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Predictive Aggregation Models in Hydrology

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A prediction model capable of aggregation with recursive unbiased minimum variance estimation algorithms based on the Kalman filter technique has been formulated and applied for predicting monthly flows such that their summation is equal to annual flow in the same year. The model represents a discrete linear stochastic system where the states are defined as monthly flows in addition to the measurement equation with time invariant measurement matrix and annual flow measurement. Provided that observed or generated annual flows are available then the proposed model can be employed to predict monthly flows so that their aggregation yields the total annual flow.

Introduction

The design and operation of water resources system can be achieved objectively by operational hydrology techniques which have been developed at an explosive rate during the last decade. Especially, in the reservoir design and operation the runoff volumes expected to occur over the future economic life of the system must be predicted in a 'best' possible way. For this purpose, in addition to the commonly used annual and seasonal Markov or ARIMA type of models, various other techniques such as the fractional Gaussian noises to generate future likely synthetic sequences have been proposed some of which are capable to account for the long term as well as the short term properties inherited from the historic record.

During the past 15 years, an extensive research effort has been spent on analysing theoretically, experimentally as well as empirically the stochastic characteristics and thus devising suitable generating mechanisms of single site hydrological variables. These researches have been initiated by the pioneering work of Thomas and Fiering (1962) with the aim of generating annual or seasonal runoff volumes. Subsequently, among others Yevjevich (1963), Quimpo (1967), Beard (1967), Mandelbrot and Wallis (1969a,b,c), O'Connell (1971), Mejia et. al. (1972), Jackson (1975) and Phien and Ruksasilp (1981) have proposed various alternatives.

Since, the information available at a single site is not sufficient, in an actual design and operation of water resources systems on a regional basis, the joint analysis of all the sites concerned within the project area become indispensible requiring multivariate generating schemes. Therefore, in the mean time, several multivariate models have been deviced by Fiering (1964), Matalas (1967), Matalas and Wallis (1971) and Mejia et. al. (1974). These models preserve both long- and short-term covariance and cross-covariance properties of the hydrologic records.

However, among the multivariate models the one proposed by Valencia and Schaake (1973) and referred to as the disaggregation model, deserves a special attention due to its capability to maintain relevant statistics at all aggregation levels. However, it requires a great amount of computer memory and time for parameters estimation. A common point to all of the models proposed heretofore is that, their parameter estimations have been achieved through the application of either moments or least squares or maximum likelihood techniques all of which require the available historic data to reside in the computer memory as a whole.

The parameters estimation of the proposed model are obtained by the application of the Kalman filter (Kalman 1960) which executes the data as it becomes available; hence, giving rise to no problem as far as the computer memory is concerned. The model is of predictive type with recursive estimation of parameters and state variables. Hereafter, the model will be referred to as the predictive aggregation model. The necessary equations both for parameter and state estimations have been developed leading to unbiased and minimum variance estimations. The main advantages of this model are that, (i) it is computationally very efficient; (ii) only the current data are needed to be stored in the computer memory; (iii) the computation time is short; (iv) the estimations are updated with the coming of new data.

The Model Description

The predictive aggregation model has two components which take the simple implicit mathematical forms as

$$\underline{x}_k = A_k \underline{x}_{k-1} + B_k \underline{e}_k \qquad (1)$$

and

$$z_k = \underline{h}_k \underline{x}_k + v_k \tag{2}$$

where capital letters denote matrices and vectors are shown by lowercase letter with an underbar. The meanings of notations are as follows:

$\underline{x}_k$ is a (12×1) vector of states at time k, (monthly flows).
A_k is a (12×12) time invariant transition matrix representing dynamics of the system.
B_k is a (12×12) time invariant matrix representing the effect of the noise input on the state vector, $\underline{x}_k$.
$\underline{e}_k$ is a (12×1) vector of independent random variables which are independent of $\underline{x}_{k-1}$.
z_k is a measured value corresponding to the total annual flow.
$\underline{h}_k$ is a (1×12) time invariant vector of measurement dynamics.
v_k is the measurement noise.

The model in Eq. (1) is a first order discrete linear expression and the problem is to estimate the unknown parameter matrices A_k and B_k from a given historic data for $\underline{x}_k$. As will be evident from the subsequent sections the historic data need neither to be transformed into Gaussian nor to have zero means. Hereafter, Eqs. (1) and (2) will be referred to as the system and measurement equations, respectively.

