Tidal Oscillations in Rotating Rectangular Basins of Uniform Depth.  
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§ 1. The problem of the free oscillations of water in a closed rotating rectangular basin of uniform depth has been solved by G. I. Taylor * in 1920. He first gave a method for dealing with the reflection of a "Kelvin-wave" at the closed end of a rectangular channel and then adapted the method to the solution of the problem of the free oscillations in a closed rectangular basin. The latter problem, in the case where the angular speed of rotation is small, had been solved by Lord Rayleigh † in 1903, while further details were supplied by him ‡ in 1909.

The first part of the present paper is mainly devoted to the more general problem of the forced tides in a rotating rectangular basin, and a method of analysis, slightly different from that used by Taylor, is introduced.

In § 4 approximate expressions are given, applicable to deep basins, for the forced tides of a rectangular lake, and also for the co-oscillations of a rectangular gulf when there is a uniform periodic current across the mouth. Similar approximations for the forced tides in deep semi-

† "On the Vibrations of a Rectangular Sheet of Rotating Liquid," Phil. Mag., 5, 297–301, 1903; Papers, 5, 93–97.
circular and elliptic basins have been given respectively by J. Proudman * and S. Goldstein.†

Finally, a comparison is made, for a deep rectangular lake, between the expression for the forced tides derived in § 4 and that obtained by the application of "narrow-sea" methods in the longitudinal and transverse directions. It is found that both expressions provide a small "correction" to the equilibrium-form, but that the form of the "correction" differs considerably in the two cases. The conclusion is drawn that the success which has attended various applications of "narrow-sea" methods to explain the tidal motions of non-elongated bodies of water is mainly due to the fact that the equilibrium-forms predominate, so that small, though erroneous, forms when superposed on these have little resultant effect. Such an application was given by R. Sterneck $ in 1922 in his discussion on the spring tides of the Black Sea.

The author is indebted to and wishes to thank Professor J. Proudman for advice.

§ 2. The notation to be used will include the following:

\[ g \] the acceleration of gravity,

\[ h \] the depth of the basin,

\[ x, y \] the rectangular co-ordinates of a point in the mean horizontal surface of the water,

\[ u, v \] the current-components in the directions of increasing \( x, y \) respectively,

\[ \zeta \] the elevation of the water-surface,

\[ \xi \] the equilibrium-form of \( \zeta \),

\[ \zeta' = \zeta - \xi \],

\[ \sigma \] the "speed" of the harmonic constituent considered,

\[ \omega \] the angular speed of rotation of the basin about the vertical.

On restricting consideration to a single harmonic constituent and using the time-factor \( e^{i\omega t} \), the differential equations, defining the tides in a rotating basin of uniform depth, may be reduced to

\[
\frac{\partial^2 \zeta'}{\partial x^2} + \frac{\partial^2 \zeta'}{\partial y^2} + \frac{\sigma^2 - 4\omega^2}{gh} \zeta = 0 . \quad \text{(2.1)}
\]

together with the boundary condition

\[
\frac{\partial \zeta'}{\partial n} + 2\omega \frac{\partial \zeta'}{\partial s} = 0 . \quad \text{(2.2)}
\]

where \( \partial / \partial n \) denotes differentiation along the outward normal to the boundary and \( \partial / \partial s \) along a line a right-angle in advance.


If $a$ be taken to denote a length of the order of the lateral dimensions of the basin, then, introducing

$$f = \alpha / \omega, \quad \beta = \alpha \beta / gh,$$

the equations (2.1) and (2.2) may be written

$$a^2 \left( \frac{\partial^2 \zeta'}{\partial x^2} + \frac{\partial^2 \zeta'}{\partial y^2} \right) + \beta \left( 1 - \frac{1}{f^2} \right) \zeta = 0 \quad \ldots \quad (2.3)$$

$$\frac{i \partial \zeta'}{\partial n} + \frac{\partial \zeta'}{\partial \xi} = 0 \quad \ldots \quad (2.4)$$

The corresponding expressions for the current-components $u, v$ are

$$\frac{i \sigma u}{g} \left( 1 - \frac{1}{f^2} \right) = - \frac{\partial \zeta'}{\partial x} + i \frac{\partial \zeta'}{\partial y},$$

$$\frac{i \sigma v}{g} \left( 1 - \frac{1}{f^2} \right) = - \frac{\partial \zeta'}{\partial y} - i \frac{\partial \zeta'}{\partial x}.$$

If the basin is sufficiently deep, $\beta$ will be small. On neglecting $\beta$ the only solution of which is $\zeta' = 0$, i.e. $\zeta = \bar{\zeta}$, the equilibrium-form. On retaining the first power of $\beta$ only, we have

$$a^2 \left( \frac{\partial^2 \zeta'}{\partial x^2} + \frac{\partial^2 \zeta'}{\partial y^2} \right) = - \beta \left( 1 - \frac{1}{f^2} \right) \bar{\zeta} \quad \ldots \quad (2.5)$$

$$\frac{i \partial \zeta'}{\partial n} + \frac{\partial \zeta'}{\partial \xi} = 0 \quad \ldots \quad (2.6)$$

### Tidal Motions in a Rectangular Basin.

§ 3. For the potential of the disturbing forces we shall take

$$\bar{\zeta} = A(x - iy/f) + B(y + ix/f) \quad \ldots \quad (3.1)$$

since this may be made to correspond to any uniform force. Further, from the possibility of rotating the axes through a right-angle, it is seen that consideration can primarily be restricted to the case in which $B = 0$.

