Nordic Hydrology, 9, 1978, 187-196

Determination of Aquifer Diffusivity from Annual Fluctuations of Ground Water Levels

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Annual fluctuations of ground water levels of an aquifer are described as a consequence of water recharging periodically over a part of the aquifer. These areas can be depicted by drawing maps showing the areal amplitude distribution of the oscillation. Considering these areas as periodic (in time) surface sources, the damping and the phase of the oscillation are found analytically as functions of distance, the recharge area being an infinite strip or a circular disc. From these solutions the aquifer diffusivity can be calculated by the use of typecurves.

Introduction

Periodic oscillations of ground water levels have not drawn much attention among ground water hydrologists as a means to investigate aquifer characteristics. Several similar problems have been solved for heat conduction in solids. Carslaw and Jaeger (1959) give solutions for the semi-infinite solid with prescribed periodic conditions over the whole of its surface, corresponding to an aquifer being in full contact with the sea or a river with periodically varying water level. They also give solutions for periodic point, line and plane sources, the line source being similar to a fully penetrating well.

This paper deals with an aquifer of finite thickness being supplied by water recharging periodically over a part of its surface. Throughout the paper, periodic means periodic in time.

Hydrological Considerations

Recharge to a deep aquifer usually is not evenly distributed over the whole of its surface. In certain areas one or several upper aquifers with higher piezometric level and limited areal extent may be found. These upper aquifers can be infiltrated by rainfall and the downward pressure gradient will cause a downward flow of water to the deep regional aquifer.

The infiltration to the upper aquifer(s) can be approximated by a periodic infiltration with period one year. This periodicity will be refound in the deep aquifer, but the amplitude will be damped and there will be a phase lag. In the deep aquifer the oscillation will spread out, and the resulting further damping and phase lag will be a function of the aquifer hydraulic properties. The amplitude will be of a maximum at the center of recharge and vanishes at large distances.

Thus, recharge to deep aquifers are caused mainly by the average difference in head between the upper and deep aquifer, superposed by a periodic recharge usually of much smaller amount. The steady part of the recharge due to the difference in head will cause a peak to occur in the piezometric surface of the deep aquifer, and since the recharge has its maximum value here, so will the amplitude of the oscillation.

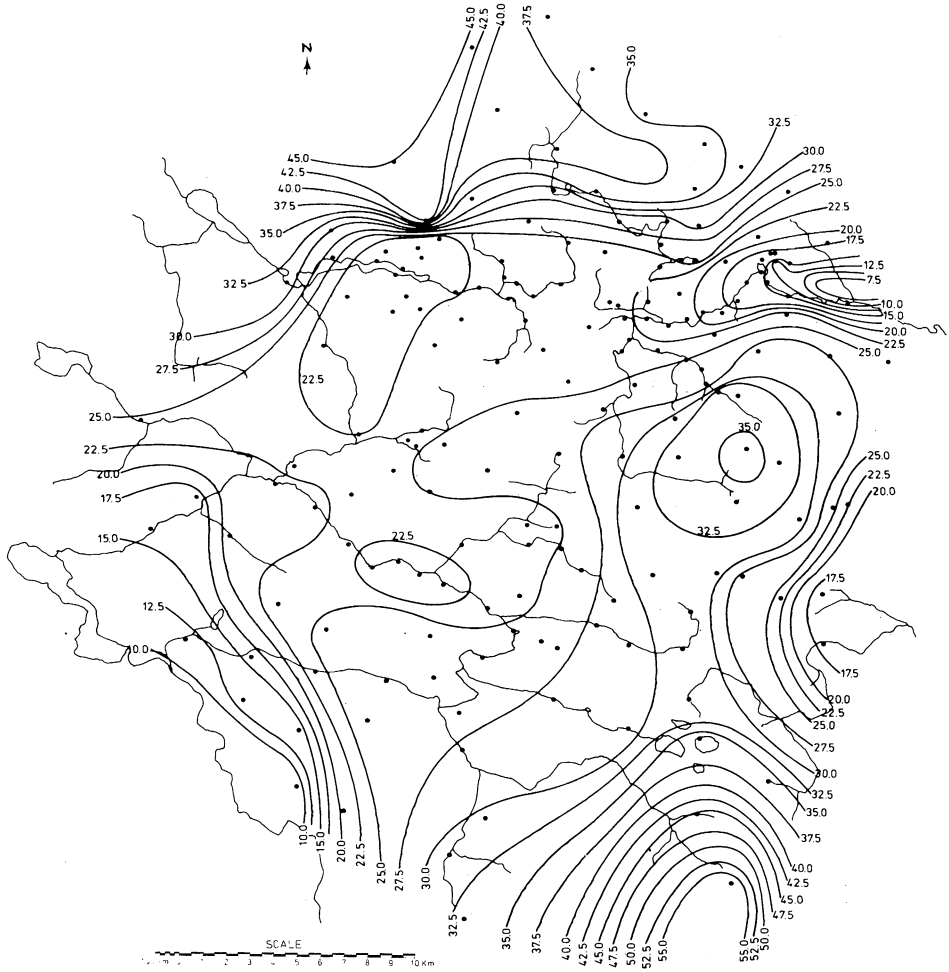
This can be seen from Figs. 1 and 2. Here, the piezometric surface and the amplitude distribution in the Suså catchment area are shown. It will be seen that areas of maximum piezometric levels are more or less coinciding with areas of maximum amplitudes. The same is true for minima, which are caused by the draining effect of streams. The deep aquifer here is of Paleocene or Danien age and the upper aquifer(s) are Quarternary deposits.

