

# A grey-based method for evaluating the effects of rating curve uncertainty on frequency analysis of annual maxima

S. Alvisi and M. Franchini

## ABSTRACT

A grey-based technique for characterizing the rating curve uncertainty due to discharge measurement errors and its effect on flood frequency analysis is here presented. On the basis of river stage and discharge measurements, the grey parameters of the rating curve are estimated by using a grey non-linear regression. Commencing with this grey rating curve and a set of annual maximum stages, we show how the probability distribution (here assumed of EV1 type) of the grey annual maximum discharges can be estimated. The grey EV1 distribution can be estimated through two approaches, the first of which directly exploits the grey discharges corresponding to the annual maximum stages, whereas with the second approach two different sets of extreme (crisp) discharges, and therefore two EV1 distributions of extreme (crisp) values which delimit the grey discharges of a given return period, are obtained by considering the lower and upper limits of the grey parameters of the rating curve. The methodology is illustrated using data pertaining to a gauged section of the River Po (Italy). The results show that the first approach yields a wider grey EV1 distribution with respect to that resulting from the second approach: physical justification of this is given.

**Key words** | extreme discharges, frequency analysis, grey numbers, rating curve, uncertainty

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## INTRODUCTION

Accurate estimation of the peak flood in a given return period in a waterway section is a fundamental requisite when planning hydraulic engineering works, particularly with a view to protecting the surrounding area (Petersen-Øverleir & Reitan 2009). Estimation of the peak discharge of a given frequency in a particular section is based on the probabilistic treatment of a series of 'observed' values of annual maximum peak discharge. It is important to emphasize that these values are rarely measured directly, but are usually estimated indirectly by means of a rating curve and measured stages. The rating curve used, however, is merely an 'estimate' of the 'real' one, as it is derived from a fairly limited series of measured stages and discharges affected (particularly the latter) by measurement errors (Di Baldassarre & Montanari 2009). Consequently, the annual maximum discharge values that constitute the starting point for flood frequency analysis are, themselves, affected

by uncertainty; this uncertainty is further amplified by the fact that the annual maximum peak discharges are typically out of the range of values used to estimate the rating curve (Petersen-Øverleir & Reitan 2009).

The uncertainty in estimating these annual maximum peak discharge values affects the estimates of the discharges of a given frequency, adding to the effects of other forms of uncertainty, namely those related to (a) the choice of probability distribution, and (b) the estimation of its parameters. These latter two forms of uncertainty have been the subject of much interest, and have been explored in numerous scientific studies (see, for example, Martins & Steindinger 2000; Laio *et al.* 2009, or Maidment 1993, pp. 18.22–18.33 for a general description of uncertainty related to the choice of probability distribution and the estimation of its parameters), whereas less attention has been paid to the former, i.e. that related to the use of data (annual maximum

peak discharge values) affected by uncertainty (being taken from a rating curve that is itself uncertain).

Using these considerations as a starting point, this study considers the issue of characterizing uncertainty in the estimation of a rating curve, and how this uncertainty is 'passed on' to extreme discharge frequency analysis.

In particular, the issue of characterizing rating curve uncertainty has been the subject of various studies based on both probabilistic (see, for example, [Moyeed & Clarke 2005](#); [Reitan & Petersen-Øverleir 2008, 2009](#); [Petersen-Øverleir \*et al.\* 2009](#)) and non-probabilistic techniques, such as fuzzy set theory (see, for example, [Shrestha \*et al.\* 2007](#); [Shrestha & Simonovic 2010a](#); [Westerberg \*et al.\* 2011](#)). It is important to remark that these studies have only focused on the effect of uncertainty on the generation of rating curves and on the estimation of a generic discharge, but not on the effect it can have on the characterization of annual maximum discharge values ([Petersen-Øverleir & Reitan 2009](#)).

In contrast, as previously mentioned, only a few studies have investigated the effect of rating curve uncertainty on frequency analysis of extreme discharges. Among these studies, those by [Kuczera \(1996\)](#), [Petersen-Øverleir & Reitan \(2009\)](#) and [Di Baldassarre \*et al.\* \(2011\)](#) are particularly significant; in particular, in [Kuczera \(1996\)](#) it is assumed that the rating curve is affected by incremental error, which is proportional to the discharge, represented by a random variable featuring lognormal probability distribution of mean equal to 1 and a given variance. [Kuczera \(1996\)](#) demonstrates the significant effect of this error on estimation of the discharge of a given return period. [Petersen-Øverleir & Reitan \(2009\)](#), on the other hand, investigate the joint effect of sample variability and rating curve error on the estimate of peak discharge of a given return period; to this end, they combined the rating curve and the extreme value probability distribution in a single likelihood function that consents estimation of both the parameters of the extreme value probability distribution and the uncertainty on the discharge value due to sample variability and rating curve error. The [Petersen-Øverleir & Reitan \(2009\)](#) study considers the rating curve represented in the logarithmic plane and the corresponding error is assumed to be a random variable distributed according to a normal with mean 0 and given variance. Finally, [Di Baldassarre \*et al.\*](#)

(2011) analyse the effect of the different types of rating curve error afflicting the estimation of the peak discharge of a given return period; in particular, they assume that the discharge data error can be represented by a Gaussian random variable with zero mean and standard deviation proportional to the discharge or, according to the results reported by [Di Baldassarre & Montanari \(2009\)](#), as a combination of two different errors, one represented through a Gaussian random variable and the other through a binary random variable.

In all the above-mentioned studies, however, the rating curve uncertainty, and consequently the effect on extreme discharge frequency analysis, is dealt with using a probabilistic approach. In contrast, this article features a non-probabilistic approach based on grey numbers ([Deng 1982](#)), which allows characterization not only of the uncertainty afflicting the rating curve, but also the way that this uncertainty affects estimation of the peak discharge of a given frequency (or return period). More precisely, the target of this study is to present a procedure which allows for the 'greyification' of the following steps: (a) set up of a rating curve starting from couples of simultaneous measurements of stage and discharge, (b) definition of the annual maximum discharges from the annual maximum stages by using the rating curve, and (c) mathematical treatment of the annual maximum discharges to estimate the discharge of a given return period. These steps are implicitly performed every time a discharge frequency analysis is developed.

All non-probabilistic approaches, such as grey or fuzzy approaches, are appropriate for modelling uncertainties not originating from randomness but caused by imprecise (or incomplete) knowledge of the system of interest ([Jacquin & Shamseldin 2007](#)). More precisely, these approaches allow the integration of information of different qualities, and are well suited to situations featuring uncertainty deriving mainly from a lack of detailed understanding of the phenomenon in question, a scarcity of data, or inaccuracies in the available data ([Alvisi & Franchini 2011a, b](#)); furthermore, they require no assumptions about the characteristics of the error ([Shrestha \*et al.\* 2007](#)) and can therefore be easily adapted to allow characterization of the uncertainty linked to the measurement of discharges ([Shrestha & Simonovic 2010b](#)). In our study we opted for the grey approach since, unlike the fuzzy approach, it does not

require any arbitrary definition of ‘credibility levels’ (Dubois & Prade 1980) and shape since the grey number has no shape whereas a fuzzy number does (Huang *et al.* 1995): indeed, it simply defines an interval which can be interpreted in a very intuitive way as representative of the uncertainty or imprecision related to the quantity considered.

The following sections contain a description of the method proposed to estimate a grey rating curve, and to use said rating curve, together with a set of annual maximum stage values, to estimate the grey peak discharge of a given return period. The proposed method is also applied to a real-life case study and a discussion of the results and conclusions follows.

## THE GREY RATING CURVE

The rating curve defines the relationship between discharge and stage in a given section under *stationary conditions*, and it is usually represented using a power function (Shrestha *et al.* 2007; Herschy 2009), such as:

$$Q = ah^b \quad (1)$$

where  $h$  and  $Q$  are the stage and corresponding discharge, respectively, and  $a$  and  $b$  are parameters estimated using  $n_{\text{obs}}$  measurements of the stage,  $h_{\text{obs},i}$  (where  $i = 1:n_{\text{obs}}$ ), and corresponding discharge,  $Q_{\text{obs},i}$ , via least-square regression (see, for example, Petersen-Øverleir 2004). However, as previously mentioned, discharge and stage measurements used to estimate the parameters of the rating curve are affected by measurement error, which is a function of the type of instrument and the method used to make the measurement itself.

As regards the discharge, a wide-reaching review of uncertainty affecting its measurement is provided by Pellerier (1988). In particular, the EN ISO 748 standard (2007), based on considerations by Herschy (2009), provide indications concerning the degree of error in measuring discharge by the velocity/area method in terms of confidence interval, taking into account various forms of uncertainty, namely local velocity and the effective depth at which the measurement is taken, etc., as well as several

operational parameters, i.e. the exposure time of the current meter, the number of verticals, and the number of points on each vertical upon which measurement is performed. Shrestha & Simonovic (2010b), on the other hand, starting from the assumption that randomness is not the only source of uncertainty in discharge measurements, as they can also be affected by systematic and human error, as well as subjective evaluation, pointed out that a probabilistic approach could be unsuitable for their treatment; they therefore went on to combine the guidelines in EN ISO 748 (2007) as regards characterization of discharge measurement error with an approach based on fuzzy sets theory.