Now the contribution of

$$\bar{\zeta} = A(x - iy/f)$$

to $v$ is everywhere zero, and to $u$ it is given by

$$- i \sigma u / g = - A \quad \ldots \quad (3.2)$$

For convenience in the ensuing analysis, let the basin have sides $2a$ and $\pi$ and let axes $Ox, Oy$ be taken as indicated in fig. 1.
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Consider now the expressions
\[ \zeta_0 = \cos \kappa (x - i(y - \frac{1}{2}\pi)/f), \]
\[ \zeta'_0 = \sin \kappa (x - i(y - \frac{1}{2}\pi)/f), \]
where \( \kappa^2 = \sigma^2/gh \), and also the expressions
\[ \zeta_s = s \cos \mu_x \cos sy + i\mu_x (\sin \mu_x \sin sy)/f, \]
\[ \zeta'_s = s \sin \mu_x \cos sy - i\mu_x (\cos \mu_x \sin sy)/f, \]
where \( s > 0 \) and \( \mu^2 = \kappa^2(1 - f^2) - s^2. \)

All four expressions satisfy the differential equation
\[ \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \kappa^2 \left( 1 - \frac{1}{f^2} \right) \zeta = 0; \]
both \( \zeta_0 \) and \( \zeta'_0 \) make \( v = 0 \) for all values of \( x \) and \( y \), while \( \zeta_s \) and \( \zeta'_s \) make \( v = 0 \) for \( y = \alpha \) and \( \pi \).

The contribution to \( u \) from \( \zeta_0 \) is given by
\[ -i\sigma u/g = -\kappa \sin \kappa (x - i(y - \frac{1}{2}\pi)/f) \]
\[ = \frac{1}{2}\kappa_0 + \kappa_1 \cos y + \ldots + \kappa_r \cos ry + \ldots, \]
where
\[ \kappa_r = -\frac{4}{\pi} \frac{\kappa^2/f}{r^2 + \kappa^2/f^2} \sin \kappa x \sinh \frac{\pi \kappa}{2f}, \]
\[ \text{(r even),} \]
and from \( \zeta'_0 \) it is given by
\[ -i\sigma u/g = \kappa \cos \kappa (x - i(y - \frac{1}{2}\pi)/f) \]
\[ = \frac{1}{2}\kappa'_0 + \kappa'_1 \cos y + \ldots + \kappa'_r \cos ry + \ldots, \]
June 1931. Rectangular Basins of Uniform Depth.

where

\[
\begin{align*}
\kappa' &= \frac{4}{\pi} \frac{\kappa^2/f}{\pi^2 + \kappa^2/f^2} \cos \kappa x \sinh \frac{\pi \kappa}{2f}, \quad (r \text{ even}), \\
\kappa &= -\frac{4}{\pi} \frac{\kappa^2/f}{\pi^2 + \kappa^2/f^2} \cos \kappa x \cosh \frac{\pi \kappa}{2f}, \quad (r \text{ odd}).
\end{align*}
\]

(3.4)

From \( \xi \), the contribution to \( u \) is

\[
- i \sigma u/g = - s \mu_x \sin \mu_x x \cos y + i \kappa^2 (\cos \mu_x x \sin sy)/f
\]

\[
= \frac{1}{2} \kappa_0 + \kappa_1 \cos y + \ldots + \kappa_{s+1} \cos ry + \ldots,
\]

where

\[
\kappa_{sr} = \frac{4\kappa^2}{\pi f} \frac{s}{s^2 - r^2} i \cos \mu_x x, \quad (s + r \text{ odd}),
\]

\[
= 0, \quad (s + r \text{ even}, r \neq s),
\]

\[
\kappa_{ss'} = - s \mu_x \sin \mu_x x,
\]

while from \( \xi' \), it is

\[
- i \sigma u/g = s \mu_x \cos \mu_x x \cos y + i \kappa^2 (\sin \mu_x x \sin sy)/f
\]

\[
= \frac{1}{2} \kappa_0 + \kappa_1' \cos y + \ldots + \kappa_{s+1}' \cos ry + \ldots,
\]

where

\[
\kappa_{sr'} = \frac{4\kappa^2}{\pi f} \frac{s}{s^2 - r^2} i \sin \mu_x x, \quad (s + r \text{ odd}),
\]

\[
= 0, \quad (s + r \text{ even}, r \neq s),
\]

\[
\kappa_{ss} = s \mu_x \cos \mu_x x.
\]