In the following the spreading in the deep aquifer of the periodic recharge is described mathematically, the recharge areas being considered as periodic surface sources.

Mathematical Model

Assumptions

The thickness d of the aquifer is assumed to be small compared to the width $2a$ of the strip or the diameter $2R$ of the disc. Usually $d/2a \sim d/2R \sim 0.02$-0.05. This means that the vertical component of the velocity need not be considered, so that all streamlines are assumed horizontal. The aquifer is homogeneous and isotropic and of infinite areal extent, see Fig. 3.



32.5 Piezometric surface contour (m.a.s.l.) as of August 1, 1974

observation well

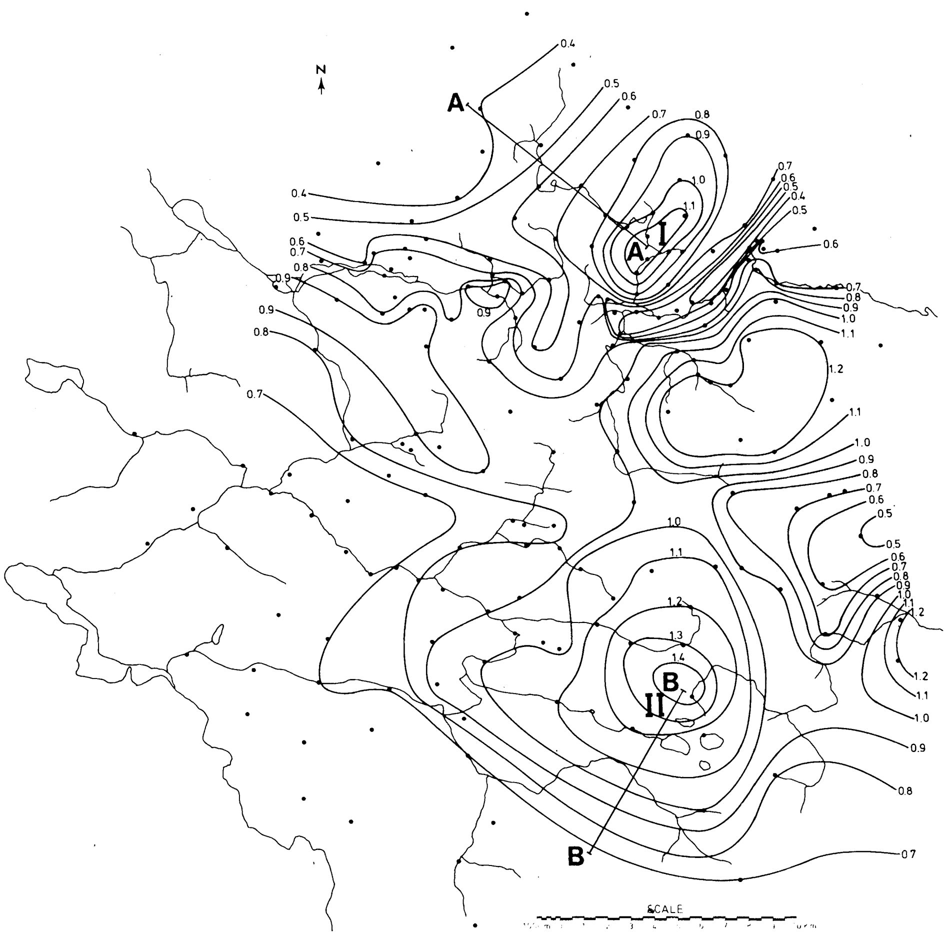
streams and lakes.