As regards error in measurement of stage, this is usually far smaller than that affecting discharge measurement (Clarke 1999); when using a float operated autographic recorder, for example, the measurement error is of the order of  $\pm 10$  mm (Hersch 2009; Shrestha & Simonovic 2010b), i.e. of the same order of magnitude of topographic errors (Di Baldassarre & Montanari 2009).

In this study we assume that the uncertainty affecting stage measurement is negligible, and, in accordance with observations made by Shrestha & Simonovic (2010b), we evaluate the uncertainty affecting discharge values by means of a non-probabilistic approach, defining the following grey number  $Q_{\text{obs}}^{\pm}$ , which represents the discharge measured with uncertainty:

$$Q_{\text{obs}}^{\pm} = Q_{\text{obs}} \cdot (1 + \Delta^{\pm}) \quad (2)$$

where  $\Delta^{\pm}$  represents the grey uncertainty in the measurement of discharge ( $\Delta^{\pm}$  corresponds to the percentage fractions  $\Delta^{-}$  and  $\Delta^{+}$ , where the central value of the interval is equal to 0) quantifiable through the approach proposed by Shrestha & Simonovic (2010b) considering the type of instrumentation and method used to perform the measurement of discharge  $Q_{\text{obs}}$ , as illustrated in the numerical application. Incidentally,  $\Delta^{\pm}$  may vary with  $Q_{\text{obs}}$  according to the precision related to all the measurements necessary to estimate a discharge when using, for instance, the velocity-area method (see Herschy 2009).

For a definition of grey numbers and considerations on the mathematics used, see Appendix A (available online at <http://www.iwaponline.com/jh/015/127.pdf>). Here, on the other hand, it is sufficient to understand that a grey

number represents a number of unknown value, but falling within an interval of known extremes. In other words, the grey value of the discharge  $Q_{obs}^\pm = [Q_{obs}^-, Q_{obs}^+]$  represents an ‘imprecise’ discharge that falls between the known extremes,  $Q_{obs}^-$  and  $Q_{obs}^+$  of the interval within which it can vary, but no information regarding its distribution within this interval is known.

In summary, we assume that the observed stages represent crisp values, while the corresponding discharge values are grey numbers. Based on these assumptions, the parameters of the rating curve of Equation (1) are estimated by means of a grey non-linear regression that represents a variant of the *fuzzy linear regression* proposed by Hojati *et al.* (2005). It is worth noting that several different formulations of the fuzzy linear regression have been proposed in the scientific literature, like, for example, that proposed by Tanaka *et al.* (1982), subsequently developed further by Bardossy *et al.* (1990) or that proposed by Peters (1994). In these formulations for an assigned credibility level, the predicted interval must contain (Tanaka *et al.* 1982) or intersect (Peters 1994) the corresponding observed interval; indeed these constraints may result in quite large predicted intervals and, in fact, with specific reference to the original formulation by Tanaka *et al.* (1982) some methods have been proposed in order to ‘control’ the width of the predicted intervals, like considering a reference point (Bardossy *et al.* 1990) or splitting the regression curve into two (or more) separate curves connecting at specific point(s) (Shrestha *et al.* 2007). Instead, the formulation proposed by Hojati *et al.* (2005), taken as reference in this paper, requires neither the inclusion nor the intersection of the observed intervals; in fact it is basically aimed at obtaining a regression band *wide enough to contain as many (grey) observations as possible and narrow enough to be of use*. It is worth noting that this latter formulation has been applied also by Westerberg *et al.* (2011) to develop a fuzzy rating curve to be used within the framework of rainfall runoff modelling.

In detail, with reference to Figure 1, the rating curve parameters are assumed to be grey numbers, that is  $a^\pm$  and  $b^\pm$ , and consequently the rating curve takes on the form:

$$Q^\pm = a^\pm \cdot h^{b^\pm} \tag{3}$$

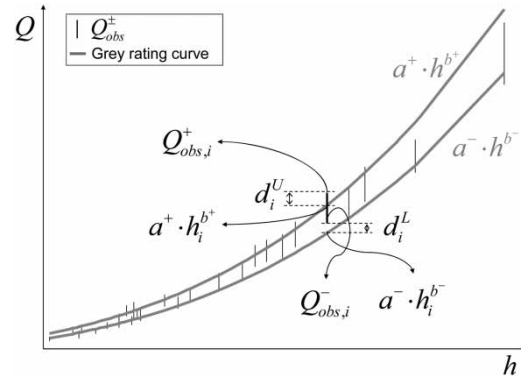


Figure 1 | Diagram for the estimation of the grey parameters of the rating curve via non-linear regression.

In particular, the lower extreme of the grey discharge  $Q^-$  represents the minimum value of discharge that can be obtained for a given stage  $h$  using grey parameters  $a^\pm$  and  $b^\pm$ , and can be calculated based on the grey mathematics outlined in Appendix A, i.e. by determining the combination of crisp values  $a^- \leq a_j \leq a^+$  and  $b^- \leq b_j \leq b^+$  that minimize the function (3) for a given value of  $h$ . Similarly,  $Q^+$  represents the upper extreme. However, due to the fact that the values of the rating curve parameters are typically positive, and therefore the function is monotonically increasing, the lower discharge limit is given by:

$$Q^- = a^- \cdot h^{b^-} \tag{4}$$

and, likewise, the upper discharge limit is given by:

$$Q^+ = a^+ \cdot h^{b^+} \tag{5}$$

Using the method proposed by Hojati *et al.* (2005) for linear regression, in this instance for non-linear conditions, estimation of the grey parameters  $a^\pm$  and  $b^\pm$  is performed by searching for the upper and lower limits  $a^-, a^+, b^-$  and  $b^+$  of said grey numbers such that:

$$\sum_{i=1}^{n_{obs}} d_i^L + d_i^U \tag{6}$$

is minimum, where  $d_i^L$  represents the distance between the lower extreme of the  $i^{th}$  observed grey discharge  $Q_{obs,i}^-$  and

the lower extreme of the grey rating curve estimated at stage  $h_i$ , i.e. (see also Figure 1):

$$d_i^L = \left| Q_{obs,i}^- - a^- \cdot h_i^{b^-} \right| \tag{7}$$

and, likewise,  $d_i^U$  represents the distance between the upper extreme of the  $i^{th}$  observed grey discharge  $Q_{obs,i}^+$  and the upper extreme of the grey rating curve estimated at stage  $h_i$ , i.e. (see also Figure 1):

$$d_i^U = \left| Q_{obs,i}^+ - a^+ \cdot h_i^{b^+} \right| \tag{8}$$

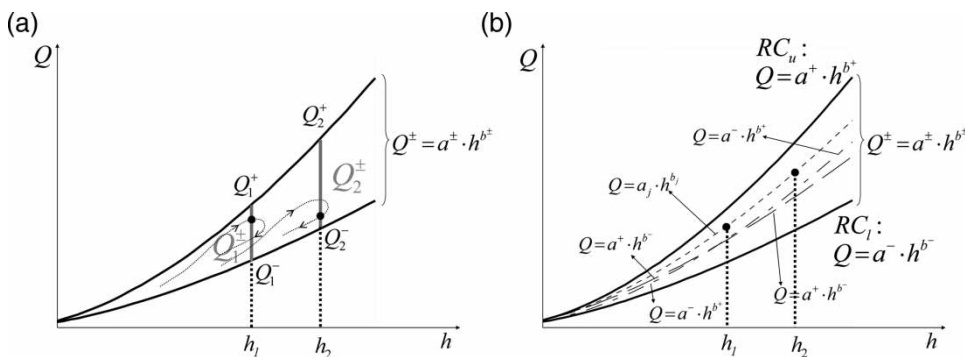
It is worth emphasizing that Equation (3), once parameterized using the above procedure, can be interpreted in two different ways, according to whether attention is focused on (a) the grey discharge values  $Q^\pm$ , or (b) the grey parameters  $a^\pm$  and  $b^\pm$ , as the stage  $h$  is fixed in both cases.

On the one hand (Interpretation (a)), for a given stage  $h$ , the grey rating curve (see Equation (3)) yields an imprecise grey discharge  $Q^\pm$ , which takes into account the measurement error of the discharges used to estimate the grey rating curve, and, adopting the definition of a grey number outlined in Appendix A, any value of crisp discharge  $Q_j$ , where  $Q^- \leq Q_j \leq Q^+$ , is possible. This latter crisp discharge  $Q_j$  may also be thought of as belonging to an unknown loop rating curve related to the specific flood condition occurring when the measurement was performed. In other words, Interpretation (a) is compatible with the idea that the grey

rating curve given by Equation (3) is a sort of envelope of *unsteady* rating curves where each discharge value is linked to its own measurement error.

On the other hand (Interpretation (b)), also adopting the definition of grey number outlined in Appendix A, reveals that the same grey rating curve parameters  $a^\pm$  and  $b^\pm$  define two grey numbers, in which it is possible to select two crisp parameters,  $a_j$  and  $b_j$ , of any value (with  $a^- \leq a_j \leq a^+$  and  $b^- \leq b_j \leq b^+$ ); each pair of crisp parameters selected from the corresponding grey numbers corresponds to a precise crisp *steady* rating curve. In practice, according to this interpretation, Equation (3) defines a *sheaf of curves*, each one representative of a different crisp steady rating curve. In particular, the lower limit of this sheaf is a ‘lower’ steady rating curve, hereinafter referred to as  $RC_l$ , described by Equation (4), i.e. the steady rating curve derived from crisp parameters represented by the lower limits of the grey parameters  $a^-$  and  $b^-$ , and the upper limit is an ‘upper’ steady rating curve, hereinafter referred to as  $RC_u$ , described by Equation (5), i.e. by the steady rating curve derived from crisp parameters represented by the upper limits of the grey parameters  $a^+$  and  $b^+$ .