Now \( \xi_0, \xi_{s \text{ even}}, \xi_{s \text{ odd}} \) are symmetrical about the centre of the basin, and in their contributions to \( u \) the coefficient of \( \cos ry \) is an odd function of \( x \) when \( r \) is even and an even function of \( x \) when \( r \) is odd. Also \( \xi_0', \xi_{s \text{ even}}', \xi_{s \text{ odd}} \) are asymmetrical about the centre of the basin, and in their contributions to \( u \) the coefficient of \( \cos ry \) is an even function of \( x \) when \( r \) is even and an odd function of \( x \) when \( r \) is odd. It is seen, therefore, that if the symmetrical and asymmetrical elevations are considered separately, it will be sufficient to choose such combinations which will make \( u = 0 \) along \( x = a \), for then necessarily \( u = 0 \) also along \( x = -a \). Further, from (3.1) it may be noted that the forced tides will be asymmetrical.

(a) Free Symmetrical Tides.

Take

\[
\xi = \xi_0 + \sum_{s \text{ even}} \lambda_2 \xi_s + \sum_{s \text{ odd}} \lambda_2' \xi_s',
\]

and equate to zero the coefficient of \( \cos ry \) in the resulting expression for \( - i \sigma u/g \) along \( x = a \). It follows that

\[
\kappa_r + \lambda_r \kappa_{rr} + \sum_{s \text{ odd}} \lambda_r \kappa_{sr'} = 0, \quad (r \text{ even}),
\]

\[
\kappa_r + \lambda_r \kappa_{rr'} + \sum_{s \text{ even}} \lambda_r \kappa_{sr} = 0, \quad (r \text{ odd}),
\]
where \( x = a \) is to be substituted in the expressions (3.3) to (3.6). On writing out for \( r = 0, 1, 2, \ldots \), we have

\[
- \frac{4 \kappa^2}{\pi f} \sin k_0 \sinh \frac{\pi \kappa}{2f} \sum_{s} \lambda_s \sin \mu_s a = 0,
\]

\[
- \frac{4 \kappa^2}{\pi f} \sum_{s} \lambda_s \sin \mu_s a = 0,
\]

\[
- \frac{4 \kappa^2}{\pi f} \sum_{s} \lambda_s \sin \mu_s a = 0,
\]

and, on neglecting all terms in which either \( r \) or \( s \) is greater than \( N \) and then eliminating

\[
- \frac{s \lambda_s \cos \mu_s a}{\cos k_0 \cosh \frac{\pi \kappa}{2f}} \quad (s \text{ even}), \quad - \frac{i \lambda_s \sin \mu_s a}{\sin k_0 \sinh \frac{\pi \kappa}{2f}} \quad (s \text{ odd}),
\]

we obtain

\[
\Delta(N) = \begin{pmatrix}
-1 & 1 & 0 & 1/2 & 0 & \ldots \\
1 & 1 & 0 & 1/2 & 0 & \ldots \\
1^2 + \kappa^2/f^2 & L_1 & 1 & 2^2 - 1^2 & 0 & \ldots \\
1 & 1^2 - 2^2 & L_2 & 3^2 - 2^2 & 0 & \ldots \\
1 & 1^2 + \kappa^2/f^2 & 0 & 1/2 - 3^2 & L_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

(3.7)

where \( \Delta(N) \) has \( N + 1 \) rows and columns, in which

\[
L_s = \frac{\pi \mu_s}{4 \kappa^2} \tan k_0 \cot \mu_s a \tanh \frac{\pi \kappa}{2f}, \quad (s \text{ odd}),
\]

\[
M_s = \frac{\pi \mu_s}{4 \kappa^2} \cot k_0 \tan \mu_s a \coth \frac{\pi \kappa}{2f}, \quad (s \text{ even}).
\]

It may be noted that if \( \mu_s \) becomes a pure imaginary, the values of \( L_s \) and \( M_s \) remain real.

The above results agree with those of Taylor (loc. cit.) except that he gives

\[
L_s = - \frac{\pi \mu_s}{4 \kappa^2} \tan k_0 \tan \mu_s a \tanh \frac{\pi \kappa}{2f}, \quad (s \text{ odd}),
\]

\[
M_s = - \frac{\pi \mu_s}{4 \kappa^2} \cot k_0 \cot \mu_s a \coth \frac{\pi \kappa}{2f}, \quad (s \text{ even}).
\]
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(b) Asymmetrical Tides.

Take

\[ \zeta = \lambda_0 \xi_0' + \sum_s \lambda_s \xi_s' + \sum_s \lambda_s \xi_s, \]  

so that, taking account of (3.2) and equating to zero the resulting expression for \(-isu/g\) along \(x = a\), it follows that

\[ \lambda_0 \xi_0' + \sum_s \lambda_s \xi_s = 2A, \]
\[ \lambda_0 \xi_r + \sum_s \lambda_s \xi_{sr}' = 0, \quad (r = 1, 3, 5, \ldots), \]
\[ \lambda_0 \xi_r' + \sum_s \lambda_s \xi_{sr} = 0, \quad (r = 2, 4, 6, \ldots), \]

where \(x = a\) is to be substituted in the coefficients. On writing out for \(r = 0, 1, 2, \ldots\), we have

