of flow rate when the deviation of viscosity of a non-Newtonian lubricant is of the order of ten per cent. The foregoing method is applicable also when the lubricant is a dilatant fluid.

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DISCUSSION

J. C. Bell[4]

The theory for the lubrication of bearings by non-Newtonian lubricants has proved to be awkward for mathematical analysis because of the nonlinear character of the equations. For treatment of nonlinear equations, numerical methods of solution are often convenient, but they often leave something to be desired in either the generality or the comprehensibility of the solution. Therefore, the authors are to be commended for producing a theory for slider bearings which is both analytic and reasonably simple, so that it can be understood fairly readily.

The perturbation technique underlying this theory does, however, leave some questions unanswered. Most noticeable is that it is hard for the reader to know how far the theory can be pushed. As long as ϵ is small, and perhaps also as long as some other parameters are properly limited, the solution should depict departures from the solution for the Newtonian lubricant. However, when so many functions are being perturbed simultaneously, one may feel fairly justifiably that the valid range of the non-Newtonian part of the solution is quite limited. A clearer picture of this limitation would be desirable.

Another bothersome aspect of this treatment arises in the rheological model assumed for the lubricant. The problem is simply that the added term is merely non-Newtonian, without saying what it is. Such a criticism may be applied to many other non-Newtonian models for lubricants. One may develop a sense for the meaning of the viscosity μ, but what is the meaning of ϵK? As we develop the theory using non-Newtonian lubricants, it seems desirable to seek physical sense in the rheological models that we use, and not to think merely about their failure to be Newtonian.

[4] Battelle Memorial Institute, Columbus, Ohio.

H. H. Horowitz[5] and E. H. Okrent[5]

Drs. Ng and Saibel are to be commended for a gallant effort at an analytic solution to a hydrodynamic lubrication problem involving a non-Newtonian lubricant. We have a number of general and specific questions to ask, however.

Is an approximate analytical solution with the length and complexity of the authors' equation (13) really an advantage over a numerical solution? It is true that a numerical solution exists in the form of a deck of punched cards, thereby making it impossible to visualize the effect of key factors on the answer. However, it is equally difficult to visualize the effect of variable changes from equation (13), a complex polynomial.

Second, the cubic form of the shear stress-shear rate relation seems well adapted to a slider bearing where adding a linear variation of film height with distance yields integrable polynomial forms. Would the same cubic expression yield integrable forms for a journal bearing, where a cosine function enters? Or would a new form of the stress-strain rate function be required?

Third, the shear stress-shear rate relation has a negative cubic term which gives the apparent drop-off in viscosity with shear. At a given shear rate the shear stress passes through zero and then becomes negative, giving a zero viscosity followed by a negative viscosity. Are the authors certain that this cannot occur anywhere within the bearing calculations? The shear rates and stresses vary considerably across and along the oil film. If at some point the viscosity has dropped, say, to two thirds of its low shear value due to the cubic term, the shear rate has to change only by a factor of two or so before this danger region of negative shear stresses is reached. How can one be sure this does not occur?

Fourth, can the cubic shear stress-strain rate relation be used in a range where it duplicates experimental results or is it restricted to very small values of epsilon? (Incidentally there appears to be an error in the equation for the experimental variation of viscosity with shear stress taken from reference [7]. The factor 0.03 in the brackets should be deleted. This may affect the answer to this question.) A measure of the non-Newtonianism of a lubricant is the shift in the location of the pressure peak away from the point of minimum clearance. The authors' curve in Fig. 2 shows an extremely slight shift indicating very slight non-Newtonianism. Does this mean that the method is restricted to very slightly non-Newtonian materials? It might be pointed out that our numerical methods have now been applied to greases, where the viscosity can vary from point to point by a factor of thousands.

Finally, all of us in this field must continually remind ourselves that polymer solutions have other properties that may upset the predictions made on the basis of nonlinear behavior alone. Viscoelasticity, the ability to store energy elastically, is obvious in concentrated polymer solutions and is said to be present in dilute polymer solutions at high shear rates. This effect may conceivably account for the results of Dubois, Ocvirk, and Wehe, who found that the load carrying ability at a given eccentricity was not reduced by the drop in viscosity with shear. This experimental result has never been confirmed, however. Also in automobile engines Savage and Bowman and Okrent found that polymer thickened oils gave reductions in wear and friction which could not be explained on the basis of reduction in viscosity at high shear alone. These effects may be due to the rheology of the polymer solutions or their surface activity. In any case, after many years of talking about the behavior of non-Newtonian lubricants in bearings, it is certainly about time for someone to

[5] Products Research Division, Esso Research and Development Company, Linden, N. J.

make definitive experimental measurements in this field. We all ought to redouble our efforts to see that this work is done.

Authors' Closure

The authors wish to thank the discussers for bringing up several points which need amplification and clarification. As Dr. Bell points out, the method of perturbation which is standard in treating nonlinear problems does not in itself give bounds on the error. Although some mathematical work exists along this line it is not yet sufficiently advanced to answer the question. However, we feel fairly confident in stating that if the a posteriori evaluation of the quantity ϵp_1 is of the order of 0.1 p_0 the results will be reliable. This of course limits us to small deviations from Newtonian behavior.

The second point mentioned by Dr. Bell brings up a very important matter, that is, does the added term have any physical meaning. In this particular case the answer is, yes. In a paper by Tanner [11] it is shown that a certain constitutive equation for a slightly non-Newtonian liquid reduces to

$$\tau = \mu \frac{\partial u}{\partial y}\left[1 - T^2\left(\frac{\partial u}{\partial y}\right)^2\right]$$

in the notation used in our paper, where T is a relaxation time in a Maxwell liquid. This is Equation (16) of Tanner's paper.

The assumptions used in reducing the general constitutive equation to the above are precisely the same as we have used so that an interpretation of our constants is apparent when the above equation is compared with Equation (1) of our paper. In particular, $T^2 \propto K/\mu$. However, the question goes much deeper than that. When a liquid is non-Newtonian, what does viscosity mean? We can no longer think in terms of the simple concepts of a Newtonian fluid and when the constitutive equation is a general one, not only is there at present no answer to the physical meaning of the parameters involved, but the manner of obtaining them is not known in general, see for example [12].

In answer to some of the questions raised by Drs. Horowitz and Okrent, first we might say that we were more interested in showing the effects of parameters or groups of parameters which entered into the problem rather than emphasizing the numerical aspect of the solution. After all, it is quite apparent that had we done a numerical analysis right from the start, the connection with the work of Tanner which in turn stems from a very basic formulation would never have shown up. Secondly, the journal bearing without side leakage is amenable to similar treatment and a solution of this problem will appear elsewhere. Third and fourth, the law used is meant to apply over the range for which the strain rate is less than the order of U/h, where h is the minimum film thickness. Also as pointed out above the solution is meant for fluids having small deviations from Newtonian behavior. Fifth, we have not attempted to relate the hydrodynamics of lubrication to the phenomenon of wear. Since no theoretical relationship of such a sort has been established and since there is considerable scatter in the experimental results we do not feel that we are in a position at this time to make any speculations on this topic.

Finally we wish to point out the further difficulties which will be attendent on obtaining solutions of lubrication problems with non-Newtonian lubricants when the flow is three dimensional. Even though computers are available for numerical solutions, correct and proper constitutive relationships will first have to be established and the values of the parameters which enter into those relationships found. Up to now this has not been done.

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