Fig. 1. Map showing the piezometric surface of Paleocene and Danien deposits. The Suså catchment area, Zealand, Denmark. From Environmental Projects No.3 (1977).

The Periodic Strip Source

The differential equation governing the flow is

$$\frac{\partial^2 h(x,t)}{\partial x^2} + \frac{I(t)}{T} = \alpha \frac{\partial h(x,t)}{\partial t} \qquad (1)$$



0.9 Amplitude contour. Numbers denote the value of twice the amplitude (m)

• observation well

streams and lakes.

Fig. 2. Map showing the areal distribution of twice the amplitude of the annual oscillation in wells screened in Paleocene and Danien deposits. The Suså catchment area, Zealand, Denmark, From Environmental Projects No.3 (1977).

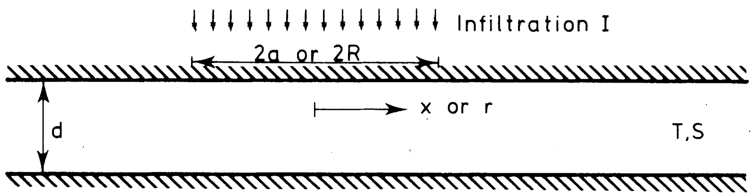


Fig. 3. Schematic representation of aquifer.

where

h = amplitude of the oscillation
$1/\alpha$ = T/S = aquifer diffusivity
S = coefficient of storage
T = transmissivity

$$I \equiv \text{recharge} = \begin{cases} q\,e^{i\omega t}, & |x| < a \\ 0, & |x| > a \end{cases}$$

q = amplitude of periodic recharge
ω = cyclic frequency (here $\omega = 2\pi/\tau$, τ = 1 year)
x = distance from the centerline of the strip
t = time
i = $\sqrt{-1}$

We are not interested in the transient period but want a solution when steady periodic conditions prevail, that is, a solution is sought of the form

$$h(x,t) = \varphi(x)e^{i\omega t} \tag{2}$$

Substitution of Eq. (2) into Eq. (1) yields

$$\frac{d^2\varphi}{dx^2} + \frac{I}{T}e^{-i\omega t} - i\alpha\omega\varphi = 0 \tag{3}$$

Since the problem is symmetric, only $x>0$ is considered. Eq. (3) can easily be solved using the Fourier cosine transform with respect to x. The result is

$$\varphi(x) = \frac{2q}{T}\int_0^\infty \frac{\sin(\lambda a)\,\cos(\lambda x)}{\lambda(\lambda^2 + i\alpha\omega)}\,d\lambda$$

Using Gradshteyn and Ryzhik (1965), the solution is

$$\varphi(x) = \begin{cases} \dfrac{q}{i\alpha\omega T}\left(1 - e^{-a\sqrt{i\alpha\omega}}\cosh(x\sqrt{i\alpha\omega})\right) & 0 \le x \le a \\[2ex] \dfrac{q}{i\alpha\omega T}\, e^{-x\sqrt{i\alpha\omega}}\sinh(a\sqrt{i\alpha\omega}) & x > a \end{cases}$$

Here $\varphi(x)$ is given by a complex function which contains the amplitude $|\varphi(x)|$ and the phase $\theta(x)$.

In Fig. 4, dimensionless amplitude $|\varphi(x)/\varphi(0)|$, and the relative phase $\theta(x)-\theta(0)$ is plotted against dimensionless distance x/a.

Eq. (1) has been solved for the special case $\omega = 0$ by Polubarinova-Kochina (1962).