\[ \begin{align*}
\frac{4}{\pi} \kappa^2/f \lambda_0 \cos \kappa a \sinh \frac{\pi \kappa}{2f} &+ \frac{4i\kappa^2}{\pi f} \sum_s \lambda_s^j \cos \mu_s a = 2A, \\
- \frac{4}{\pi} \kappa^2/f \lambda_0 \sin \kappa a \cosh \frac{\pi \kappa}{2f} - \lambda_1 \mu_1 \sin \mu_1 a &+ \frac{4i\kappa^2}{\pi f} \sum_s \lambda_s^j \frac{s}{\kappa^2 - s^2} \sin \mu_s a = 0, \\
\frac{4}{\pi} \frac{\kappa^2/f}{2 + \kappa^2/f^2} \lambda_0 \cos \kappa a \sin \frac{\pi \kappa}{2f} + 2\lambda_2 \mu_2 \cos \mu_2 a &+ \frac{4i\kappa^2}{\pi f} \sum_s \lambda_s^j \frac{s}{\kappa^2 - s^2} \cos \mu_s a = 0,
\end{align*} \]

On neglecting again all terms in which \(r\) or \(s\) is greater than \(N\) and then solving the equations, it is found that

\[ \lambda_0 = - \Delta_0(N)/\Delta(N), \]
\[ \frac{is\lambda_i}{\cos \kappa a \sinh \frac{\pi \kappa}{2f}} = \frac{\Delta_s(N)}{\Delta(N)}, \quad (s \text{ odd}), \]
\[ \frac{s\lambda_i}{\sin \kappa a \cosh \frac{\pi \kappa}{2f}} = \frac{\Delta_s(N)}{\Delta(N)}, \quad (s \text{ even}), \]

where \(\Delta(N)\) is as in (3.7), except that now

\[ \begin{align*}
I_s &= \pi f \mu_s \cot \kappa a \tan \mu_s a \tanh \frac{\pi \kappa}{2f}, \quad (s \text{ odd}), \\
M_s &= \pi f \mu_s \tan \kappa a \cot \mu_s a \coth \frac{\pi \kappa}{2f}, \quad (s \text{ even}), 
\end{align*} \]

and \(\Delta_s(N)\) differs only from \(\Delta(N)\) in having

\[ \frac{\pi f}{4\kappa^2} \frac{2A}{\cos \kappa a \sinh \frac{\pi \kappa}{2f}} \circ, \circ, \circ, \ldots \]

in the \((s + 1)\)th column.

For free tides, putting \(A = 0\), results identical with those of Taylor are obtained.
Approximate Forms for a Deep Rectangular Basin.

§ 4. Considering the forced tides in a deep basin, so that $\kappa^2$ may be regarded as small, and retaining only the principal parts of the various expressions, we may write

$$\Delta(N) = -\frac{f_0^2}{\kappa^2} L_1 M_2 L_3 \ldots,$$

$$\Delta_0(N) = \frac{\pi f}{2\kappa^2} \cos \kappa a \sinh \pi \kappa f L_1 M_2 L_3 \ldots,$$

$$\Delta_s(N) = \frac{\pi f}{2\kappa^2} \cos \kappa a \sinh \pi \kappa f \frac{1}{s^2} L_1 M_2 \ldots L_{s-1} L_{s+1} \ldots, \text{ (s even),}$$

$$\Delta_s(N) = -\frac{\pi f}{2\kappa^2} \cos \kappa a \sinh \pi \kappa f \frac{1}{s^2} L_1 M_2 \ldots M_{s-1} M_{s+1} \ldots, \text{ (s odd),}$$

so that, neglecting $\kappa^3$ and higher powers,

$$\lambda_0 = \frac{A}{\kappa} \left\{ 1 + \kappa^2 \left( \frac{1}{2} a^2 - \frac{\pi^2}{24 f^2} \right) \right\},$$

$$\lambda_s \text{ even} = 2iA \frac{\kappa^2 }{ f^2 } \frac{1}{ s^2 \cosh sa },$$

$$\lambda_s \text{ odd} = 4iA \frac{\kappa^2 a }{ \pi f } \frac{1}{ s^3 \sinh sa }.$$

Hence, substituting in (3.8), the following expression for $\zeta$ is obtained,

$$\frac{\zeta}{A} = \left\{ 1 + \kappa^2 \left( \frac{1}{2} a^2 - \frac{\pi^2}{24 f^2} \right) \right\} \left\{ x - i \left( y - \frac{1}{2} \pi \right) \right\} - \frac{\kappa^2}{2 f^2} \sum_{s=1}^{even} \frac{1}{s^2 \cosh sa} \left( \sinh sx \cos sy - i \frac{f}{2} \cosh sx \sin sy \right)$$

$$+ \frac{4i\kappa^2}{\pi f^2} \sum_{s=1}^{odd} \frac{1}{s^3 \sinh sa} \left( \cosh sx \cos sy - i \frac{f}{2} \sinh sx \sin sy \right).$$

For $\kappa^2 = 0$, this gives

$$\zeta = A \{ x - i(y - \frac{1}{2} \pi) \} / f = \tilde{\zeta},$$

the equilibrium-form. Retaining $\kappa^2$ and introducing $\tilde{\zeta} = \zeta - \tilde{\zeta}$, it may be verified by direct substitution that the expression for $\tilde{\zeta}$ satisfies the equations (2.5) and (2.6).