These two interpretations are exemplified in Figure 2, which shows an example of a grey rating curve obtained by means of two generic grey values of parameters  $a^\pm$  and  $b^\pm$  (specifically  $a^\pm = [1, 1.35]$  and  $b^\pm = [3/2, 5/3]$ ). In particular, Figure 2(a) shows two grey  $Q^\pm$  numbers that can be estimated from the grey rating curve at given stages  $h_1$  and  $h_2$ , evidencing that the values of crisp discharge highlighted are not



**Figure 2** | Example of grey rating curve obtained from two generic grey values of parameters  $a^\pm$  and  $b^\pm$ : (a) grey discharges  $Q_1^\pm$  and  $Q_2^\pm$  obtained for given stages  $h_1$  and  $h_2$  (the crisp discharge values marked with the symbol • each belong to the corresponding grey discharge but do not necessarily belong to a crisp steady rating curve, instead, they can be imagined as positioned on two different loop/unsteady rating curves), and (b) sheaf of crisp rating curves obtained for different pairs of crisp parameters included in the corresponding grey values  $a^\pm$  and  $b^\pm$  (the two crisp discharge values marked with the symbol • belong to a crisp rating curve derived from a pair of crisp parameters  $a_j$  and  $b_j$ , each belonging to the corresponding grey parameters  $a^\pm$  and  $b^\pm$ ).

necessarily positioned on a ‘crisp steady rating curve’ positioned ‘inside’ the grey rating curve (Interpretation (a)): instead, they can be thought as positioned on two different loop/unsteady rating curves: the first value relative to the arrival phase of the corresponding flood and the second value relative to the depletion phase of the corresponding flood.

Figure 2(b) shows several of the possible crisp steady rating curves that are obtained using different pairs of crisp values for the parameters  $a_j$  and  $b_j$ , where  $a^- \leq a_j \leq a^+$  and  $b^- \leq b_j \leq b^+$ ; in this case, having established two generic stages  $h_1$  and  $h_2$  and a pair of parameters  $a_j$  and  $b_j$ , where  $a^- \leq a_j \leq a^+$  and  $b^- \leq b_j \leq b^+$ , two values of discharge that sit firmly on a crisp steady rating curve (Interpretation (b)) are obtained. This latter interpretation is thus useful for steady and quasi-steady flow conditions but also when floods occur in a kinematic way.

All previous aspects have effects on the frequency analysis of grey discharge, as explained in the numerical application.

taking into account the uncertainty of the discharge values represented by the grey-parameter rating curve. To this end, it is assumed that the maximum stages observed at the section in question over a period of  $n_y$  years are known, i.e.  $h_{max,k}$ , where  $k = 1:n_y$ . Then, starting from these maximum stages and using the grey rating curve, the grey annual maximum discharges are computed. However, in order to better explain how the frequency analysis of these grey discharges can be performed, a typical frequency analysis based on crisp discharges is initially described. The formulae obtained for the crisp case will then be generalized to the grey case on the basis of the extension principle (Zadeh 1965). In this latter case, both Interpretation (a), focussing only on the grey discharges supplied by the grey rating curve at the given annual maximum stages, and Interpretation (b), focussing on the grey parameters  $a^\pm$  and  $b^\pm$  and on the discharge values derived from the corresponding grey rating curve, will be considered.

### ANALYSIS OF ANNUAL MAXIMUM DISCHARGE VALUES ESTIMATED BY GREY-PARAMETER RATING CURVE

Let us now consider the issue of probabilistic treatment of annual maximum discharge in a given section in order to estimate the (grey) discharge of a given return period, thus

### Crisp treatment of annual maximum discharge

In this case, considering a number of  $n_y$  values of annual maximum stages, the crisp-parameter rating curve yields  $n_y$  crisp annual maximum discharge values  $Q_{max,k}$ , where  $k = 1:n_y$  (see Figure 3(a)). Incidentally, this operation perfectly reflects what is done in practice, where the (steady) rating curve is assumed known (estimated through a least square

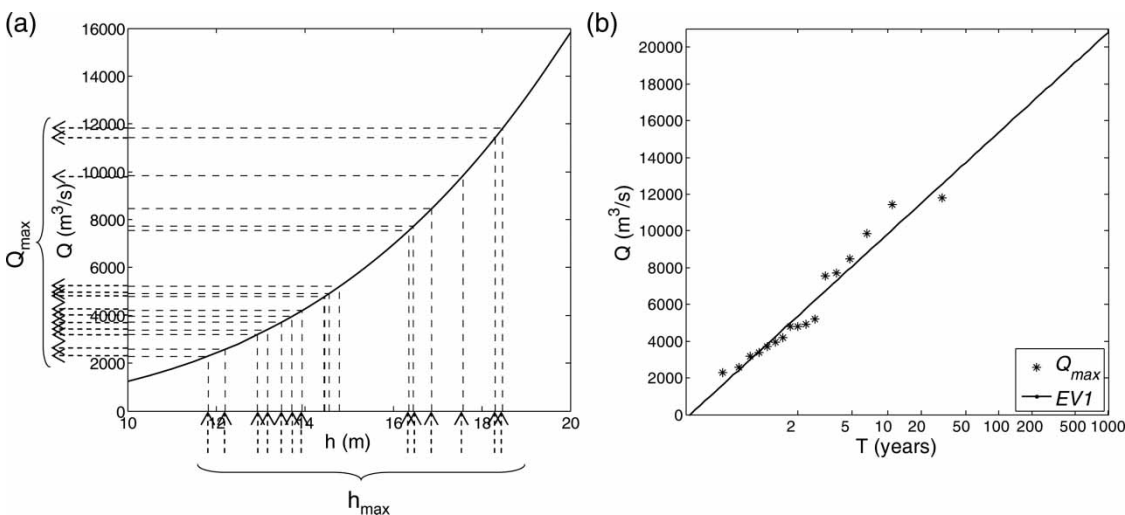


Figure 3 | (a) Calculation of the annual maximum discharge values in a given section performed from a generic set of annual maximum stage values, adopting a crisp-parameter rating curve, and (b) corresponding EV1 distribution on a Gumbel probability paper.

method applied to simultaneous measurements of crisp stages and discharges). Annual maximum discharge values  $Q_{\max}$  are assumed here to be distributed according to an EV1 probability distribution  $F$  (Gumbel 1954), i.e.:

$$F(Q_{\max}) = \exp \left[ - \exp \left( - \frac{Q_{\max} - u}{\alpha} \right) \right] \quad (9)$$

The EV1 assumption is not binding for the procedure we are going to present and is made here for its simple structure which makes it easy to perform any mathematical development.

The parameters  $u$  and  $\alpha$  can be estimated using the method of moments as follows:

$$\alpha = \frac{\sqrt{6} \cdot \sigma}{\pi} \quad (10)$$

$$u = \mu - 0.5772\alpha$$

$\mu$  and  $\sigma$  being the mean and standard deviation of the population, respectively (Benjamin & Cornell 1970), which can be estimated, in turn, using the mean  $m$  and standard deviation  $s$  of the data sample as follows:

$$\hat{\mu} = m = \frac{1}{n_y} \sum_{k=1}^{n_y} Q_{\max,k} \quad (11)$$

$$\hat{\sigma} = s = \sqrt{\frac{1}{n_y - 1} \sum_{k=1}^{n_y} (Q_{\max,k} - m)^2}$$

Introducing the reduced variate  $y$ , determined by:

$$y = \frac{Q_{\max} - u}{\alpha} \quad (12)$$

replacing in (9) and developing further yields:

$$y = - \ln \left[ \ln \left( \frac{1}{F(Q_{\max})} \right) \right] \quad (13)$$

or rather, as  $F(Q_{\max}(T)) = \frac{T-1}{T}$ , where  $T$  is the return period, we obtain:

$$y_T = - \ln \left[ \ln \left( \frac{T}{T-1} \right) \right] \quad (14)$$

and the annual maximum discharge  $Q_{\max}(T)$  over a given return period  $T$  is therefore:

$$Q_{\max}(T) = u + \alpha y_T \quad (15)$$

In summary,  $n_y$  values of *crisp* annual maximum discharge  $Q_{\max,k}$  yield *one* EV1 probability distribution represented, in a Gumbel probability paper, by the straight line described in Equation (15) (see Figure 3(b)).