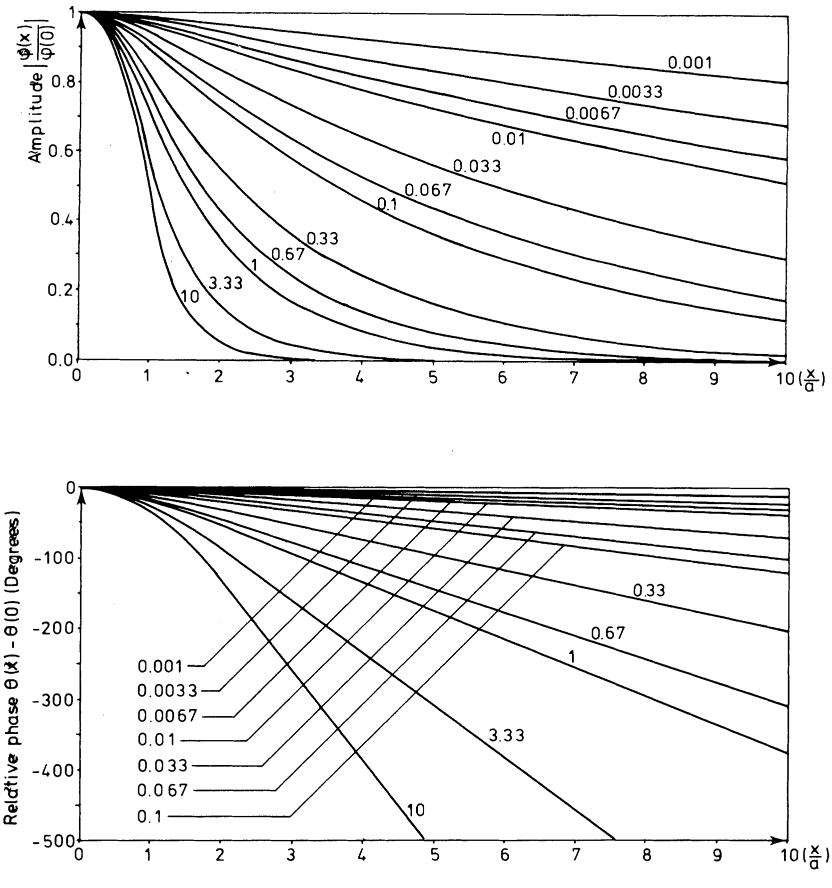


Fig. 4. The periodic infinite strip source. Amplitude and phase. Numbers on the curves are values of $\alpha \omega a^2$.

For an aquifer of infinite depth the term $\partial^2 h/\partial z^2$ should be included in the left hand side of Eq. (1), z being the vertical coordinate. The steady periodic solution would read

$$\varphi(x,z) \doteq \frac{2q}{\pi K} \int_0^\infty \frac{\sin(\lambda a)\ \cos(\lambda x)\ e^{-z\sqrt{\lambda^2 + i\alpha\omega}}}{\lambda\sqrt{\lambda^2 + i\alpha\omega}}\, d\lambda \qquad (4)$$

where K is the permeability. The integral can be expressed by ker and kei functions (see McLachlan (1961) for definitions) in the plane $z=0$ (the surface of the aquifer), otherwise it must be evaluated numerically. Carslaw and Jaeger (1959) give a solution to the integral Eq. (4) for the case $\omega = 0$.

The Periodic Disc Source

In this case the differential equation reads

$$\frac{\partial^2 h(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial h(r,t)}{\partial r} + \frac{I(t)}{T} = \alpha \frac{\partial h(r,t)}{\partial t} \qquad (5)$$

where the symbols used are the same as before, the radius R of the disc replacing a. Inserting

$$h(r,t) = \varphi(r)e^{i\omega t}$$

in Eq. (5) yields the ordinary differential equation

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + \frac{I}{T}e^{-i\omega t} - i\alpha\omega\varphi = 0 \qquad (6)$$

Eq. (6) can be solved using the Hankel transform of order 0, with respect to r. The result is

$$\varphi(r) = \frac{qR}{T}\int_0^\infty \frac{J_0(\lambda r)J_1(\lambda R)}{\lambda^2 + i\alpha\omega}\,d\lambda \quad \text{where} \qquad (7)$$

J_0 and J_1 are Bessel functions of the first kind of order 0 and 1 respectively.