Transferring the origin to the centre of the rectangle and taking now the lengths of the sides to be $2a$ and $2b$ (see fig. 2) then, corresponding to

$$\zeta = A \{ x - i(y + \frac{1}{2} \pi) \} / f + B \{ y + i(x + \frac{1}{2} \pi) \},$$
the following expression for $\zeta'$, to the order which includes terms in $\kappa^2$, may be derived easily,

$$
\frac{\zeta'}{ab} = A\left\{\frac{1}{2}\left(1 - \frac{b^3}{3f^2a^3}\right)(x - iy) - \frac{1}{6}(a - f)\right\} - \frac{1}{\pi^3f^3a^3}\sum_{s}^{\text{even}}\frac{1}{s^3\cosh s\pi a/2b}\left(\sinh s\pi x/2b \cos s\pi(y + b)/2b - \frac{i}{f}\cosh s\pi x/2b \sin s\pi(y + b)/2b\right)

+ \frac{16i\pi^2b^2}{\pi^3f^2a^3}\sum_{s}^{\text{odd}}\frac{1}{s^3\sinh s\pi b/2a}\left(\cosh s\pi x/2b \cos s\pi(y + b)/2b - \frac{i}{f}\sinh s\pi x/2b \sin s\pi(y + b)/2b\right) + \frac{16i}{\pi^3f^2a^3}\sum_{s}^{\text{even}}\frac{1}{s^3\cosh s\pi b/2a}\left(\sinh s\pi y/2a \cos s\pi(x + a)/2a + \frac{i}{f}\cosh s\pi y/2a \sin s\pi(x + a)/2a\right)

- \frac{16i\pi b}{\pi^3f^3a}\sum_{s}^{\text{odd}}\frac{1}{s^3\sinh s\pi b/2a}\left(\cosh s\pi y/2a \cos s\pi(x + a)/2a + \frac{i}{f}\sinh s\pi y/2a \sin s\pi(x + a)/2a\right)ight\} \tag{4.1}

A numerical illustration, in connection with this solution for the tides of a deep rectangular basin, will be given in the following section.

Considering next the case of a deep rectangular gulf, sides $2a$ and $\pi$, open across $x = a$ (see fig. 1), it is proposed to obtain an expression for $\zeta$ which corresponds to a uniform current $U\text{e}^{i\sigma t}$ across $x = a$.

For this take

$$
\zeta_1 = l_0\zeta_0 + \sum_{s}^{\text{even}}l_s\zeta_s + \sum_{s}^{\text{odd}}l_s\zeta_s',
$$

and for $x = a$ equate the constant term to $-i\sigma U/g$ and the coefficient of $\cos ry$ to zero. The elevation $\zeta_1$ will then correspond, omitting the time-factor, to uniform currents $\frac{1}{2}U$ across $x = a$ and $-\frac{1}{2}U$ across $x = -a$. 

![Fig. 2.](https://academic.oup.com/gsmnras/article-abstract/2/8/385/562124/28/385562124)
Next, taking

\[ \zeta_2 = m_0 \zeta_0' + \sum_{s \text{ even}} m_s \zeta_s' + \sum_{s \text{ odd}} m_s \zeta_s, \]

and for \( x = a \) equating the constant term to \(-i \sigma U/g\) and the coefficient of \( \cos \pi y \) to zero, the elevation obtained will correspond to uniform currents \( \frac{1}{2} U \) across \( x = \pm a \).

Hence

\[ \zeta = \zeta_1 + \zeta_2 \]

is the expression required, i.e. corresponding to the current \( U \) across the mouth of the gulf.

For \( \zeta_1 \) the following equations to determine \( l_s \) are obtained:

\[ \begin{align*}
- \frac{4 \kappa^2}{\pi} & f l_0 \sin \pi \kappa / 2f \\
- \frac{4 \kappa^2}{\pi^2} & f_1 \cos \pi \kappa / 2f + l_1 \mu_1 \cos \pi \kappa / 2f + 4 \kappa^2 \sum_{s \text{ odd}} l_s \frac{s}{s^2 - 1} \cos \pi \kappa / 2f = 0, \\
- \frac{4 \kappa^2}{\pi^2} & f_2 \sin \pi \kappa / 2f - 2 l_2 \mu_2 \sin \pi \kappa / 2f + 4 \kappa^2 \sum_{s \text{ even}} l_s \frac{s}{s^2 - 1} \sin \pi \kappa / 2f = 0,
\end{align*} \]

and, neglecting all terms in which either \( r \) or \( s \) is greater than \( N \) and then solving the equations, we have

\[ l_0 = \Delta_0 (N)/\Delta (N), \]