Finally, it is useful, for the ends of the following grey treatment, to observe that replacing the expression of the parameters in Equation (15) with those of Equation (10), i.e.:

$$\begin{aligned} Q_{\max}(T) &= u + \alpha y_T = \mu - 0.5772\alpha + \alpha y_T \\ &= \mu - 0.5772 \frac{\sqrt{6} \cdot \sigma}{\pi} + \frac{\sqrt{6} \cdot \sigma}{\pi} y_T \\ &= \mu + \frac{\sqrt{6} \cdot \sigma}{\pi} (y_T - 0.5772) \end{aligned} \quad (16)$$

and replacing the mean  $\mu$  and standard deviation  $\sigma$  of the population with the mean  $m$  and standard deviation  $s$  of the data sample of Equation (11), the estimated annual maximum discharge  $Q_{\max}(T)$  over a given return period  $T$  may be expressed as:

$$\begin{aligned} \hat{Q}_{\max}(T) &= \frac{1}{n_y} \sum_{k=1}^{n_y} Q_{\max,k} \\ &+ \sqrt{\frac{1}{n_y - 1} \sum_{k=1}^{n_y} \left( Q_{\max,k} - \frac{1}{n_y} \sum_{k=1}^{n_y} Q_{\max,k} \right)^2} \\ &\cdot \frac{\sqrt{6} (y_T - 0.5772)}{\pi} \end{aligned} \quad (17)$$

### Analysis of extreme discharge values estimated using a grey rating curve: Interpretation (a)

Let us now consider the case in which the rating curve parameters are grey numbers. According to Interpretation (a) of the rating curve, explained in The Grey Rating Curve section above, i.e. the attention is focussed on the grey discharge per given stage,  $n_y$  annual maximum stages from the grey rating curve of Equation (3) yield  $n_y$  values of grey annual

maximum discharge  $Q_{\max,k}^\pm$ , where  $k = 1:n_y$ :

$$Q_{\max,k}^\pm = a^\pm \cdot h_{\max,k}^{b^\pm} \tag{18}$$

Replacing in Equation (17) the annual maximum discharges  $Q_{\max,k}$  obtained from the crisp rating curve with the grey annual maximum discharge values  $Q_{\max,k}^\pm$  obtained from the grey rating curve yields, according to the extension principle:

$$\hat{Q}_{\max}^\pm(T) = \frac{1}{n_y} \sum_{k=1}^{n_y} Q_{\max,k}^\pm + \sqrt{\frac{1}{n_y - 1} \sum_{k=1}^{n_y} \left( Q_{\max,k}^\pm - \frac{1}{n_y} \sum_{k=1}^{n_y} Q_{\max,k}^\pm \right)^2} \cdot \frac{\sqrt{6}(y_T - 0.5772)}{\pi} \tag{19}$$

In practice, Equation (19) represents a function of  $n_y$  grey numbers  $Q_{\max,k}^\pm$  where  $k = 1:n_y$ ; according to the definition of grey function provided in Appendix A, the lower limit  $\hat{Q}_{\max}^-(T)$  of said function can be obtained by searching for the set of  $n_y$  crisp values  $(Q_{\sim \max,1}^-, Q_{\sim \max,2}^-, \dots, Q_{\sim \max,k}^-, \dots, Q_{\sim \max,n_y}^-)$ , where  $Q_{\max,k}^- \leq Q_{\sim \max,k}^- \leq Q_{\max,k}^+ \forall k = 1:n_y$ , which minimizes the function and the upper limit  $\hat{Q}_{\max}^+(T)$  can be obtained by searching for the set of  $n_y$  crisp values  $(\tilde{Q}_{\max,1}^+, \tilde{Q}_{\max,2}^+, \dots, \tilde{Q}_{\max,k}^+, \dots, \tilde{Q}_{\max,n_y}^+)$ , where  $Q_{\max,k}^- \leq \tilde{Q}_{\max,k}^+ \leq Q_{\max,k}^+ \forall k = 1:n_y$ , which maximizes the function. It is worth noting that the values  $(Q_{\sim \max,1}^-, Q_{\sim \max,2}^-, \dots, Q_{\sim \max,k}^-, \dots, Q_{\sim \max,n_y}^-)$  and  $(\tilde{Q}_{\max,1}^+, \tilde{Q}_{\max,2}^+, \dots, \tilde{Q}_{\max,k}^+, \dots, \tilde{Q}_{\max,n_y}^+)$  can be thought to be positioned on different loop rating curves, once the annual maximum stages are defined, thus highlighting that  $\hat{Q}_{\max}^\pm(T)$  is estimated when unsteady flow conditions and measurement errors are considered when reading the grey rating curve.

The  $\hat{Q}_{\max}^\pm(T)$  values yielded by Equation (19) upon variation of the return period  $T$  represent a grey EV1 distribution in the Gumbel probability paper.

**Analysis of extreme discharge values estimated using a grey rating curve: Interpretation (b)**

Let us now again consider the case in which the rating curve parameters are grey numbers, this time according to Interpretation (b) of the rating curve, explained in The Grey Rating Curve section above, i.e. the attention is focussed on the grey parameters  $a^\pm$  and  $b^\pm$  of the rating curve of Equation (3).

Replacing the  $Q_{\max,k}^\pm$  in Equation (19) with Equation (18) yields:

$$\hat{Q}_{\max}^\pm(T) = \frac{1}{n_y} \sum_{k=1}^{n_y} a^\pm \cdot h_{\max,k}^{b^\pm} + \sqrt{\frac{1}{n_y - 1} \sum_{k=1}^{n_y} \left( a^\pm \cdot h_{\max,k}^{b^\pm} - \frac{1}{n_y} \sum_{k=1}^{n_y} a^\pm \cdot h_{\max,k}^{b^\pm} \right)^2} \cdot \frac{\sqrt{6}(y_T - 0.5772)}{\pi} \tag{20}$$

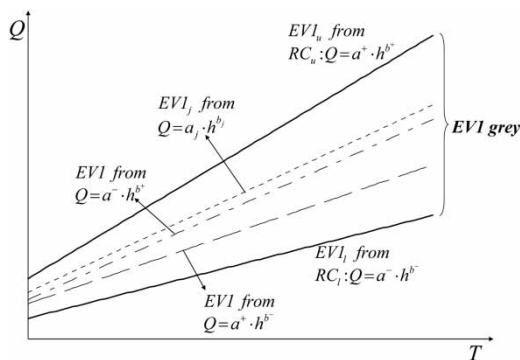
Equation (20), like Equation (19), provides the estimate of the grey discharge per given return period, and is itself a function of grey numbers, but, unlike Equation (19), which is a function of grey numbers of the discharges  $Q_{\max,k}^\pm$ , where  $k = 1:n_y$ , Equation (20) is a function of the two grey numbers  $a^\pm$  and  $b^\pm$ , representative of the grey rating curve parameters. Hence, in theory, according to the definition of grey function, the lower limit  $\hat{Q}_{\max}^-(T)$  of said function can be obtained by searching for the pair of crisp parameters  $a^-$  and  $b^-$ , with  $a^- \leq a^- \leq a^+$  and  $b^- \leq b^- \leq b^+$ , which minimize the function, and the upper limit  $\hat{Q}_{\max}^+(T)$  can be obtained by searching for the pair of crisp parameters  $\tilde{a}$  and  $\tilde{b}$ , with  $a^- \leq \tilde{a} \leq a^+$  and  $b^- \leq \tilde{b} \leq b^+$ , which maximize the function. The decisional variables of the problems of minimization and maximization are no longer therefore the  $n_y$  crisp values of discharge, but only the two crisp values of the parameters from the rating curve. Specifically, given Equation (20) and considering the fact that the rating curve parameters take on positive values, in practice the lower limit  $\hat{Q}_{\max}^-(T)$  is obtained using the pair of crisp parameters  $a^-$  and  $b^-$ , and the upper limit  $\hat{Q}_{\max}^+(T)$  is obtained using the pair of crisp parameters  $\tilde{a} = a^+$  and  $\tilde{b} = b^+$ .

It is important to bear in mind that, according to Equation (20), i.e. encompassing the grey rating curve



expression in the estimate of the grey discharge of a given return period, contrary to what occurs when referring to the Interpretation (a) of the grey rating curve, the crisp discharge values upon which the estimate of the  $Q_{\max}^{\pm}(T)$  is based are constrained to belong to single and specific crisp steady rating curves. In fact, identification of, for example, the pair of crisp parameters  $\tilde{a} = a^+$  and  $\tilde{b} = b^+$  that maximize Equation (20) and therefore provide the value  $\hat{Q}_{\max}^+(T)$  independent of  $T$ , leads to the indirect identification of a set of annual maximum discharges that all lie on one of the possible *crisp* steady rating curves that, together, make up the *grey* rating curve. In summary, also in this case the  $\hat{Q}_{\max}^{\pm}(T)$  values yielded by Equation (20) upon variation in the return period  $T$  represent a grey EV1 distribution in the Gumbel probability paper. Nevertheless, unlike the preceding case (Interpretation (a) of the grey rating curve), this grey EV1 distribution, according to the previous considerations, can be seen as a sheaf of crisp EV1 distributions, each corresponding to a specific pair of parameters  $a$  and  $b$ , with  $a^- \leq a \leq a^+$  and  $b^- \leq b \leq b^+$ . In summary, Equation (20) produces an estimate  $\hat{Q}_{\max}^{\pm}(T)$  when steady flow conditions and measurement errors are considered for reading the grey rating curve.