In fact, if considered individually and provided that the parameters are given the system equation gives the prediction of monthly flows from the previous year's monthly flows. However, the measurement equation aggregates these predicted monthly flows in such a way that their summation within a given error is equal to the total annual flow in the same year. Furthermore, a joint use of the two equations through a Kalman filter procedure gives rise to adaptive prediction of monthly flows and simultaneously their aggregation to the total annual flow. Contrarily, the same procedure can be viewed as a disaggregation procedure capable of disaggregating given sequence of total annual flows into the respective monthly flows. Especially, when the measurement error is equal to zero then the summation of monthly flows become equal to the annual flow.

Various physical interpretations can be attached to variables in Eqs. (1) and (2) in the hydrological context. For instance, in the rainfall-runoff studies $\underline{x}_k$ might represent rainfall and z_k the resulting runoff and vice versa. In fact, Eq. (1) subsumes most of the models employed in the hydrological studies concerned with simulation. In the case of one dimensional studies it may represent either the stationary lag-one Markov process or the Thomas-Fiering model with seasonal parameters. On the other hand, an interesting property of Eq. (1) is that it leads to Matalas (1967) multivariate autoregressive model.

As for the statistical properties, the system noise e_k is assumed to have zero mean vector and (12×12) covariance matrix, Q_k, as,

$$E(\underline{e}_k) = \underline{0} \quad \text{and} \quad E(\underline{e}_k \, \underline{e}_k^T) = Q_k \tag{3}$$

The same assumptions are also valid for the measurement noise, v_k, therefore

$$E(v_k) = 0 \quad \text{and} \quad E(v_k\, v_k^T) = R_k \tag{4}$$

where R_k is the measurement error variance. It is further assumed that the system and measurement noises are uncorrelated, that is

$$E(\underline{e}_k\, v_k^T) = \underline{0} \tag{5}$$

where superscript T denotes the matrix transpose. On the other hand, Eqs. (1) and (2) can be written explicitly as,

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_i \\ \cdot \\ x_{12} \end{bmatrix}_k = \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1i} & \cdot & a_{112} \\ a_{21} & a_{22} & \cdot & a_{2i} & \cdot & a_{212} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ a_{i1} & a_{i2} & \cdot & a_{ii} & \cdot & a_{i12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{121} & a_{122} & \cdot & a_{12i} & \cdot & a_{1212} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_i \\ \cdot \\ x_{12} \end{bmatrix}_{k-1} + \begin{bmatrix} b_{11} & b_{12} & \cdot & b_{1i} & \cdot & b_{112} \\ b_{21} & b_{22} & \cdot & b_{2i} & \cdot & b_{212} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{i1} & b_{i2} & \cdot & b_{ii} & \cdot & b_{i12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{121} & b_{122} & \cdot & b_{12i} & \cdot & b_{1212} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ e_i \\ \cdot \\ e_{12} \end{bmatrix}_k \tag{6}$$

and

$$z_k = [h_1\ h_2\ \ldots\ h_{12}] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_{12} \end{bmatrix} + v_k \tag{7}$$

where $[h_1\, h_2 \cdot \cdot h_{12}]$ are the weights attached to the monthly flows so that their linear combination with flows in a year yields the total annual flow in this year plus an error. However, if the summation is required to be equal to the total annual flow in the same year as proposed by Valencia and Schaake (1973) then in Eq. (7) one has to assume that $h_1 = h_2 = \cdot \cdot = h_{12} = 1.0$ and $v_k = 0$ which implies $R_k = 0$. In such a case Eqs. (6) and (7) considered together give monthly flows the summation of which is equal to the total annual flow. Contrarily, given the total annual flow, one is interested in finding the constituent monthly flows as is possible through disaggregation model proposed by Valencia and Schaake (1973). The main interest in this paper is to disaggregate the total annual flows into respective monthly flows through an adaptive procedure based on the Kalman filter.