\[ \frac{is}{\sin \pi \kappa / 2f} \frac{\sin \mu a}{\sin \kappa a \sinh \pi \kappa / 2f} = \frac{\Delta_i (N)}{\Delta (N)}, \quad (s \text{ odd}), \]

\[ \frac{s}{\cos \pi \kappa / 2f} \frac{\cos \mu a}{\cos \kappa a \sinh \pi \kappa / 2f} = \frac{\Delta_s (N)}{\Delta (N)}, \quad (s \text{ even}), \]

where \( \Delta (N) \) is as in (3.7) and \( \Delta_s (N) \) differs only from \( \Delta (N) \) in having

\[ -\frac{\pi f}{4 \kappa^2 \sin \kappa a \sinh \pi \kappa / 2f}, \quad \sigma, \sigma, \sigma, \ldots \]

in the \((s + 1)\)th column.

On approximating for \( \kappa^2 \) small, it is noted that the order of magnitude of the coefficients \( l_s \) is such that it may be sufficient in applications to neglect \( \kappa^2 \) and higher powers. Thus

\[ \Delta (N) = - \frac{f^2}{\kappa^2} L_1 M_2 L_3 \ldots, \]

\[ \Delta_0 (N) = - \frac{\pi f}{4 \kappa^2 \sin \kappa a \sinh \pi \kappa / 2f} L_1 M_2 L_3 \ldots, \]

\[ \Delta_s (N) = - \frac{\pi f}{4 \kappa^2 \sin \kappa a \sinh \pi \kappa / 2f} \frac{1}{s^2} L_1 M_2 \ldots L_{s-1} L_{s+1} \ldots, \quad (s \text{ even}), \]

\[ \Delta_s (N) = \frac{\pi f}{4 \kappa^2 \sin \kappa a \sinh \pi \kappa / 2f} \frac{1}{s^2} L_1 M_2 \ldots L_{s-1} M_{s+1} \ldots, \quad (s \text{ odd}), \]
so that
\[
  l_0 = \frac{i\sigma U}{g} \frac{1}{2\kappa^2 a} \left( 1 + \frac{i}{4\kappa^2 \left( a^2 - \frac{\pi^2}{4f^2} \right) \right),
\]
and to this order
\[
  l_{\text{even}} = 0, \quad l_{\text{odd}} = \frac{i\sigma U \left( 1 \right)}{g \pi f \frac{1}{\sinh sa} + 2i \sum_{s} \frac{1}{\sinh sa}},
\]
so that
\[
  \frac{aG_1}{isU} = \frac{1}{2\kappa^2} + \frac{1}{4\kappa^2} \left( a^2 - \frac{\pi^2}{4f^2} \right) - \frac{1}{f} \left( x - i(y - \frac{\pi}{2}) \right)^2.
\]

The corresponding expression for \( \zeta_2 \) is, from the analysis for the forced tides in a deep rectangular basin
\[
  \frac{ag\zeta_2}{isU} = \frac{1}{2} \left( x - i(y - \frac{\pi}{2}) \right),
\]
so that
\[
  \frac{ag\zeta}{isU} = \frac{1}{2\kappa^2} + \frac{1}{12} \left( a^2 - \frac{\pi^2}{f^2} \right) - \frac{1}{2} \left( x - i(y - \frac{\pi}{2}) \right)^2
\]
and to this order
\[
  \frac{2i}{\pi f} \sum_{s} \frac{1}{\sinh sa} \left( \sinh sx \cos sy - \frac{1}{f} \cosh sx \sin sy \right).
\]

It may be verified by direct substitution that this expression for \( \zeta \) satisfies, to the order mentioned, the equations
\[
  \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \kappa^2 \left( 1 - \frac{1}{f^2} \right) \zeta = 0,
\]
\[
  \frac{2\omega}{\rho \kappa^2 \left( 1 - \frac{1}{f^2} \right)} \left( i \frac{\partial \zeta}{\partial n} + \frac{\partial \zeta}{\partial s} \right) = U \text{ on } x = a,
\]
\[
  = 0 \text{ on } x = -a, \quad y = 0 \text{ and } y = \pi.
\]

For the rectangle, sides 2a and 2b, and for axes as indicated in fig. 2, the expression for the elevation becomes
\[
  \frac{g\zeta}{isUa} = \frac{1}{2b} + \frac{1}{12} \left( 1 - \frac{b^2}{a^2} \right) - \frac{1}{2} \left( x - i(y - \frac{\pi}{2}) \right)^2
\]
\[
  + \frac{8ib^2}{\pi^2 fa^2} \sum_{s} \frac{1}{\sinh sa} \left( \sin \frac{s\pi x}{2b} \cos \frac{s\pi(y + b)}{2b} - \frac{i}{f} \cosh \frac{s\pi x}{2b} \sin \frac{s\pi(y + b)}{2b} \right).
\]

Numerical Considerations.