Figure 4, with reference to Interpretation (b) of the grey rating curve, shows an example of a few of the possible EV1 distributions that would be obtained using *one* generic set of annual maximum stage values and the grey-parameter rating curve illustrated in Figure 2(b). It can be observed that the sheaf of straight lines (in the Gumbel probability paper) representative of the grey EV1 distribution has as its lower



**Figure 4** | Grey EV1 distribution estimated according to Interpretation (b) of the grey rating curve. In particular, the crisp EV1 distributions obtained from *one* generic set of annual maximum stage values and the crisp rating curves shown in Figure 2(b) are illustrated.

and upper limits the crisp EV1<sub>l</sub> and EV1<sub>u</sub> distributions, respectively: EV1<sub>l</sub> is the crisp distribution whose parameters are estimated using the annual maximum values of crisp discharge  $Q_{\max,l,k}$  (where  $k = 1:n_y$ ) obtained from the lower RC<sub>l</sub> rating curve (see Equation (4)). EV1<sub>u</sub> is the crisp distribution whose parameters are estimated using the annual maximum values of crisp discharge  $Q_{\max,u,k}$  (where  $k = 1:n_y$ ) obtained from the upper rating curve RC<sub>u</sub> (see Equation (5)).

In summary, the grey annual maximum discharge  $Q_{\max}^{\pm}(T)$  of a given return period yielded by the grey EV1 distribution thereby defined has its lower limit as  $Q_{\max}^-(T)$ , obtained using the (crisp) EV1<sub>l</sub> distribution, i.e.:

$$Q_{\max}^-(T) = \frac{1}{n_y} \sum_{k=1}^{n_y} a^- \cdot h_{\max,k}^{b^-} + \frac{\sqrt{\frac{1}{n_y - 1} \sum_{k=1}^{n_y} \left( a^- \cdot h_{\max,k}^{b^-} - \frac{1}{n_y} \sum_{k=1}^{n_y} a^- \cdot h_{\max,k}^{b^-} \right)^2}}{\sqrt{6}(y_T - 0.5772)} \cdot \pi \tag{21}$$

and its upper limit is  $Q_{\max}^+(T)$ , obtained using the (crisp) EV1<sub>u</sub> distribution, i.e.:

$$Q_{\max}^+(T) = \frac{1}{n_y} \sum_{k=1}^{n_y} a^+ \cdot h_{\max,k}^{b^+} + \frac{\sqrt{\frac{1}{n_y - 1} \sum_{k=1}^{n_y} \left( a^+ \cdot h_{\max,k}^{b^+} - \frac{1}{n_y} \sum_{k=1}^{n_y} a^+ \cdot h_{\max,k}^{b^+} \right)^2}}{\sqrt{6}(y_T - 0.5772)} \cdot \pi \tag{22}$$

### CASE STUDY

The procedure for the treatment of the annual maximum discharge values whose uncertainty is characterized by means of a grey rating curve was applied to the Pontelagoscuro (Ferrara) section of the Po river. The Po is the longest river in Italy, at a total length of roughly 652 km. The basin at the Pontelagoscuro section has an area of approximately 70,000 km<sup>2</sup>.

The data available for estimating the (grey) rating curve at the Pontelagoscuro section comprise  $n_{\text{obs}} = 19$  pairs of values measured for stage  $h_{\text{obs},i}$  (where  $i = 1:n_{\text{obs}}$ ) and the

corresponding discharges  $Q_{\text{obs},i}$  (Franchini *et al.* 1999). Annual maximum stages measured between 1990 and 2008 were then used to analyse the extreme values (i.e. the annual maximum peak discharges), giving a total of  $n_y = 19$  values. Incidentally, to avoid any misunderstanding, these latter  $n_y = 19$  stage values are different from the  $n_{\text{obs}} = 19$  stage values used to set up the (grey) rating curve.

The estimate of the grey rating curve parameters at the Pontelagoscuro section featuring uncertainty was performed assuming that the uncertainty concerning stage measurement is negligible, while the uncertainty regarding the discharge is represented by means of Equation (2). The grey uncertainty  $\Delta^\pm$  pertaining to the discharge measurement was estimated according to the method proposed by Shrestha & Simonovic (2010b), taking into consideration the indications provided in EN ISO 748 (2007) regarding the quantification, at a given confidence level (e.g. 68 or 95%), of each component of uncertainty that affects discharge measurement by means of the velocity/area method.

In more detail, the grey uncertainty  $\Delta^\pm$  pertaining to the discharge measured was evaluated by means of the following expression (see EN ISO 748 2007, p. 25):

$$\Delta^\pm = \pm \sqrt{(\delta_m^\pm)^2 + \frac{1}{n_m} \left( (\delta_b^\pm)^2 + (\delta_d^\pm)^2 + (\delta_p^\pm)^2 + \frac{1}{n_p} \left( (\delta_c^\pm)^2 + (\delta_e^\pm)^2 \right) \right)} \quad (23)$$

where  $\delta_m^\pm$  is the uncertainty in the measurement of mean velocity due to a limited number  $n_m$  of verticals,  $\delta_b^\pm$  is the

uncertainty in width,  $\delta_d^\pm$  the uncertainty in depth,  $\delta_p^\pm$  the uncertainty in the measurement of mean velocity due to a limited number  $n_p$  of points in the vertical,  $\delta_c^\pm$  the uncertainty in point velocity measurement due to current-meter rating error, and  $\delta_e^\pm$  the uncertainty in point velocity measurements due to limited exposure times. For each of these single grey components of uncertainty, the upper/lower limits are assumed to be equal to  $+/-$  the corresponding value at 95% confidence level, as per EN ISO 748 (2007) (see Table 1), assuming a number of verticals used in gauging  $n_m = 20$ , a number of points taken in the vertical  $n_p = 5$ , an average velocity in measuring section above 0.5 m/s and a current meter exposure time of 2 min. Of course these assumptions imply that  $\Delta^\pm$  is the same for all the discharges measured: unfortunately we do not have enough information to differentiate between the 19 discharge measurements used to set up the (grey) rating curve. However, this position does not affect the logic of the procedures described in the previous sections.

Considering Equation (23) and the values in Table 1, a value of grey uncertainty  $\Delta^\pm$  of  $[-5.3\%, +5.3\%]$  was obtained; this value is in line with the estimate of the uncertainty of the discharge measurements, though probabilistic, for the same case study reported by Di Baldassarre & Montanari (2009).

The SCE-UA algorithm (Shuffled Complex Evolution – University of Arizona; Duan *et al.* 1992) was used to estimate the two grey rating curve parameters  $a^\pm$  and  $b^\pm$  (i.e. to minimize the objective function in Equation (6)) and to solve the

**Table 1** | Values of the uncertainty components at the 95% confidence level, provided by EN ISO 748 (2007), used to estimate the corresponding components of grey uncertainty (see Annex E of EN ISO 748 (2007))

Symbol	Component uncertainty	Assumptions	Percentage value
$\delta_b$	Uncertainty in width		1%
$\delta_d$	Uncertainty in depth		1%
$\delta_m$	Uncertainty in measurement of mean velocity due to limited number $n_m$ of verticals	$n_m = 20$	5%
$\delta_p$	Uncertainty in measurement of mean velocity due to limited number $n_p$ of points in the vertical	$n_p = 5$	5%
$\delta_e$	Uncertainty in point velocity measurements due to limited exposure times	Times of exposure: 2 min; $n_p = 5$	6% for each point taken in the vertical: $\sqrt[3]{5 \cdot 6^2} = 13\%$
$\delta_c$	Uncertainty in point velocity measurement due to current-meter rating error	Individual rating; velocity above 0.5 m/s	1%

grey function in Equation (19), performing a final refinement by means of the function MATLAB<sup>TM</sup> *fmincon* based on sequential quadratic programming (Powell 1983; Schittowski 1985). The estimation process performed can be considered sufficiently reliable since the ratio between the number of parameters to be estimated and the number of data available is around 10.

The results yielded by the application of the procedure for estimating, with uncertainty, the rating curve parameters, and statistical analysis of the extreme values according to both Interpretations (a) and (b) of the grey rating curve are presented and discussed below.

## ANALYSIS AND DISCUSSION OF RESULTS

Estimation of the grey rating curve parameters at the Pontelagoscuuro section yielded the following values:  $a^\pm = [143.4, 158.4]$  and  $b^\pm = [1.5947, 1.5948]$ . It can be seen that, following calibration, parameter  $a$  is affected by greater uncertainty with respect to parameter  $b$ . In fact, the latter tends to collapse at crisp numbers close to 1.595. The reasons behind this phenomenon could be summarized as follows: in the rating curve expressed by Equation (1), parameter  $a$  summarizes the dimensions and roughness in the section in question, as well as the slope friction, while parameter  $b$  takes into account the shape of the section (Herschly 1999). In fact, assuming, for example, uniform flow, according to the Manning equation, it follows that:

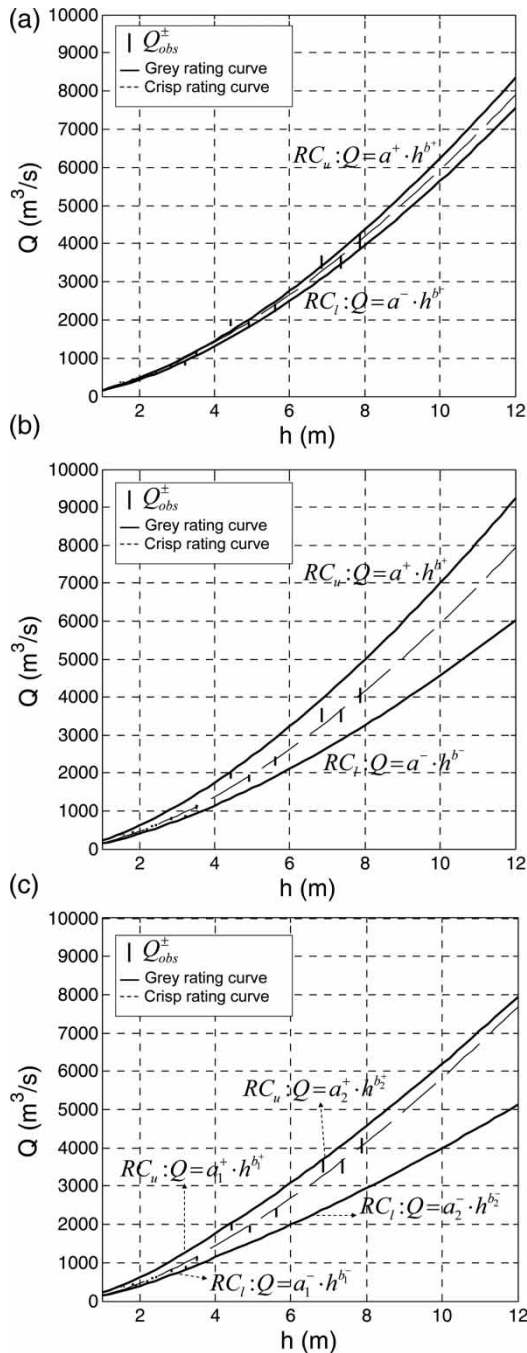
$$Q = \frac{1}{n_{\text{man}}} AR^{2/3} S_0^{1/2} \quad (24)$$

where  $n_{\text{man}}$  is the Manning resistance coefficient,  $A$  the cross-sectional area,  $R$  the hydraulic radius and  $S_0$  the bed slope coincident with the friction slope. Assuming, for example, a very wide rectangular section, and approximating therefore the cross-sectional area  $A$  by means of  $B \cdot h$  ( $B$  being the width of the section), and the hydraulic radius  $R$  with stage  $h$ , yields:

$$Q = \frac{1}{n} B h^{5/3} S_0^{1/2} \quad (25)$$

which can be referred to Equation (1), encompassing the factors relative to the bed slope, width and roughness in parameter  $a$  and assuming  $b = 5/3$  (Marchi & Rubatta 1981, pp. 579–580). Therefore, taking into account the physical significance of the parameters  $a$  and  $b$ , if the shape of the section does not vary over time, it is reasonable to assume a *single* value for parameter  $b$ ; if, on the other hand, the section shape varies considerably (see, for example, the case study reported by Westerberg *et al.* 2011) it is reasonable to assume that the resulting rating curve will be characterized by different values of parameter  $b$ . Thus, the fact that, in the case examined, parameter  $b$  tends to collapse at a crisp number can be explained considering that the section in question is one of the terminal sections of the river whose shape can be considered as relatively stable over time, especially as regards flood discharges.

Now let us move on to discussion of the effects of the grey rating curve of parameters  $a^\pm = [143.4, 158.4]$  and  $b^\pm \approx 1.595$  (see Figure 5(a)) on the probabilistic assessment of the annual maximum discharge. Specifically, Figure 5(a) shows the ‘lower’ rating curve  $RC_l$  of Equation (4), the ‘upper’ rating curve,  $RC_u$ , of Equation (5) and, by means of the dashed line, the rating curve that would be obtained by ‘conventional’ estimation of crisp parameters from crisp discharge and stage values via the least square method. Incidentally, Figure 5(b) shows the same grey rating curve obtained by minimizing Equation (6) (see The Grey Rating Curve section) using constraints such that the (grey) observed discharges are bounded by the grey rating curve and Figure 5(c) the same grey rating curve obtained still considering the same constraints but also splitting the rating curve into two parts connecting at a boundary point ( $h_{\text{bound}} = 4$  m) and considering a ‘reference point’ which coincides with this boundary point, as suggested by Shrestha *et al.* (2007) (see also Bardossy *et al.* 1990). It is evident that: (a) when the (grey) observed discharges are systematically bounded by the grey rating curve, a clear over-estimation of the uncertainty, related to the generic discharge, is observed, and (b) using a reference point and splitting the rating curve into two parts only partially reduces this problem (similar results are obtained when different boundary points are considered). Instead, when Equation (6) is minimized without such constraints (Figure 5(a)), some grey observed discharges are not completely covered by the

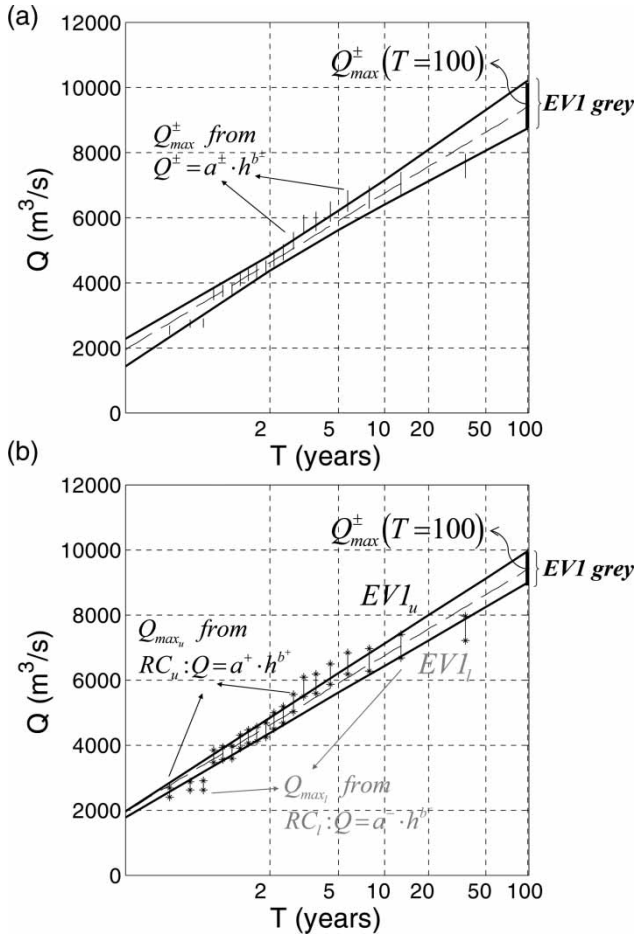


**Figure 5** | Grey-parameter rating curve for the Pontelagoscuro section estimated (a) by minimizing Equation (6), (b) by minimizing Equation (6) using constraints such that the grey observed discharges are bounded by the grey rating curve and (c) by minimizing Equation (6) using constraints such that the grey observed discharges are bounded by the grey rating curve while splitting the rating curve into two parts connecting at a boundary point ( $h_{\text{bound}} = 4$  m) and considering a reference point which coincides with this boundary point. (The black continuous line shows the grey rating curve; the dashed line shows the rating curve whose crisp parameters are estimated by means of the least square method, using crisp discharge values – i.e. not considering the error (see Equation (2)) – and stage values).

grey rating curve. These latter discharges may be due to particular measurement conditions/errors or hysteresis effects and thus could be considered as outliers when very far from the grey rating curve; in this case the whole variability is not represented, but overall the grey rating curve well represents the amplitude of the (observed) grey discharges when the stage increases. For this reason, reference to the grey rating curve shown in Figure 5(a) is made below. In particular, it is worth noting that the grey rating curve shown in Figure 5(a) is drawn up to discharge values of about  $8,000 \text{ m}^3 \text{ s}^{-1}$  even though measurements reach discharge values of about  $4,000 \text{ m}^3 \text{ s}^{-1}$ . This is consistent with the fact that discharges of the order of  $8,000 \text{ m}^3 \text{ s}^{-1}$  can flow through the cross section in Pontelagoscuro without overtopping the embankments and at the same time without expanding over a flood plain since no flood plains are present in the cross section considered.

Starting with  $n_y$  values of annual maximum stage, the probabilistic treatment of the annual maximum values of (grey) discharge was developed, based on the two different Interpretations ((a) and (b)) of the grey rating curve above. Figure 6(a) shows the grey EV1 distribution obtained considering Interpretation (a) of the grey rating curve in conjunction with the annual maximum grey discharge values  $Q_{\text{max},k}^{\pm}$  obtained by Equation (18), considering the  $n_y$  annual maximum values of stage  $h_{\text{max},k}$  (where  $k = 1:n_y$ ). Likewise, Figure 6(b) illustrates the grey EV1 distribution obtained considering Interpretation (b) of the grey rating curve, showing in particular the  $\text{EV1}_l$  and  $\text{EV1}_u$  distributions obtained when considering the crisp values of annual maximum discharge calculated, respectively, using rating curves  $\text{RC}_l$  and  $\text{RC}_u$  of Figure 5(a). Both Figures 6(a) and (b) show return periods up to 100 years since, given the reduced length of the annual maximum (grey) discharge sample used, higher return periods would be unreliably estimated (Benson 1962).