It is well established that in any modeling problem there are three stages, namely, identification, parameters estimation and diagnostic checking (Box and Jenkins 1970). By hypothesing that the model in Eqs. (1) and (2) fits suitably the generating mechanism of the phenomenon concerned one has to find the parameters estimation from the available historic data. For the purpose of this paper, the general approach to the state estimates of model in Eqs. (1) and (2) will be summarized on the basis of Kalman filters. However, the fundamental of such a filter can be found in the pioneering works of Kalman (1960) and Kalman and Bucy (1961). In addition, practical aspects of the Kalman filters have been presented by Gelb (1974) who gave an algorithm to perform an optimal estimation. The state estimate extrapolation $\hat{\underline{x}}_{k/k-1}$ to time k prior to any measurement can be obtained by exploiting the information available at time (k-1) through the system equation as,

$$\hat{\underline{x}}_{k/k-1} = A_{k/k-1}\,\hat{\underline{x}}_{k-1/k-1} \tag{8}$$

where $\hat{\underline{x}}_{k-1/k-1}$ denotes all of the available information by all means up to time (k-1). The error covariance extrapolation is given as,

$$P_{k/k-1} = A_{k/k-1}\,P_{k-1/k-1}\,A^{T}_{k/k-1} + B_{k/k-1}\,Q_{k}\,B^{T}_{k/k-1} \tag{9}$$

where by definition the prediction covariance is,

$$P_{k/k} = E(\underline{x}_{k/k} - \hat{\underline{x}}_{k/k})(\underline{x}_{k/k} - \hat{\underline{x}}_{k/k})^{T} \tag{10}$$

The updated estimations of state vector and covariance matrix after the measurement can be obtained by incorporating the measurement equation leading to,

$$\hat{\underline{x}}_{k/k} = \hat{\underline{x}}_{k/k-1} + \underline{K}_{k}[z_{k} - \underline{h}_{k}\,\hat{\underline{x}}_{k/k-1}] \tag{11}$$

and

$$P_{k/k} = P_{k/k-1} - \underline{K}_{k}\,\underline{h}_{k}\,P_{k/k-1} \tag{12}$$

respectively, herein $\underline{K}_k$ is known as the Kalman filter gain given by,

$$\underline{K}_{k} = P_{k/k-1}\,\underline{h}^{T}_{k}\,[\underline{h}_{k}\,P_{k/k-1}\,\underline{h}^{T}_{k} + R_{n}]^{-1} \tag{13}$$

In the following calculations with no loss of generality, the system and measurement error covariances will be taken as identity matrices.

Parameter Estimation

Since the objective is to find the parameter estimations by the Kalman filter applications, first of all the relevant system and measurement equations for the parameters similar to Eqs. (1) and (2) must be deviced. Therefore, original Kalman filter state estimation shortly summarized in the previous section need to be modified. An important point to be noticed at this stage is that parameter matrices A_k and B_k, exist in the system equation only and therefore, their estimation need to be based on the monthly flow data hence their estimation is independent of the total annual flow.

The parameters in Eq. (1) are assumed to be invariant and therefore the system equation for A can be written as,

$$A_k^T = A_{k-1}^T + W_k^T \tag{14}$$

where W_k is (12×12) system noise component of parameters with $E(W_k^T) = 0$ and $E(W_k^T W_k) = S_k^T$; S_k being the covariance matrix. For a given sequence of observed monthly state vector the relevant measurement equation A_k^T can be written as,

$$\underline{x}_k^T = \underline{x}_{k-1}^T A_k^T + V_k^T \tag{15}$$

V_k is (1×12) error vector of measurement equation. Here, $E(V_k^T)=0$ and $E(V_k^T V_k) = R_k$. However, it is assumed that $R_k=0$ which means to say that all of the available monthly flow information is exploited in estimating matrix A_k. The reason for employing transpozes in Eqs. (14) and (15) is entirely due to get a consistency with Eqs. (1) and (2). The sample estimation of matrix A_k^T can be evaluated by Kalman filter provided that an observation sequence of monthly state vector $\underline{x}_k$ is given for a finite time period. In Eq. (15) $\underline{x}_k^T$ plays the role of currently measured vector as z_k in Eq. (2) and $\underline{x}_k^T{}_{-1}$ corresponds to measurement matrix $\underline{h}_k$. Since, the transition matrix representing system's dynamics in Eq. (14) is (12×12) identity matrix it has not been shown explicitly. After some simple algebra in the light of Eqs. (8)-(13) necessary steps in the recursive estimation procedure of matrix A^T can be written as,