§ 5. It is proposed in this present section to test the validity of the application of "narrow-sea" methods in two or more directions to account for the tides of non-elongated enclosed seas, and in this connection particular reference is made to a discussion on the spring tides of the Black Sea by Sterneck (loc. cit.). He replaced the basin by a rectangular one, having respectively a length and breadth approxi-
Dr. S. F. Grace, Tidal Oscillations in Rotating

2. 8,

mately equal to the mean length and breadth of the sea and a uniform
depth which gave the longest free period of the Black Sea; he then
calculated the forced longitudinal and transverse oscillations on the
hypothesis of no rotation and on these superposed oscillations due to
the earth's rotation in accordance with the "narrow-sea" theory.

Taking Sterneck's values for a rectangular basin having the approxi-
mate dimensions and position of the Black Sea, it is assumed that

\[ a = 500 \text{ km.}, \quad b = 250 \text{ km.}, \quad h = 1259 \text{ m.}, \]

\[ \lambda_0 = \text{mean latitude} = 43^\circ, \quad g = 980.4 \text{ cm./sec.}^2, \]

so that for the spring tides

\[ \sigma = 1.419 \times 10^{-4} \text{ sec.}^{-1}, \quad f = 1.427, \quad \beta = \sigma^2 a^2 / gh = 0.4077. \]

The equilibrium-form is *

\[ \xi = \frac{2H a \cos \lambda_0 (x/a - \sin \lambda_0 y/a)}{R} e^{i \omega t} \]

\[ = 4.081 \left( \frac{x}{a} - 0.682 \cos \lambda_0 \frac{y}{a} \right) e^{i \omega t} \text{ cm.,} \]

so that in the expression (3.1) it is necessary to substitute

\[ A = \frac{2H \cos \lambda_0}{R} \frac{1}{1 - \frac{1}{f^2} \left( 1 + \sin \lambda_0 \right)}, \]

\[ B = -\frac{2H \cos \lambda_0}{R} \frac{1}{1 - \frac{1}{f^2} \left( 1 + \sin \lambda_0 \right)}. \]

For a deep lake Sterneck's expression for \( \zeta' \), to the same order of
magnitude as (4.1), is equivalent to

\[ \zeta' = \frac{H \cos \lambda_0}{a \beta} \left[ \frac{x}{a} - \frac{1}{3} \frac{x^3}{a^3} - \frac{\sin \lambda_0}{4f} \frac{x}{a^2} \left( 1 - \frac{4y^2}{a^2} \right) \right] \]

\[ - \frac{H \cos \lambda_0}{R} \left( \sin \lambda_0 \frac{y}{4a^2} - \frac{1}{3} \frac{y^3}{a^3} - \frac{y}{fa} \left( 1 - \frac{x^2}{a^2} \right) \right). \]

To effect a comparison between the expressions for \( \zeta' \) given in (4.1)
and (5.1) it will be sufficient, on account of asymmetry, to restrict
consideration to the quadrant in which \( 0 < x < a \) and \( 0 < y < b \).

First, the values of \( \zeta \), \( \zeta' \) (corresponding to (4.1)) and \( \zeta_0 \) (corresponding
to (5.1)) are given below at certain points of the quadrant, the time-
factor \( e^{i \omega t} \) for convenience being omitted.

\[
\begin{array}{ccc}
\zeta_{(cm.)} & \zeta'_{(cm.)} & \zeta_0_{(cm.)} \\
\hline
x = a, y = 0, & 4.081i, & 4.563i, & 4.536i, \\
x = 0, y = b, & -1.392, & -1.183, & -1.438, \\
x = \frac{a}{2}, y = \frac{b}{2}, & -0.696 + 2.040i, & -0.626 + 2.365i, & -0.619 + 2.384i, \\
x = a, y = b, & -1.392 + 4.081i, & -1.329 + 4.577i, & -1.439 + 4.635i. \\
\end{array}
\]

* Cf. H. Lamb, Hydrodynamics, Appendix to Chapter VIII.
June 1931.  Rectangular Basins of Uniform Depth.

It appears, therefore, that both $\zeta$ and $\zeta_5$ differ but little from $\bar{\zeta}$ so that (4.1) and (5.1) are equivalent to small "corrections" to the equilibrium-form.

In order to compare these "corrections" we have the following values:

<table>
<thead>
<tr>
<th>At $x = a, y = 0$</th>
<th>$\zeta'$ (cm.)</th>
<th>$\zeta_5'$ (cm.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = a, y = b$</td>
<td>0.482i</td>
<td>0.455i</td>
</tr>
<tr>
<td>$x = \frac{a}{2}, y = \frac{b}{2}$</td>
<td>0.209, 0.325i</td>
<td>0.244, 0.344i</td>
</tr>
</tbody>
</table>

Hence (5.1) does not differ greatly from (4.1) except near the corners of the rectangle. The difference between the two expressions is, however, seen more clearly by considering separately equilibrium-forms of the types $\bar{\zeta} = kx$ and $\bar{\zeta} = ly$, where $k$ and $l$ are constants.