Comparison of the two figures immediately reveals a different shape of grey EV1 distribution in case (a) with respect to case (b). In fact, under Interpretation (a), i.e. that focussed on grey discharge values (see also Equation (19)), the distribution at the extreme grey values assumes an hour-glass shape, characterized by an increase in uncertainty for both high and very low values of return period (see Figure 6(a)). This is due to the fact that, in order to



**Figure 6** | Grey EV1 distribution estimated from the annual maximum stage values observed at the Pontelagoscuro section and the grey rating curve shown in Figure 5(a): (a) according to Equation (19) (Interpretation (a) of the grey rating curve); (b) according to Equations (21) and (22) (Interpretation (b) of the grey rating curve). In both cases the dashed line represents the EV1 distribution obtained from the annual maximum stage values observed at the Pontelagoscuro section and the crisp-parameter rating curve shown in Figure 5(a), obtained without considering the discharge measurement error.

obtain the maximum and minimum discharge values for a given return period  $T$ , it is possible through the grey function of Equation (19) to determine sets of crisp values for the discharges (included in the corresponding grey values) which are unconstrained to lie on a single crisp steady rating curve, but instead can belong to different loop rating curves (implicitly represented by the grey rating curve according to Interpretation (a)). Effectively, for low values of  $T$ , the optimization procedure identifies sets of crisp discharge values which, when searching for the maximum values of  $Q_{max}^+(T)$ , tend to raise the intercept (i.e. the

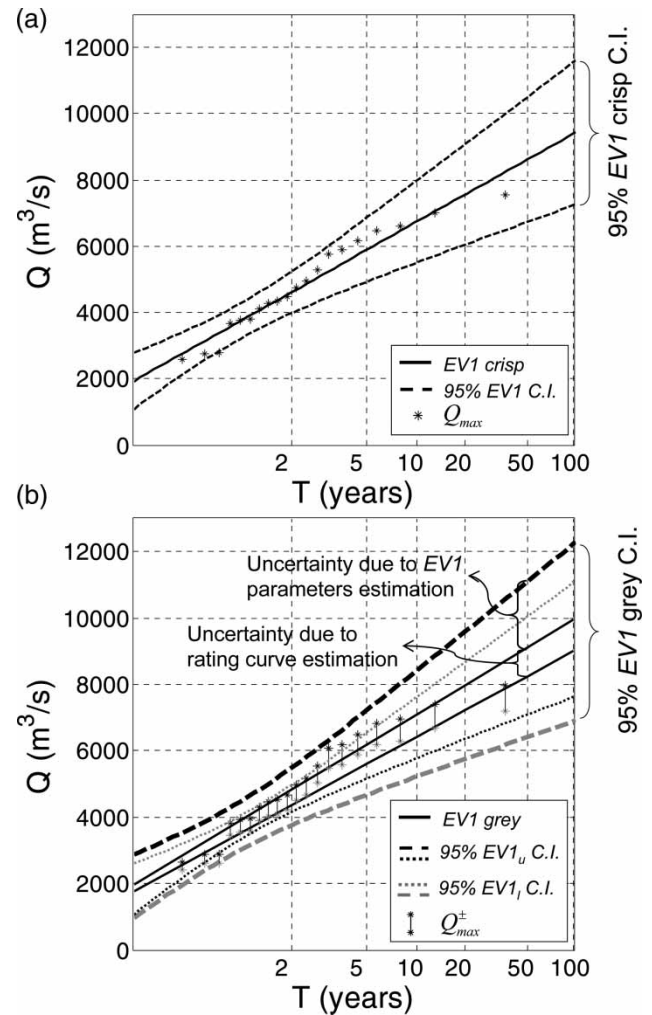
parameter of position  $u$  of the EV1 distribution) even at the expense of a smaller slope of the line (i.e. smaller value of the EV1 distribution scale parameter  $\alpha$ ) and vice versa for high values of  $T$ ; in fact, analysis of these sets of crisp values shows that for low values of  $T$ ,  $Q_{max}^+(T)$  is obtained by selecting the upper limit of the grey discharge  $Q_{max}^{\pm}$  corresponding to the lowest of the annual maximum stages observed, and the lower limit of the grey discharge  $Q_{max}^{\pm}$  corresponding to the greatest of the annual maximum stages observed, and, as previously mentioned, the upper and lower limits of grey discharge  $Q_{max,k}^{\pm}$  correspond to different crisp rating curves, i.e. those of Equations (4) and (5), respectively. Likewise, for high values of  $T$ ,  $Q_{max}^+(T)$  is obtained by selecting the lower limit of the grey number pertaining to the discharge  $Q_{max}^{\pm}$  corresponding to the smallest of the observed annual maximum stage values and the upper limit of the grey number pertaining to the discharge  $Q_{max}^{\pm}$  corresponding to the largest of the annual maximum stage values observed; also in this case, therefore, the crisp values of discharge belong to different crisp unsteady rating curves.

Application of Interpretation (b) of the grey rating curve does not produce this phenomenon, since, as previously mentioned, due to its formulation, the sets of crisp annual maximum discharges used to estimate  $Q_{max}^+(T)$  and  $Q_{max}^-(T)$  are constrained to lie on separate crisp steady rating curves. It follows that the probability distribution of the extreme grey values can be read as the envelope of probability distribution of the extreme crisp values, each associated with a specific crisp steady rating curve ‘contained’ within the grey rating curve (see, for example, Figure 2(b)).

Comparison of the grey numbers for annual maximum discharges  $Q_{max}^{\pm}(T)$  in a given return period obtained according to the two different interpretations of the grey rating curve shows that Interpretation (a) leads to an estimate of the grey annual maximum discharge of given return period of greater width with respect to the corresponding grey annual maximum discharge estimated according to Interpretation (b). For example, for a return period of  $T=100$  years, Interpretation (a) yields  $\hat{Q}_{max}^{\pm}(T=100) = [8,748, 10,218] \text{ m}^3/\text{s}$ , whereas Interpretation (b) yields  $\hat{Q}_{max}^{\pm}(T=100) = [8,993, 9,940] \text{ m}^3 \text{ s}^{-1}$ . Basically, Interpretation (a) of the grey rating curve, since

it implicitly considers unsteady flow conditions and measurement errors when reading the grey rating curve, leads to a wider estimation of the uncertainty acting on the grey annual maximum discharges for a given return period. Interpretation (b) of the grey rating curve, on the other hand, since it implicitly considers steady flow conditions and measurement errors when reading the grey rating curve, produces slightly narrower grey annual maximum discharges for a given return period. This result seems reasonable since Interpretation (a) introduces a source of dispersion (unsteady flow conditions, i.e. loop rating curves) which is not considered by Interpretation (b), with the measurement errors being the same.

Probabilistic treatment of the extreme values developed from the grey rating curve leads to the grey estimate  $\hat{Q}_{\max}^{\pm}(T)$ , which represents the uncertainty linked to the imprecision of the discharges estimated using the grey rating curve. On the other hand, as mentioned in the *Introduction*, there is also uncertainty relative to the estimate of the parameters of the probability distribution assumed for the treatment of the extreme values. Hence, as in the case of the treatment of the annual maximum discharges estimated by means of the crisp-parameter rating curve, it is possible to define, by means of the Monte Carlo method or by asymptotic estimate (NERC 1975; Maidment 1993, pp. 18.3), the confidence band of a given significance level (e.g. 95%), which represents the uncertainty component linked to estimation of the parameters of the probability distribution (see Figure 7(a)), likewise, by treating the extreme values derived from a grey rating curve, it is possible to define a confidence band of the extreme grey distribution, linked to the uncertainty on parameters of the probabilistic model. In practice, making reference, for instance, to Interpretation (b), since the grey probability distribution can be seen as the envelope of a group of crisp probability distributions, the corresponding confidence band can also be obtained as the envelope of confidence bands, each relative to a different crisp probability distribution obtained from a pair of crisp rating curve parameters  $a$  and  $b$  with  $a^- \leq a \leq a^+$  and  $b^- \leq b \leq b^+$ . Figure 7(b) shows, as an example, the 95% confidence bands of the crisp distributions  $EV1_u$  and  $EV1_l$ , which define the upper and lower limits of the grey probability distribution (see also Figure 6(b)). Each confidence band is here obtained by



**Figure 7** | (a) 95% confidence band of the crisp EV1 distribution of the annual maximum discharges estimated by means of the crisp rating curve, obtained without considering the discharge measurement error (see dashed line in Figure 5(a)) and (b) 95% confidence band of the grey EV1 distribution estimated according to Equations (21) and (22) (Interpretation (b) of the grey rating curve): the grey dotted and dashed lines show the upper and lower limits of the EV1 distribution confidence band, respectively, and the black dotted and dashed lines show the upper and lower limits of the  $EV1_u$  distribution confidence band, respectively.

using the asymptotic estimation of the variance  $\sigma^2$  of the corresponding maximum discharge  $Q_{\max}(T)$  of a given return period  $T$  as follows (Maidment 1993, pp. 18.31):

$$\sigma^2(Q_{\max}(T)) = \frac{\alpha^2(1.11 + 0.52y_T + 0.61y_T^2)}{n_y} \quad (26)$$

where  $\alpha$  is the scale parameter of the crisp EV1 distribution considered (see Equation (9)) estimated through the method

of moments. Since the quantile estimator  $\hat{Q}_{\max}(T)$  (Equation (17)) is asymptotically normally distributed (see for instance, Kottegoda & Rosso 2008), it is easy to calculate the 95% confidence band.

The envelope of said bands defines the 95% confidence band of the grey probability distribution, the lower limit of which, with reference to Figure 7(b), is therefore represented by the grey dashed line, representing the lower limit of the 95% confidence band of  $EV1_l$ , and the upper limit of which is the black dashed line, representing the upper limit of the 95% confidence band of  $EV1_u$ .

