations and also analytic approximations. The propagated uncertainty is a function of the measurement errors of the a factor and the magnetic fabric as well as the scatter in the corrected directions.

The additional uncertainty caused by the ~15 per cent error/variability of the factor f of the Shepody Fm. caused an ~0.31° increase of the confidence cone and a 0.42° steepening of the corrected mean inclination. Moderate errors of the f-factor therefore do not substantially affect the estimate of the mean direction of the formation in the context of a typical palaeomagnetic study. Both the bootstrap technique and the analytic expressions where found to be in accordance within the reasonable assumptions of small angles and moderate within-site errors of the f-factor.

The propagation of the error was evaluated on other haematite-bearing rock formations by using both the bootstrap technique and the analytic approximations. End-member errors of the f-factor 0 and 30 per cent were chosen to provide a realistic range. For all the four rock formations (Shepody included for comparison) the increase of the confidence ellipses and the steepening of the corrected mean inclinations were ~1°.

A comparison with the elongation/inclination technique of Tauxe & Kent (2004) indicates that if sufficient directions are available (>100) then the corrections and their uncertainties are comparable. However, for studies with smaller sample numbers the anisotropy-based technique appears to give a better constrained correction.

For rock formations for which anisotropy data do not exist it is possible to evaluate possible inclination shallowing by using the smallest f factors that have been observed for haematite- and magnetite-bearing rocks (Bilardello & Kodama 2010b). When this is done one must bear in mind that the true corrected inclination may lie anywhere between the measured inclination and that obtained by the f factor used.

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REFERENCES


Bilardello, D. & Kodama, K.P., 2010a. Paleomagnetism and magnetic anisotropy of Carboniferous red beds from the Maritime Provinces of