For $\bar{\zeta} = kx$ we obtain

\[
\begin{align*}
\zeta'^{(1 - 1/f^2)}_{\beta a k} & = 0.1696, \\
x = a, y = 0 & = 0.1696, \\
& = 0.0323i, \\
x = a, y = b & = 0.0212, \\
& = 0.0212 |
\end{align*}
\]

and corresponding to $\bar{\zeta} = ly$

\[
\begin{align*}
\zeta'^{(1 - 1/f^2)}_{\beta a l} & = 0.1696, \\
x = a, y = 0 & = 0.0323i, \\
& = 0.0212, \\
x = a, y = b & = 0.0212 + 0.0264i, \\
& = 0.0212 |
\end{align*}
\]

while for a square basin of side $2a$ and $\bar{\zeta} = kx$

\[
\begin{align*}
\zeta'^{(1 - 1/f^2)}_{\beta a k} & = 0.1696, \\
x = a, y = 0 & = 0.1696, \\
& = 0.1396i, \\
x = a, y = a & = 0.1697 - 0.0912i, \\
& = 0.1696 |
\end{align*}
\]

It is now noted that (5.1) gives the correct value along the axis in the direction of the disturbing force, but the values are more or less in error elsewhere, the degree of error being least, as might be expected, when the disturbing force acts parallel to the longer sides of the rectangle and vice versa; the degree of error for a square basin is intermediate to these.

Hence for deep basins the equilibrium tide predominates, so that both (4.1) and (5.1) are small "corrections" which do not affect greatly the equilibrium-form. The above considerations show, however, that the "correction" (5.1) may be considerably in error in certain portions of the basin. It thus appears that the agreement between theory and observation, which was obtained by Sterneck in his discussion on the spring tides of the Black Sea, was mainly due to the fact that the equilibrium-form predominates.
In connection with the view that the agreement between Sterneck's results and the observed tides in the Black Sea is mainly due to the fact that the equilibrium tide predominates so that the differences between the correct dynamical calculation and that of Sterneck is only seen when the equilibrium tide is abstracted, it has been suggested by a referee that this point would be more forcible if figures are given showing the observed differences between the actual and the equilibrium tides in the Black Sea.

Such figures are given in the following table, which contains the observed amplitudes and phases of the spring tides at the five stations for which observations are available, together with the corresponding equilibrium values and also the differences between the two tides; the amplitudes are denoted by $H$ and the phases, referred to the mean meridian of the basin, by $\gamma'$. The observed values are taken from Sterneck's last paper on the tides of the Black Sea.*

<table>
<thead>
<tr>
<th>Station</th>
<th>Observed $H$ (cm.)</th>
<th>$\gamma'$</th>
<th>Equilibrium $H$ (cm.)</th>
<th>$\gamma'$</th>
<th>Difference $H$ (cm.)</th>
<th>$\gamma'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constanta</td>
<td>3.9</td>
<td>96</td>
<td>3.6</td>
<td>100</td>
<td>0.3</td>
<td>4</td>
</tr>
<tr>
<td>Odessa</td>
<td>5.4</td>
<td>150</td>
<td>2.9</td>
<td>135</td>
<td>2.5</td>
<td>166</td>
</tr>
<tr>
<td>Sevastopol</td>
<td>0.7</td>
<td>102</td>
<td>0.9</td>
<td>152</td>
<td>0.2</td>
<td>22</td>
</tr>
<tr>
<td>Theodosia</td>
<td>1.1</td>
<td>237</td>
<td>1.4</td>
<td>218</td>
<td>0.7</td>
<td>353</td>
</tr>
<tr>
<td>Poti</td>
<td>4.1</td>
<td>279</td>
<td>5.1</td>
<td>277</td>
<td>1.0</td>
<td>89</td>
</tr>
</tbody>
</table>

The comparatively large difference for Odessa is probably due to the fact that this station lies near the inner end of a large shallow bay.

It is of interest to compare the above values with the theoretical values for the spring tides derived by Sterneck in his last paper, where he took into account the form of the basin. He calculated the longitudinal oscillation, using the "narrow-sea" method of step-by-step integration, and then superposed on this two transverse oscillations, one given by the equilibrium theory and the other arising from the effect of the earth's rotation on the longitudinal oscillation. His values, together with their differences from the observed values, are

<table>
<thead>
<tr>
<th>Station</th>
<th>Observed $H$ (cm.)</th>
<th>$\gamma'$</th>
<th>Equilibrium $H$ (cm.)</th>
<th>$\gamma'$</th>
<th>Difference $H$ (cm.)</th>
<th>$\gamma'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constanta</td>
<td>4.5</td>
<td>102</td>
<td>0.7</td>
<td>175</td>
<td>3.8</td>
<td>315</td>
</tr>
<tr>
<td>Odessa</td>
<td>3.0</td>
<td>126</td>
<td>2.9</td>
<td>359</td>
<td>0.1</td>
<td>316</td>
</tr>
<tr>
<td>Sevastopol</td>
<td>1.1</td>
<td>141</td>
<td>0.4</td>
<td>68</td>
<td>0.7</td>
<td>316</td>
</tr>
</tbody>
</table>

Thus Sterneck's values appear to be inferior to those given by the equilibrium theory.