$$\hat{A}_{k/k-1} = \hat{A}_{k-1/k-1} \tag{16}$$

and the error covariance update can be evaluated simply as,

$$P_{k/k-1} = P_{k-1/k-1} + S_k^T \tag{17}$$

where the diagonal elements in matrix $P_{k/k-1}$ give the total variance of weights attached to months and off diagonal elements are the total covariances between pairs of weight vectors. For instance, a location (i,j) in the matrix has the following explicit form,

$$(p_{ij})_{k/k-1} = \sum_{j=1}^{12} \mathrm{Cov}\,(a_{ij}, a_{jj}) \qquad (i,j = 1,2, \ldots 12)$$

In other words, the Kalman estimation of matrix A_k is based on the minimization of the total covariance between pairs of the matrix elements. Subsequently, parallel to Eqs. (11), (12) and (13) one can obtain the relevant estimation equations as,

$$\hat{A}_{k/k}^T = \hat{A}_{k/k-1}^T + \underline{K}_k[\underline{x}_k^T - \underline{x}_{k-1}^T \hat{A}_{k/k-1}^T] \tag{18}$$

$$P_{k/k} = P_{k/k-1} - \underline{K}_k \underline{x}_{k-1} P_{k/k-1} \tag{19}$$

and

$$\underline{K}_k = P_{k/k-1} \underline{x}_{k-1} [\underline{x}^T_{k-1} P_{k/k-1} \underline{x}_{k-1}]^{-1} \tag{20}$$

The resultant matrix multiplication in square brackets of Eq. (20) is in fact a scalar value the inverse of which is simple to take. Therefore, the successive executions of Eqs. (16)-(20) digest available monthly data through simple matrix algebra. This is an important advantage over other widely used estimation procedures where the inverse of a matrix is unavoidable. More explicitly Eqs. (16)-(20) can be written in terms of elements at i-th row and j-th column leading to,

$$(\hat{a}_{ji})_{k/k-1} = (\hat{a}_{ji})_{k-1/k-1} \tag{21}$$

$$(p_{ij})_{k/k-1} = (p_{ij})_{k-1/k-1} \tag{22}$$

$$(\hat{a}_{ji})_{k/k} = (\hat{a}_{ji})_{k/k-1} + \left\{ \sum_{j=1}^{12} (p_{ij})_{k/k-1} (x_j)_{k-1} \times \times [(x_j)_k - \sum_{i=1}^{12} (\hat{a}_{ji})_{k/k-1} (x_i)_{k-1}] \right\} \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (x_i)_{k-1} (p_{ij})_{k/k-1} (x_j)_{k-1} \tag{23}$$

$$(p_{ij})_{k/k} = (p_{ij})_{k/k-1} - \sum_{j=1}^{12} (x_j)_{k-1} (p_{ij})_{k/k-1} \sum_{i=1}^{12} (p_{ij})_{k/k-1} (x_i)_{k-1} \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (x_i)_{k-1} (p_{ij})_{k/k-1} (x_j)_{k-1} \tag{24}$$

and

$$k_{i1} = \sum_{j=1}^{12} (p_{ij})_{k/k-1} (x_j)_{k-1} \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (x_i)_{k-1} (p_{ij})_{k/k-1} (x_j)_{k-1} \tag{25}$$

After having evaluated the estimate of A_k the residuals can be found from Eq. (1) as,

$$B_k \underline{e}_k = \underline{x}_k - A_k \underline{x}_{k-1}$$

or by defining a new residuals vector as

$$\underline{u}_k = B_k \underline{e}_k$$

where $\underline{u}_k$ may be regarded as a vector of apparent 'error' in the historic sample function of monthly flows. The vector of independent random variables, $\underline{e}_k$, can be generated as a sequence of white noise with zero mean and unit variance; hence it can be considered as a known quantity. In the light of this information the state and measurement equations for matrix B_k can be written as,

$$B_k^T = B_{k-1}^T \tag{26}$$

and

$$\underline{u}_k^T = \underline{e}_k^T B_k^T \tag{27}$$

where $\underline{e}_k^T$ plays the role of the measurement dynamics matrix similar to $\underline{h}_k$ in Eq. (2). The filtering equations for the recursive estimation of B_k can be found in their implicit forms as,

$$B_{k/k-1}^T = B_{k-1/k-1}^T \tag{28}$$

$$P_{k/k-1} = P_{k-1/k-1} \tag{29}$$

Here again the elements of error covariance extrapolation includes the total covariances of error weight vectors.