From Gradshteyn and Ryzhik (1965) the integral can be expressed as

$$\varphi(r) = \frac{qR}{T(i\alpha\omega)^{\frac{1}{2}}} I_1(R\sqrt{i\alpha\omega})\,K_0(r\sqrt{i\alpha\omega}) \qquad r > R \qquad (8)$$

For $r<R$, the integral Eq. (7) cannot be expressed in closed form.

I_1 and K_0 are modified Bessel functions of the first and second kind of order 1 and 0 respectively.

From the theory of Bessel functions, the amplitude $|\varphi(r)|$ and the phase $\theta(r)$ can be found from Eq. (8). The result is

$$|\varphi(r)| = \frac{qR^2}{T\sqrt{\alpha\omega R^2}} M_1(\sqrt{\alpha\omega R^2})\;N_0\left(\frac{r}{R}\sqrt{\alpha\omega R^2}\right) \qquad r > R$$

$$\theta(r) = \theta_1(\sqrt{\alpha\omega R^2}) + \varphi_0\left(\frac{r}{R}\sqrt{\alpha\omega R^2}\right) - \frac{3\pi}{4} \qquad r > R$$

For definitions of M_1, N_0, θ_1 and φ_0, see McLachlan (1961).

In Fig. 5, the dimensionless amplitude $|\varphi(r)/\varphi(R)|$ and the relative phase $\theta(r)-\theta(R)$ are plotted against dimensionless distance r/R.

Eq. (5) has been solved for the special case $\omega = 0$ by Polubarinova-Kochina (1962).

For an aquifer of infinite depth, like before the term $\partial^2 h/\partial z^2$ should be added to the left hand side of Eq. (5). The steady periodic solution would then read

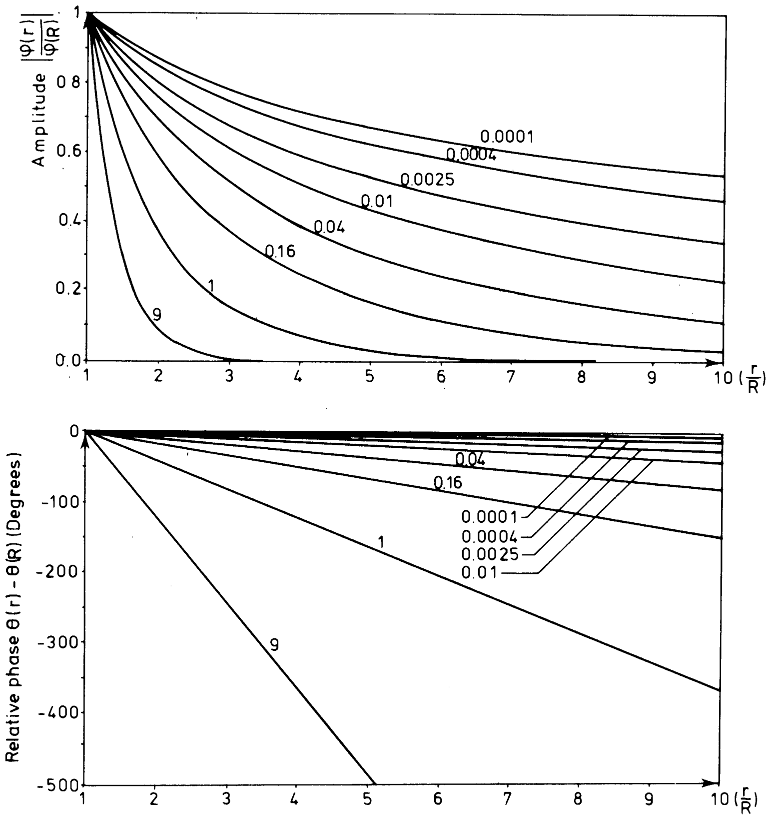


Fig. 5. The periodic disc source. Amplitude and phase. Numbers on the curves are values of $\alpha\omega R^2$.

$$\varphi(r,z) = \frac{qR}{K}\int_0^\infty \frac{J_0(\lambda r)J_1(\lambda R)e^{-z\sqrt{\lambda^2+i\alpha\omega}}}{\sqrt{\lambda^2+i\alpha\omega}}\,d\lambda$$

This integral must be evaluated numerically. For $\omega = 0$, the solution is given by Carslaw and Jaeger (1959). Their solution can be expressed by the hypergeometric function in the plane $z=0$.