Finally, comparison of the EV1 crisp (Figure 7(a)) and grey (Figure 7(b), grey and black dashed lines) confidence bands shows that for a given return period, the band in Figure 7(b) is characterized by a greater width, which is understandable considering that it corresponds to a grey probability distribution where the uncertainty of parameter estimation and the uncertainty of discharge measurement combine (see Figure 7(b)).

## CONCLUSIONS

This paper presents a non-probabilistic approach, based on the use of grey numbers, for characterizing the uncertainty inherent in a rating curve, which is, in turn, due to the error/uncertainty in measurement of discharge. It is demonstrated herein that this uncertainty has a knock-on effect on estimation of the peak discharge of given return period.

Starting from stage measurement and the corresponding discharge measurement affected by uncertainty, the grey parameters of the rating curve are estimated through a grey regression approach. The approach used allows us to obtain a grey rating curve narrow enough to be of use and wide enough to contain as many (grey) observations as possible, even though some 'outliers' may not be fully included, and thus not all the observed variability may be represented.

Two different ways of reading the grey rating curve have been presented. The former (Interpretation (a) – based on the grey nature of the discharge per given stage) enables the grey rating curve to be read as an envelope of loop/unsteady rating curves while the latter (Interpretation (b) – based on the grey nature of the rating curve parameters)

enables the rating curve to be read as an envelope of steady rating curves. Adopting these two interpretations leads to two respective estimates of grey annual maximum discharge of given return period. The estimate linked to the former interpretation produces a grey discharge of a given return period whose amplitude is greater than that produced by the latter interpretation. This is due to the fact that Interpretation (a) introduces a source of dispersion (unsteady flow conditions, i.e. loop rating curves) which is not considered by Interpretation (b), with the measurement errors being the same in both cases.

It was also observed, during statistical analysis of the extreme values, that beyond the discharge measurement uncertainty, represented by the grey rating curve, there is also a component of uncertainty linked to estimation of the parameters of the probability distribution EV1 considered, consenting the construction of the corresponding confidence band. The width of this band, for a given return period, is greater than the width that would be yielded by 'conventional' analysis of the extreme crisp discharge values (estimated by means of a crisp rating curve) as, due to its construction, it also takes into account the uncertainty of discharge measurement.

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## REFERENCES

- Alvisi, S. & Franchini, M. 2011a Fuzzy neural networks for water stage and discharge forecasting with uncertainty. *Environmental Modelling and Software* **26**, 523–537.
- Alvisi, S. & Franchini, M. 2011b Grey neural networks for river stage forecasting with uncertainty. *Physics and Chemistry of the Earth* **42–44**, 108–118.

- Bardossy, A., Bogardi, I. & Duckstein, L. 1990 Fuzzy regression in hydrology. *Water Resources Research* **26**, 1497–1508.
- Benjamin, J. R. & Cornell, C. A. 1970 *Probability, Statistics and Decision for Civil Engineers*. McGraw-Hill, New York.
- Benson, M. A. 1962 Evaluation of methods for evaluating the occurrence of floods, Water Supply Paper 1550-A. US Geological Survey, Reston, VA.
- Clarke, R. T. 1999 Uncertainty in the estimation of mean annual flood due to rating curve indefiniton. *Journal of Hydrology* **222**, 185–190.
- Deng, J. L. 1982 Control problems of grey systems. *Systems and Control Letters* **1**, 288–294.
- Di Baldassarre, G. & Montanari, A. 2009 Uncertainty in river discharge observations: a quantitative analysis. *Hydrology and Earth System Sciences* **13**, 913–921.
- Di Baldassarre, G. D., Laio, F. & Montanari, A. 2011 Effect of observation errors on the uncertainty of design floods. *Physics and Chemistry of the Earth* **42–44**, 85–90.
- Duan, Q., Sorooshian, S. & Gupta, V. K. 1992 Effective and efficient global optimization for conceptual rainfall runoff models. *Water Resources Research* **24**, 1163–1173.
- Dubois, D. & Prade, H. 1980 *Fuzzy sets and Systems: Theory and Application*. Academic, San Diego, CA.
- European ISO EN Ruthe 748 2007 Hydrometry-Measurement of liquid flow in open channels using current meters or floats. Reference number ISO 748:2007 (E). International Standard.
- Franchini, M., Lamberti, P. & Giammarco, P. 1999 Rating curve estimation using local stages, upstream discharge data and a simplified hydraulic model. *Hydrology and Earth System Sciences* **3**, 541–548.
- Gumbel, E. J. 1954 *Statistic Theory of Extreme Values and Some Practical Applications*. Columbia University Press, New York.
- Hersch, R. W. 1999 Flow measurements. In: *Hydrometry Principles and Practices* (R. W. Hersch, ed.). John Wiley & Sons, Chichester.
- Hersch, R. W. 2009 *Streamflow Measurements*, 3rd edition. E. & F. N. Spon, London.
- Hojati, M., Bector, C. R. & Smimou, K. 2005 A simple method for computation of fuzzy linear regression. *European Journal of Operational Research* **166** (1), 172–184.
- Huang, G. H., Baetz, B. W. & Patry, G. G. 1995 Grey integer programming: an application to waste management planning under uncertainty. *European Journal of Operational Research* **83**, 594–620.
- Jacquin, A. P. & Shamseldin, A. Y. 2007 Development of a possibilistic method for the evaluation of predictive uncertainty in rainfall-runoff modelling. *Water Resources Research* **43**, W04425.
- Kottogoda, T. & Rosso, R. 2008 *Applied Statistics for Civil and Environmental Engineers*. Blackwell Publishing Ltd, London.
- Kuczera, G. 1996 Correlated rating curve error in flood frequency inference. *Water Resources Research* **32**, 2119–2127.
- Laio, F., Di Baldassarre, G. & Montanari, A. 2009 Model selection techniques for the frequency analysis of hydrological extremes. *Water Resources Research* **45**, W07416.
- Maidment, D. R. 1993 *Handbook of Hydrology*, Ch 18. McGraw-Hill Inc., New York.
- Marchi, E. & Rubatta, A. 1981 *Fluids Mechanics*. UTET, Torino, Italy. (In Italian.)
- Martins, E. S. & Stedinger, J. R. 2000 Generalized maximum-likelihood generalized extreme value quantity estimates for hydrologic data. *Water Resources Research* **36**, 737–744.
- Moyeed, R. A. & Clarke, R. T. 2005 The use of Bayesian methods for fitting rating curves, with case studies. *Advances in Water Resources* **28**, 807–818.
- Natural Environmental Research Council 1975 *Flood Studies Report, Vol.1, Hydrological Studies*. NERC, London.
- Pelletier, M. P. 1988 Uncertainties in the determination of river discharge: a literature review. *Canadian Journal of Civil Engineering* **15**, 834–850.
- Peters, G. 1994 Fuzzy linear regression with fuzzy intervals. *Fuzzy Sets and Systems* **63**, 45–55.
- Petersen-Øverleir, A. 2004 Accounting for heteroscedasticity in rating curve estimates. *Journal of Hydrology* **292**, 173–181.
- Petersen-Øverleir, A. & Reitan, T. 2009 Accounting for rating curve imprecision in flood frequency analysis using likelihood-based methods. *Journal of Hydrology* **366**, 89–100.
- Petersen-Øverleir, A., Soot, A. & Reitan, T. 2009 Bayesian rating curve inference as a streamflow data quality assessment tool. *Water Resources Management* **23**, 1835–1842.
- Powell, M. J. D. 1983 Variable metric methods for constrained optimization. In: *Mathematical Programming: The State of the Art* (A. Bechem, M. Grotschel & B. Korte, eds). Springer Verlag, Germany, pp. 288–311.
- Reitan, T. & Petersen-Øverleir, A. 2008 Bayesian power-law regression with a location parameter, with application for construction of discharge rating curves. *Stochastic Environmental Research and Risk Assessment* **22**, 351–365.
- Reitan, T. & Petersen-Øverleir, A. 2009 Bayesian methods for fitting multi-segment stage-discharge rating curves. *Stochastic Environmental Research and Risk Assessment* **23**, 627–642.
- Schittowski, K. 1985 NLQPL: a Fortran-subroutine solving constrained nonlinear programming problems. *Operations Research* **5**, 485–500.
- Shrestha, R. R., Bardossy, A. & Nestmann, F. 2007 Analysis and propagation of uncertainties due to the stage-discharge relationship: a fuzzy set approach. *Hydrological Sciences Journal—Journal Des Sciences Hydrologiques* **52**, 595–610.
- Shrestha, R. R. & Simonovic, S. P. 2010a Fuzzy nonlinear regression approach to stage-discharge analyses: case study. *Journal of Hydrologic Engineering* **15**, 49–56.
- Shrestha, R. R. & Simonovic, S. P. 2010b Fuzzy set theory based methodology for the analysis of measurements uncertainties



- in river discharge and stage. *Canadian Journal of Civil Engineering* **37**, 429–439.
- Tanaka, H., Uejima, S. & Asai, K. 1982 Linear regression analysis with fuzzy model. *IEEE Transactions on Systems, Man and Cybernetics* **12**, 903–907.
- Westerberg, I., Guerrero, J. L., Seibert, J., Beven, K. J. & Halldin, S. 2011 Stage-discharge uncertainty derived with a non-stationary rating curve in the Choluteca River, Honduras. *Hydrological Processes* **25**, 603–613.
- Zadeh, L. A. 1965 Fuzzy sets. *Information and Control* **8**, 338–353.

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