$$B_{k/k} = B_{k/k-1} + \underline{K}_k[\underline{u}_k - \underline{e}_k B_{k/k-1}] \tag{30}$$

$$P_{k/k} = P_{k/k-1} - \underline{K}_k \underline{e}_k P_{k/k-1} \tag{31}$$

and

$$\underline{K}_k = P_{k/k-1} \underline{e}_k [\underline{e}_k^T P_{k/k-1} \underline{e}_k]^{-1} \tag{32}$$

whereas the explicit formulations are,

$$(b_{ji})_{k/k-1} = (b_{ji})_{k-1/k-1} \tag{33}$$

$$(p_{ij})_{k/k-1} = (p_{ij})_{k-1/k-1} \tag{34}$$

$$(b_{ji})_{k/k} = (b_{ji})_{k/k-1} + \left\{ \sum_{j=1}^{12} (p_{ij})_{k/k-1} (\varepsilon_j)_{k-1} \times \times [(u_j)_k - \sum_{i=1}^{12} (b_{ji})_{k/k-1} (\varepsilon_i)_{k-1}] \right\} \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (\varepsilon_i)_{k-1} (p_{ij})_{k/k-1} (\varepsilon_j)_{k-1} \tag{35}$$

$$(p_{ij})_{k/k} = (p_{ij})_{k/k-1} - \sum_{j=1}^{12} (\varepsilon_j)_{k-1} (p_{ij})_{k/k-1} \sum_{i=1}^{12} (p_{ij})_{k/k-1} (\varepsilon_i)_{k-1} \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (\varepsilon_i)_{k-1} (p_{ij})_{k/k-1} (\varepsilon_j)_{k-1} \qquad (36)$$

$$k_{i1} = \sum_{j=1}^{12} (p_{ij})_{k/k-1} (\varepsilon_j)_{k-1} \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (\varepsilon_i)_{k-1} (p_{ij})_{k/k-1} (\varepsilon_j)_{k-1} \qquad (37)$$

where ε_i's denote elements of the white noise vector. It can be shown likewise to A_k that the estimation procedure given for B_k is based on the monthly flow data only.

State Estimation

With the parameters known or estimated by procedures presented in the previous sections Eqs. (1) and (2) can be used jointly for predicting the monthly flows so that their summation is equal to the total annual flow. In this part of the estimation only the total annual flow values are needed as data. The main purpose is to find the monthly flows of the same year which means a disaggregation. The best least square unbiased and minimum variance estimate of the unknown state vector $\underline{x}_k$ can be obtained by applying the linear Kalman filter which after some algebra leads to,

$$\hat{\underline{x}}_{k/k-1} = A_k \hat{\underline{x}}_{k-1/k-1} \qquad (38)$$

$$P_{k/k-1} = A_k P_{k-1/k-1} A_k^T + B_k B_k^T \qquad (39)$$

$$\hat{\underline{x}}_{k/k} = \hat{\underline{x}}_{k/k-1} + \underline{K}_k [z_k - \underline{h}_k \hat{\underline{x}}_{k/k-1}] \qquad (40)$$

$$P_{k/k} = P_{k/k-1} - \underline{K}_k \underline{h}_k P_{k/k-1} \qquad (41)$$

and

$$\underline{K}_k = P_{k/k-1} h_k^T [\underline{h}_k P_{k/k-1} h_k^T]^{-1} \qquad (42)$$

The corresponding explicit formulations are,

$$(x_i)_{k/k-1} = \sum_{j=1}^{12} a_{ij} (x_j)_{k-1/k-1} \qquad (43)$$

$$(p_{ij})_{k/k-1} = \sum_{k=1}^{12} a_{jk} \sum_{l=1}^{12} a_{il} (p_{lk})_{k/k-1} + \sum_{k=1}^{12} b_{ij}\, b_{jk} \qquad (44)$$

$$(x_i)_{k/k} = (x_i)_{k/k-1} + \sum_{j=1}^{12} (p_{ij})_{k/k-1} [z_k - \sum_{j=1}^{12} (x_j)_{k/k-1}] \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (p_{ij})_{k/k-1} \qquad (45)$$