Case Study

As an example of recharge over a strip the area marked I in Fig. 2 is used. Amplitudes are measured along the line A-A, and from a geological map the width of the strip is assumed to be 0.5 km. The amplitudes are plotted and matched with the typecurves, Fig. 4, as shown in Fig. 6. From this $\alpha\omega a^2$ is found to be

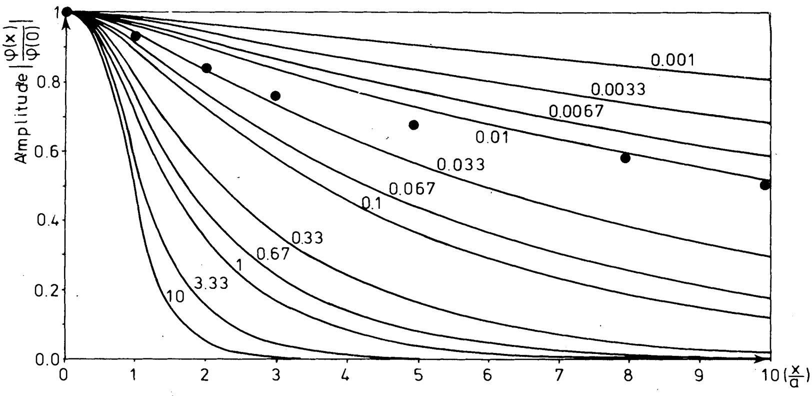


Fig. 6. Example of periodic recharge over a strip. Line A-A in area I of Fig.2 is used. Numbers on the curves are values of $\alpha\omega a^2$.

approximately 0.02 and accordingly $1/\alpha = T/S \simeq 2.5$ m²/s, which is within the range of values found from wells in the area.

The case of the disc source is illustrated using the area marked II in Fig. 2. Amplitudes are measured along the line B-B. The radius of the area is estimated to be 3 km. When the amplitudes are compared with typecurves as shown in Fig. 7, $\alpha\omega R^2$ is found to be approximately 0.01 and the diffusivity $T/S \simeq 180$ m²/s, which is 3-4 times larger than would be expected. However, the condition of homogeneity is not fulfilled in this area, which is situated on a geological boundary between Paleocene and Danien deposits with different transmissivities. Furthermore nearby pumpage and the existence of lakes and streams in the area may limit the use of the method here.

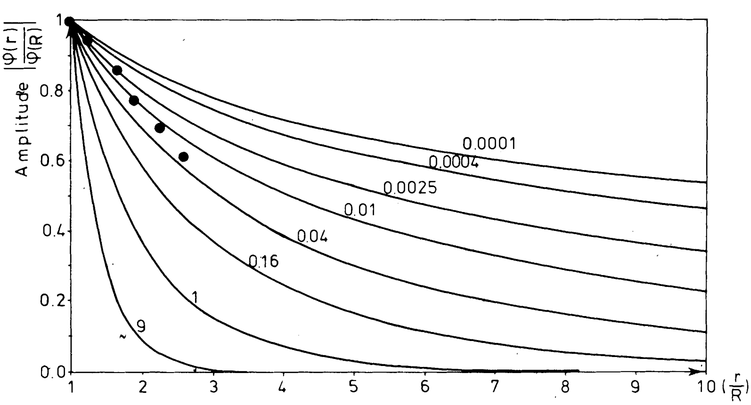


Fig. 7. Example of periodic recharge over a disc. Line B-B in area II of Fig.2 is used. Numbers on the curves are values of $\alpha\omega R^2$.

References

Carslaw, H.S., and Jaeger, J.C. (1959) *Conduction of Heat in Solids,* 2. Ed. Oxford.
Gradshteyn, I.S., and Ryzhik, I.M. (1965) *Table of Integrals, Series and Products,* New York.
Polubarinova-Kochina, P.Ya. (1962) Theory of Ground Water Movement. Princeton University.
McLachlan, N.W. (1961) *Bessel Functions for Engineers,* 2. Ed. Oxford.
Environmental Projects No. 3 (1977) Hydrological investigations in the Suså catchment applying hydrological models. Institute of Hydrodynamics and Hydraulic Engineering, Techn. Univ. of Denmark for the Danish Agency of Environmental Protection. (In Danish).

Received: 18 May, 1978

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