$$(p_{ij})_{k/k} = (p_{ij})_{k/k-1} - \sum_{i=1}^{12} (p_{ij})_{k/k-1} \sum_{j=1}^{12} (p_{ij})_{k/k-1} \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (p_{ij})_{k/k-1} \qquad (46)$$

and

$$k_{i1} = \sum_{j=1}^{12} (p_{ij})_{k/k-1} \Big/ \sum_{i=1}^{12} \sum_{j=1}^{12} (p_{ij})_{k/k-1} \qquad (47)$$

the summation of each one of the terms in Eq. (45) for 12 states leads to,

$$z_k = \sum_{i=1}^{12} (\hat{x}_i)_{k/k} \qquad (48)$$

which proves that the summation of lower-level (short-time) events yields the corresponding higher-level (long-time) events. For instance, the summation of monthly flows in a year gives the total annual flow. On the other hand, Eq. (48) shows that a linear relation will be preserved exactly by the predictive aggregation model between historical data at successive levels of aggregation.

Model Application

The aforementioned model is applied to St. Lawrence and Göta rivers. The characteristics of these rivers are given in Table I.

Table 1 – Drainage basin characteristics

River name	Station name	Drainage area (km^2)	Observation period
Göta (Sweden)	Vanesborg	46,480	1807-1964
St. Lawrence (USA)	St. Lawrence	614,000	1879-1964

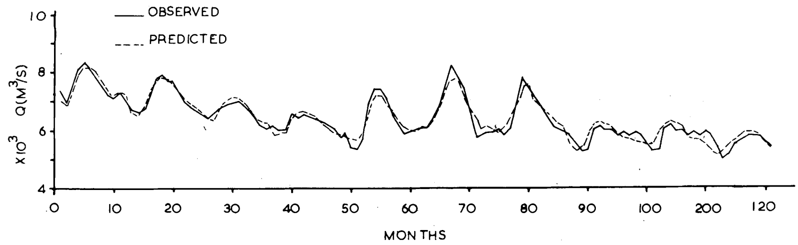

Fig. 1. Observed and predicted monthly river flows of the St. Lawrence river.

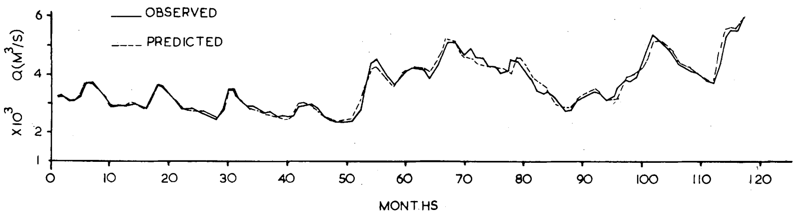

Fig. 2. Observed and predicted monthly river flows of the Göta river.

The primary concern is to disaggregate the total annual flows into monthly flows. To initiate the recursive estimation of matrix A_k its initial form is arbitrarily taken as units along the main diagonal and 0.5 values at other locations. The initial measurement error covariance. $P_{0/0}$, is assumed to possess 100's along the main diagonal and 10's at other locations. With these initial values processing of the available monthly data through Eqs. (21)-(25) gives sequences of matrices two of which are $A_{i/i}$ and $P_{i/i}$ where $i=1,2, \cdot\cdot N$; N being the number of years. As the recursive calculations advance these matrices approach to their respective asymptotic values with steadily decreasing error covariance matrix. It has been observed that the diagonal elements of $A_{k/k}$ are the greatest compared to the elements in the same row and column. This is an expected result showing the fact that the major contribution to a month comes from the same month in the previous year.

In the estimation procedure of matrix B_k the same initial values as for the A_k matrix are assumed and Eqs. (33)-(37) are executed recursively. During the execution a vector of independent random variables, $\underline{e}_k$, with zero mean and unit variance. The final B matrix in the recursive parameters estimation is employed in the state estimation.

After having estimated A_k and B_k from the monthly data the historic total annual flows are calculated. The problem is now to predict monthly flows such that their summation is equal to the total annual flow. The initial estimate $\underline{x}_{0/0}$, is taken equal to the first year's historical monthly flows. Then recursively by applying Eqs. (43)-(47) one year ahead predictions of monthly flows are obtained. Figs. 1 and 2 show the observed and predictions of monthly flows. It is evident from

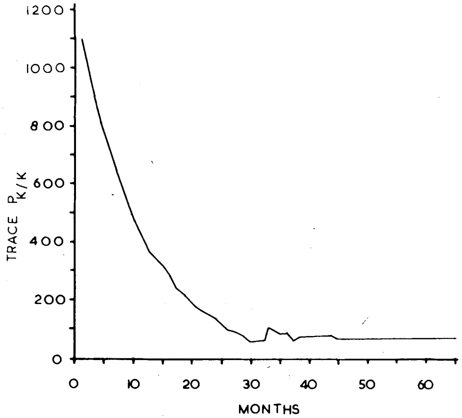

Fig. 3. Trace of the estimation error covariance update matrix for the St. Lawrence river.

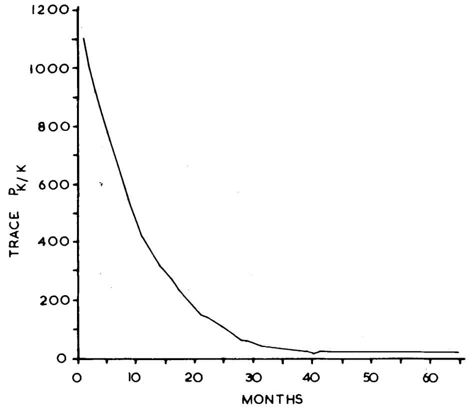

Fig. 4. Trace of the estimation error covariance update matrix for the Göta River.

these figures that the predicted and observed monthly flows follow each other very closely. On the other hand, the prediction errors that is the trace $P_{k/k}$ of the state estimation covariance matrices are depicted in Figs. 3 and 4 which show a continuous decrease and after some months it attains at a constant value.

In order to evaluate the goodness of the predictions there are several criteria. However, the one used here is an analysis of the innovation sequence $\nu_{k/k}$, defined by Kailath (1968) with notations relevant to this paper,

$$\nu_{k/k} = z_k - \sum_{i=1}^{12} (\hat{x}_i)_{k/k-1} \tag{49}$$

It has been shown by Mehra (1969) that the innovation sequence is a white Gaussian process for an optimal filter. Heuristically, this means that if $\nu_{k/k}$ is a white Gaussian noise process there is no information left in the measurement sequence. This can be further interpreted that all of the relevant information has been exploited during the estimation procedure and better estimations cannot be made with the proposed model. After some algebraic calculations the lag-*i* covariance at *k*-th recursion can be found as,

$$\begin{aligned}\mathrm{Cov}(i,k) = {} & A_{k/k-1}[P_{k-i/k-i-1}\, H^T_{k-i} - K_{k/k-i}\, H_{k-i}\, P_{k/k-1}\, H^T_{k-i}] \\ & \prod_{j=1}^{i-1} H_{k-j}\, A_{k/k-j}[I_n - K_{k/k-j}\, H_{k-j}]\end{aligned} \tag{50}$$

Hence, it is clear from Eq. (50) that in order to have all the Cov (i,k) =0 for i 1, it is sufficient that

$$P_{k/k-1}\, H_k^T - K_{k/k}\, H_k\, P_{k/k-1}\, H_k^T = 0 \tag{51}$$

This last expression, however, is satisfied by the Kalman gain derived in Eq. (42) which leads to the conclusion that the estimation procedure presented in this paper is optimal in the sense defined above.

Conclusions

A predictive aggregation model with recursive parameter estimations has been proposed, necessary estimation formulations have been explicitly derived and then applied to monthly flows. The presented estimation procedure does not impose any restrictions concerning neither the parameter matrices such as positive semidefiniteness nor the standardization of observations to have zero mean. The estimations are unbiased and have the minimum variance as properties of the Kalman filter.

The overall advantages of the model are: (i) the computations are based on simple matrix algebra; (ii) no need for an inverse matrix operation provided that the measurement matrix is scalar; (iii) the parameters and states are estimated on-line as the data becomes available; (iv) no need to preserve all the data in the computer memory; (v) computations can be performed in a short time.

It is possible to generate lower-order events which add up to the given higher-order events without any external intervention.

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